Numerical implementation of a one-loop tensor reduction

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(DESY, ZEUTHEN)

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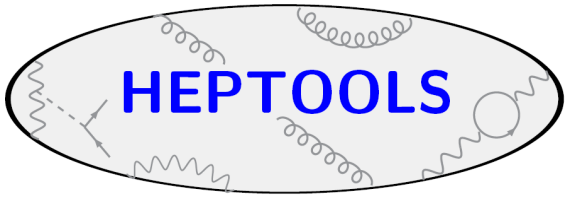
Based on [**arXiv:hep-ph/0807.2984, 0812.2134, 0901.4455, 0907.2115]**
Sponsored by EU-program:



Motivation and goals

- Recent years have seen the emergence of first results for $2 \rightarrow 4$ scattering processes

- One of the challenges posed is the need to compute one-loop tensor integrals with up to 6 legs

- We provide numerical results from compact analytic formulas for the complete reduction of tensor integrals to scalar master integrals

From bubbles to hexagons

- Implementation to Fortran and Mathematica code

Fortran

- For Tensor integrals, we have a Fortran implementation package (Th. Diakonidis & B. Tausk)

The present implementation includes:

- Six point functions up to rank four (Hexagon.F)
- Five point functions (all 5 ranks) (Pentagon.F)
- Boxes (all 4 ranks) (Box.F)
- Triangles (all 3 ranks) (Triangle.F)
- Bubbles (all 2 ranks) (Bubble.F)

It is able to output the full result for:

- The tensor integrals

- The code so far uses:

Looptools 2.2 (by Thomas Hahn)
(calculates only the finite part)
QCDLoop (R.K. Ellis and G. Zanderighi)
(Finite part and $1/\varepsilon$ and $1/\varepsilon^2$ terms)

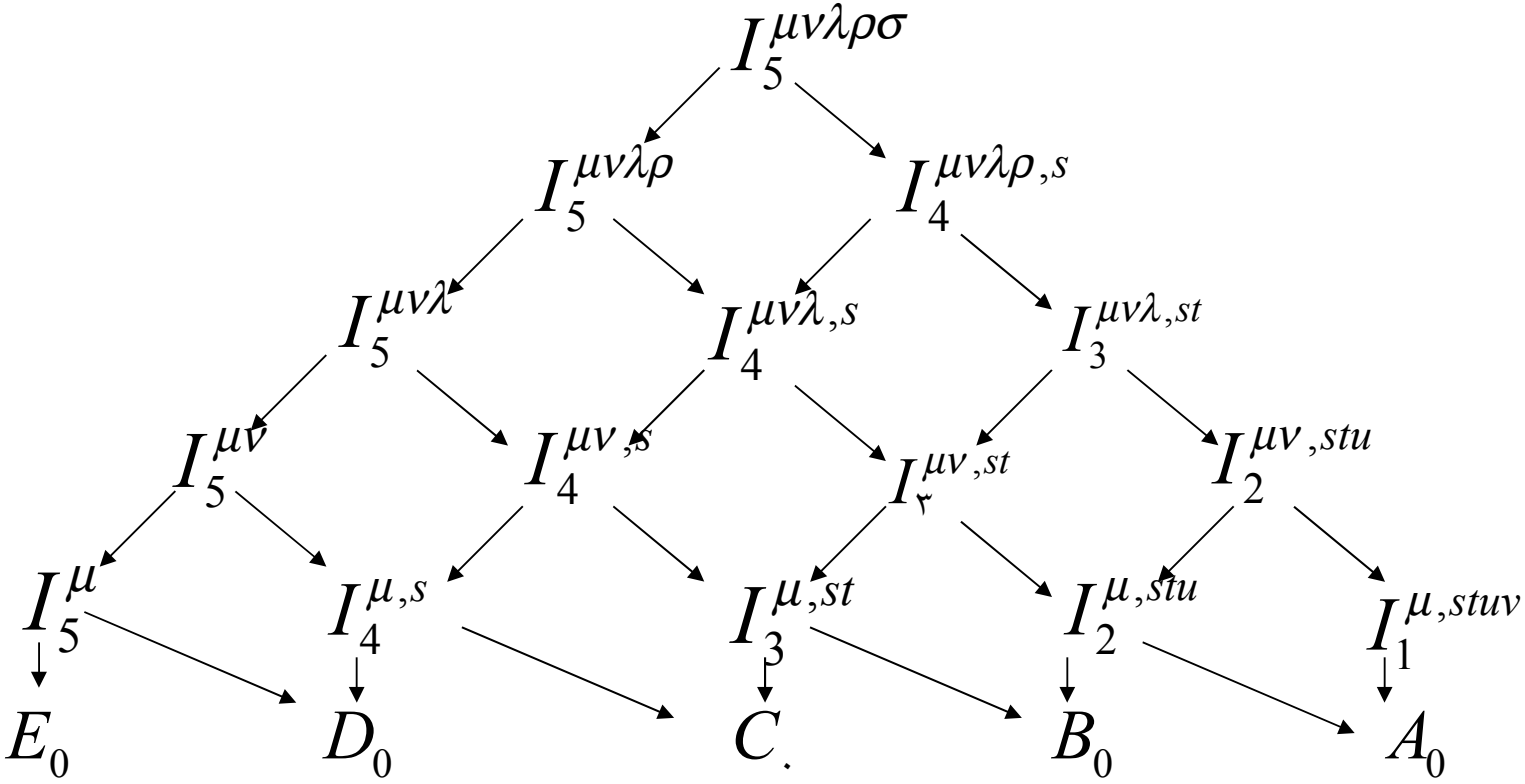
To calculate the scalar master integrals after the reduction

(The first is restricted to massive cases but the second can be implemented for massless cases too)

- It can be adapted to any Fortran package for 1,2,3,4 point functions

- A lot of cross checks have been done so far (shown after) and we also cross checked the results with an independent code by Peter Uwer

The triangle



Here we have to add some extra terms in the cases of boxes, triangles and bubbles with the exception of 1st rank (See Jochem's talk)

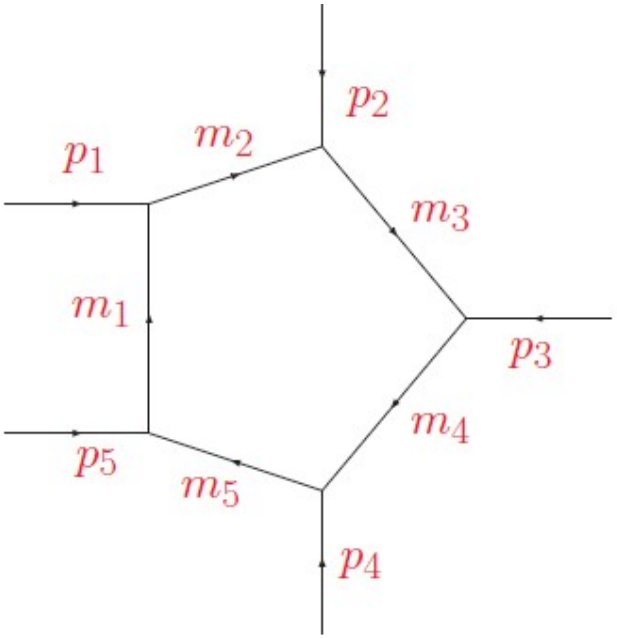
Starting from a pentagon

For the randomly chosen phase space point:

p_1	5.00000000000 E+00	0.0000000000 E+00	0.00000000000 E+00	4.00000000000 E+00
p_2	5.00000000000 E+00	0.0000000000 E+00	0.00000000000 E+00	-4.00000000000 E+00
p_3	– 0.30770034895 E+01	0.5359484673 E+00	–0.37447035150 E+00	– 0.20120057390 E+00
p_4	– 0.34048537280 E+01	0.2184763540 E-01	– 0.10479394969 E+01	0.12224460727 E+01
p_5	– 0.35181427825 E+01	– 0.5577961027 E+00	0.14224098484 E+01	– 0.10212454988 E+01
$m_1 = 0.0,$	$m_2 = 2.0,$	$m_3 = 3.0,$	$m_4 = 4.0,$	$m_5 = 5.0$

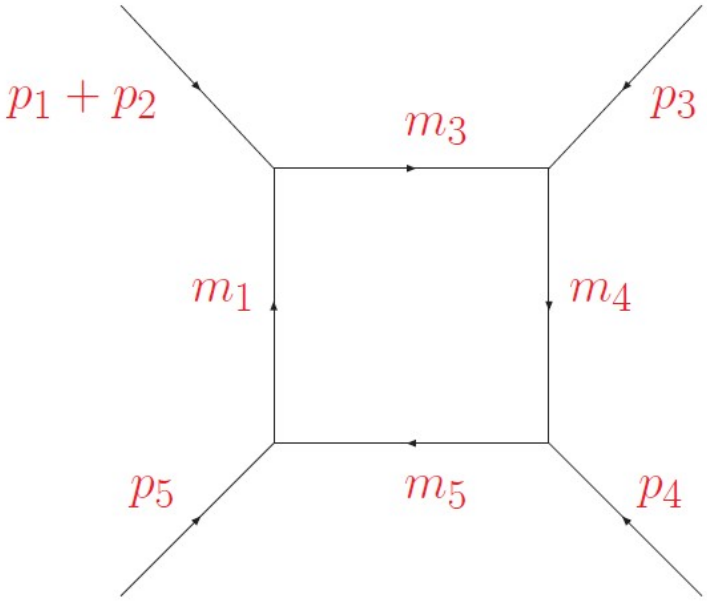
A mixed case of massless and massive particles

Pentagon



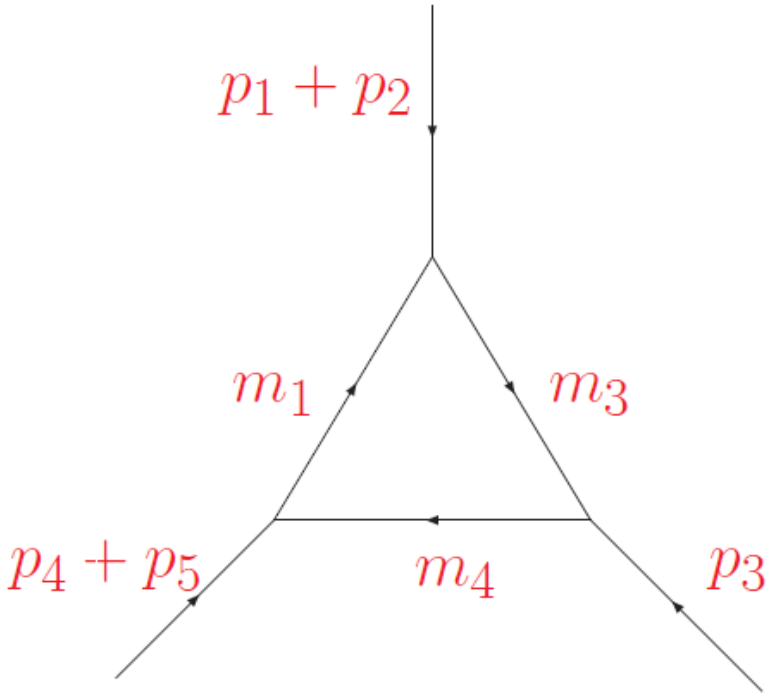
	Pentagon.F
E^2	(2.80450709388539E-05 , -1.08461817406464E-05)
E^{12}	(-5.4133397866730 1E-06 , 6.26985967678899E-06)
E^{232}	(-1.20374858970726E-04 , 4.07974751672555E-04)
E^{0321}	(-9.1119453570372 7E-06 , 4.39187998675819E-05)
E^{01230}	(4.37928367160152E-05 , -2.18183151665913E-04)

Box case



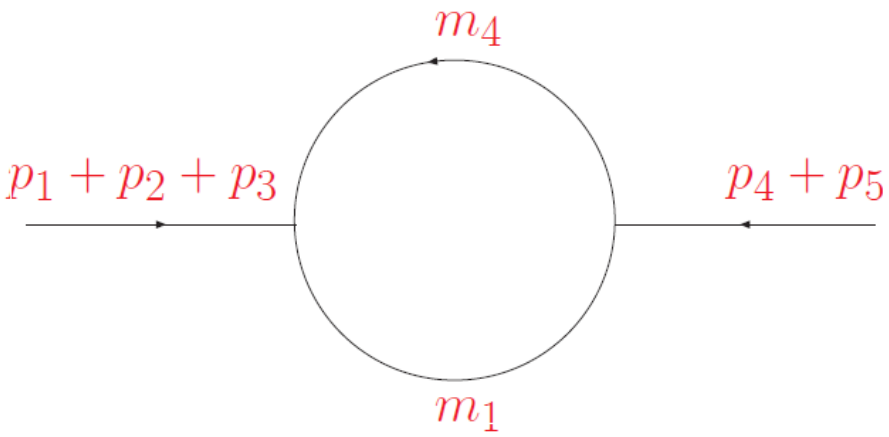
	Box.F	*LoopTools*
D^{1}	(6.81403420828588E-03 , -5.74298462683219E-03)	(6.8140342082847463E-03 , -5.7429846268324187E-03)
D^{33}	(2.40138809967981E-03 , 1.11591328775015E-02)	(2.4013880996803092E-03,1.1159132877500448E-02)
D^{212}	(-1.69702786278243E-03 , -2.83731121595478E-03)	(-1.6970278627700630E-03,-2.8373112159962330E-03)
D^{0123}	(-1.92190388316994E-04 , -4.04730302413490E-04)	(-1.9219038693301300E-04,-4.0473030187772325E-04)

Triangle



	Triangle.F	*LoopTools*
C^2	(2.44757827793318E-04 , -7.50688449850356E-03)	(2.4475782779342707E-04,-7.5068844985030472E-03)
C^{01}	(-1.28259813172255E-02 , -6.73809718907549E-02)	(-1.2825981317215014E-02,-6.7380971890795340E-02)
C^{133}	(-7.00360822297110E-02 , 7.24628606014397E-02)	(-7.0036082229746830E-02,7.2462860601566081E-02)

Bubble



	Bubble.F	*LoopTools*
B^3	(-0.141525070262337E+00 , 0.138870631815383E+00)	(-0.141525070262336,0.138870631815382)
B^{12}	(0.102490343329085E+00 , -6.12154531068256E-02)	(0.102490343329084,-6.12154531068257E-02)

Some sample results for hexagons

For the randomly chosen phase space point given by:

$$
\begin{aligned}
p_1 &= (0.21774554E+03, \quad 0, \quad 0, \quad 0.21774554E+03) \\
p_2 &= (0.21774554E+03, \quad 0, \quad 0, \quad -0.21774554E+03) \\
p_3 &= (-0.20369415E+03, \quad -0.47579512E+02, \quad 0.42126823E+02, \quad 0.84097181E+02) \\
p_4 &= (-0.20907237E+03, \quad 0.55215961E+02, \quad -0.46692034E+02, \quad -0.90010087E+02) \\
p_5 &= (-0.68463308E+01, \quad 0.53063195E+01, \quad 0.29698267E+01, \quad -0.31456871E+01) \\
p_6 &= (-0.15878244E+02, \quad -0.12942769E+02, \quad 0.15953850E+01, \quad 0.90585932E+01) \\
& m_1 = 110.0,\ m_2 = 120.0,\ m_3 = 130.0,\ m_4 = 140.0,\ m_5 = 150.0,\ m_6 = 160.0
\end{aligned}
$$

Results for scalar, vector and 2nd rank six point functions:

		RESULTS	
		REAL	IM
		F_0	
		-0.223393E-18	-0.396728E-19
μ		F^μ	
0		0.192487E-17	0.972635E-17
1		-0.363320E-17	-0.11940E-17
2		0.365514E-17	0.106928E-17
3		0.239793E-16	0.341928E-17
μ	ν	$F^{\mu\nu}$	
0	0	0.599459E-14	-0.114601E-14
0	1	0.323869E-15	0.423754E-15
0	2	-0.294252E-15	-0.375481E-15
0	3	-0.255450E-14	-0.195640E-14
1	1	-0.164562E-14	-0.993796E-16
1	2	0.920944E-16	0.706487E-17
1	3	0.347694E-15	-0.127190E-16
2	2	-0.163339E-14	-0.994148E-16
2	3	-0.341773E-15	0.818678E-17
3	3	-0.413909E-14	0.670676E-15

3rd rank 6 point functions

μ	ν	λ	$F^{\mu\nu\lambda}$
0	0	0	– 0.227754 E–11 – i 0.267244 E–12
0	0	1	0.140271 E–13 – i 0.119448 E–12
0	0	2	– 0.201270 E–13 + i 0.101968 E–12
0	0	3	0.102976 E–12 + i 0.624467 E–12
0	1	1	0.183904 E–12 + i 0.142429 E–12
0	1	2	– 0.131028 E–13 – i 0.610343 E–14
0	1	3	– 0.543316 E–13 – i 0.158809 E–13
0	2	2	0.181352 E–12 + i 0.141686 E–12
0	2	3	0.506408 E–13 + i 0.163568 E–13
0	3	3	0.600542 E–12 + i 0.130733 E–12
1	1	1	– 0.563539 E–13 + i 0.178403 E–13
1	1	2	0.210641 E–13 – i 0.584990 E–14
1	1	3	0.120482 E–12 – i 0.574688 E–13
1	2	2	– 0.201182 E–13 + i 0.620591 E–14
1	2	3	– 0.686164 E–14 + i 0.205457 E–14
1	3	3	– 0.447329 E–13 + i 0.193180 E–13
2	2	2	0.582201 E–13 – i 0.163889 E–13
2	2	3	0.119659 E–12 – i 0.570084 E–13
2	3	3	0.457464 E–13 – i 0.181141 E–13
3	3	3	0.557081 E–12 – i 0.374359 E–12

4th rank 6-point

				REAL	IM
μ	ν	λ	ρ	$F^{\mu\nu\lambda\rho}$	
0	0	0	0	0.666615D-09	0.247562D-09
0	0	0	1	-0.200049D-10	0.294036D-10
0	0	0	2	0.200975D-10	-0.237333D-10
0	0	0	3	0.645477D-10	-0.162236D-09
0	0	1	1	-0.116956D-10	-0.516760D-10
0	0	1	2	0.160357D-11	0.222284D-11
0	0	1	3	0.792692D-11	0.729502D-11
0	0	2	2	-0.111838D-10	-0.513133D-10
0	0	2	3	-0.681086D-11	-0.708933D-11
0	0	3	3	-0.804454D-10	-0.801909D-10
0	1	1	1	0.100498D-10	-0.151735D-13
0	1	1	2	-0.348984D-11	-0.195436D-12
0	1	1	3	-0.211111D-10	0.295212D-11
0	1	2	2	0.357455D-11	0.662809D-14
0	1	2	3	0.121595D-11	-0.807388D-13
0	1	3	3	0.825803D-11	-0.142086D-11
0	2	2	2	-0.958961D-11	-0.585948D-12

				REAL	IM
μ	ν	λ	ρ	$F^{\mu\nu\lambda\rho}$	
0	2	2	3	-0.209232D-10	0.289031D-11
0	2	3	3	-0.802359D-11	0.994701D-12
0	3	3	3	-0.102576D-09	0.378476D-10
1	1	1	1	-0.246426D-10	0.276326D-10
1	1	1	2	0.915670D-12	-0.660629D-12
1	1	1	3	0.303529D-11	-0.287480D-11
1	1	2	2	-0.822697D-11	0.919635D-11
1	1	2	3	-0.116294D-11	0.100024D-11
1	1	3	3	-0.146918D-10	0.183799D-10
1	2	2	2	0.908296D-12	-0.654735D-12
1	2	2	3	0.109510D-11	-0.100875D-11
1	2	3	3	0.717342D-12	-0.557293D-12
1	3	3	3	0.450661D-11	-0.485065D-11
2	2	2	2	-0.245154D-10	0.274313D-10
2	2	2	3	-0.318500D-11	0.279750D-11
2	2	3	3	-0.146317D-10	0.182912D-10
2	3	3	3	-0.477335D-11	0.477368D-11
3	3	3	3	-0.730168D-10	0.112865D-09

More results (massless case)

For the phase space point given by:

p1 =(1, 0, 0, 0)
p2 =(-0.19178191,-0.12741180,-0.08262477,-0.11713105)
p3 =(-0.33662712, 0.06648281, 0.31893785, 0.08471424)
p4 =(-0.21604814, 0.20363139,-0.04415762,-0.05710657)
p5 =-(p1+p2+p3+p4)

M1=0, M2=0, M3=0, M4=0, M5=0

Golem95: T.Binoth, J.-Ph.Guillet, G. Heinrich, E.Pilon, T.Reiter
[arXiv:hep-ph/0810.0992]

Comparisons with golem95

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
E_0	(202.168496, 3211.04072)	(1022.10601 , 972.027061)	(309.405823 ,0)
E_3	(-264.996441,303.068452)	(96.4696846,149.228472)	(47.5008979,0)
E_{44}	(1780.58042 , 2914.50734)	(927.71650 , 568.572069)	(180.982111 ,0)
E_{00}	(9.56327810 , 0)	(0 , 0)	(0,0)
E_{555}	(1035.29689 , 1422.01085)	(452.640112 , 254.226520)	(80.9228146 , 0)
E_{001}	(0.84742102 ,0)	(0,0)	(0,0)

Complete agreement to all the numbers shown

(QCDLoop was used for the scalar master integrals)

Mixed case (Hexagon)

p_1	0.21774554 E+01	0.0	0.0	0.21774554 E+01
p_2	0.21774554 E+01	0.0	0.0	– 0.21774554 E+01
p_3	– 0.20369415 E+01	– 0.47579512 E+00	0.42126823 E+00	0.84097181 E+00
p_4	– 0.20907237 E+01	0.55215961 E+00	– 0.46692034 E+00	– 0.90010087 E+00
p_5	– 0.68463308 E–01	0.53063195 E–01	0.29698267 E–01	– 0.31456871 E–01
p_6	– 0.15878244 E+00	– 0.12942769 E+00	0.15953850 E–01	0.90585932 E–01
$m_1 = 0.0,\ m_2 = 0.0,\quad m_3 = 0.0,\quad m_4 = 1.7430,\quad m_5 = 0.0,\quad m_6 = 0.0$				

It corresponds to the reaction : $gg + t\bar{t}$ $q\bar{q}$

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
F_0	0.2403558675 E+04 – i 0.2058213187 E+03	0.7315208677 E+02 – i 0.4276718518 E+02	– 0.7543148872 E+01 + i 0.0
F^2	0.1112747404 E+03 – i 0.6809282900 E+01	0.4419243474 E+01 – i 0.1201033663 E+01	– 0.1044856909 E+00 + i 0.0
F^{13}	– 0.1014018623 E+02 + i 0.1797332619 E+01	– 0.5914958485 E–01 + i 0.3275539398 E+00	0.7678550480 E–01 + i 0.0
F^{123}	– 0.5007216712 E+00 + i 0.4194342396 E–01	– 0.1642316924 E–01 + i 0.7789453935 E–02	0.1225024390 E–02 + i 0.0
F^{3210}	0.1263455978 E+00 – i 0.6509987460 E–02	0.4610567958 E–02 – i 0.1506637282 E–02	– 0.1945123881 E–03 + i 0.0

Mixed case (Pentagon)

p_1	0.21774554 E+01	0.0	0.0	0.21774554 E+01
p_2	0.21774554 E+01	0.0	0.0	– 0.21774554 E+01
p_3	– 0.20369415 E+01	– 0.47579512 E+00	0.42126823 E+00	0.84097181 E+00
p_4	– 0.20907237 E+01	0.55215961 E+00	– 0.46692034 E+00	– 0.90010087 E+00
p_5	– 0.68463308 E–01	0.53063195 E–01	0.29698267 E–01	– 0.31456871 E–01
p_6	– 0.15878244 E+00	– 0.12942769 E+00	0.15953850 E–01	0.90585932 E–01
$m_1 = 0.0,\ m_2 = 0.0,\quad m_3 = 0.0,\quad m_4 = 1.7430,\quad m_5 = 0.0,\quad m_6 = 0.0$				

Produced by adding together the external momenta : $p_1 = p_1 + p_2$

	ϵ^0	$1/\epsilon$	$1/\epsilon^2$
E_0	– 0.289852933 E+04 + i 0.228935552 E+03	– 0.945038648 E+02 + i 0.454178453 E+02	0.7112330546 E+01 + i 0.0
E^3	0.168344624 E+03 – i 0.181758172 E+02	0.4242553725 E+01 – i 0.338838829 E+01	– 0.6442770877 E+00 + i 0.0
E^{23}	– 0.79409571852 E+01 + i 0.5445326927 E+00	– 0.3008645503 E+00 + i 0.9457613783 E–01	0.1027869989 E–01 + i 0.0
E^{012}	0.2472148936 E+01 – i 0.127011969 E+00	0.9699262574 E-01 – i 0.2560545796 E–01	– 0.2331885086 E–02 + i 0.0
E^{2130}	0.2733228280 E+02 – i 0.519106421 E+02	– 0.909476582 E+01 + i 0.1744459753 E–02	0.2112313083 E–03 + i 0.0

Numerical results (Mathematica)

Mathematica package hexagon.m (by K. Kajda)

The present implementation includes:

- Six point functions up to rank four
- Five point functions up to rank three

It is able to output the full result for:

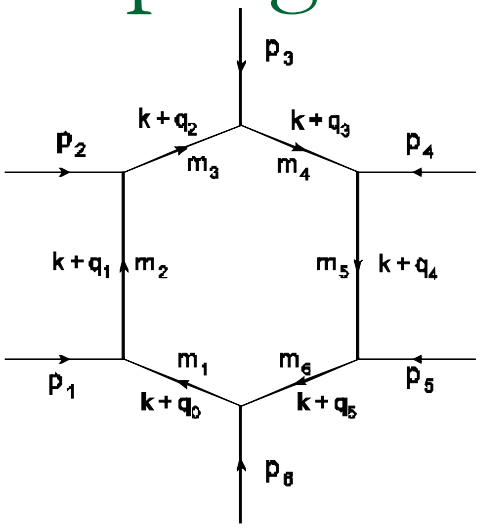
- Six or five point tensor integral
- A specific coefficient for a given rank
- A list of all coefficients of a given rank

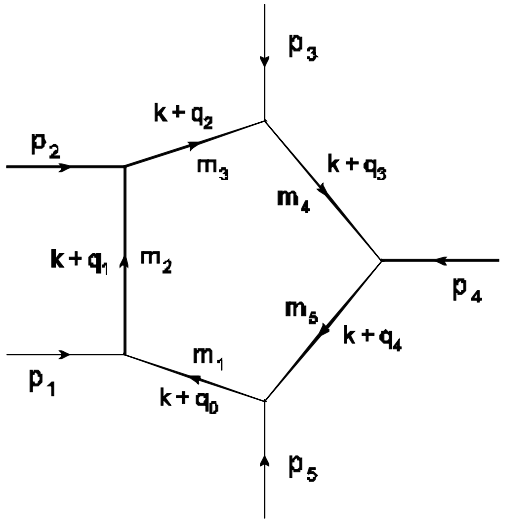
More about the Mathematica program

- They provide coefficients of Lorentz-covariant tensors, and work in a basis of $g^{\mu\nu}$ and internal momenta q_i

$$q_0 = 0, \quad q_n = \sum_{i=1}^{n} p_i$$

- In terms of these coefficients, the tensor decomposition e.g. for pentagons E and hexagons F reads:





$$
\begin{aligned}
E^{\mu} &= \sum_{i=1}^{4} q_i^{\mu} E_i, \\
E^{\mu\nu} &= \sum_{i,j=1}^{4} q_i^{\mu} q_i^{\nu} E_{ij} + g^{\mu\nu} E_{00}, \\
E^{\mu\nu\lambda} &= \sum_{i,j,k=1}^{4} q_i^{\mu} q_i^{\nu} q_k^{\lambda} E_{ijk} + \sum_{i=1}^{4} g^{[\mu\nu} q_i^{\lambda]} E_{00i}, \\
F^{\mu} &= \sum_{i=1}^{5} q_i^{\mu} F_i, \\
F^{\mu\nu} &= \sum_{i,j=1}^{5} q_i^{\mu} q_i^{\nu} F_{ij}, \\
F^{\mu\nu\lambda} &= \sum_{i,j,k=1}^{5} q_i^{\mu} q_i^{\nu} q_k^{\lambda} F_{ijk} + \sum_{i=1}^{5} g^{\mu\nu} q_i^{\lambda} F_{00i}, \\
F^{\mu\nu\lambda\rho} &= \sum_{i,j,k,l=1}^{5} q_i^{\mu} q_i^{\nu} q_k^{\lambda} q_l^{\rho} F_{ijkl} \\
&\quad + \sum_{i,j=1}^{5} q_i^{\mu} q_j^{[\nu} g^{\lambda\rho]} F_{00ij}
\end{aligned}
$$

Functions used in the package

Six point functions		Five point functions	
`RedF0`	scalar 6pt integral	`RedE0`	scalar 5pt integral
`RedF1`	vector 6pt integral	`RedE1`	vector 5pt integral
`RedF2`	rank two 6pt tensor integral	`RedE2`	rank two 5pt tensor integral
`RedF3`	rank three 6pt tensor integral	`RedE3`	rank three 5pt tensor integral
`RedF4`	rank four 6pt tensor integral		
`RedFcoef`	coefficient of given 6pt	`RedEcoef`	coefficient of given 5pt
`RedFget`	all coefficients of given 6pt	`RedEget`	all coefficients of given 5pt

The basic functions have the following arguments, here $s_{ij} = (p_i + p_j)^2$, $s_{ijk} = (p_i + p_j + p_k)^2$:

`RedF0`$[p_1^2, \ldots, p_6^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}, m_1^2, \ldots, m_6^2]$

`RedE0`$[p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2]$

Numerical cross checks

1. Comparison with AMBRE & MB.m $p_1^\mu p_2^\nu p_3^\lambda E_{\mu\nu\lambda}$

Point:

$p_1^2 = p_2^2 = p_3^2 = p_5^2 = 1,\ p_4^2 = 0,\ m_1^2 = m_3^2 = 0,\ m_2^2 = m_4^2 = m_5^2 = 1,$

$s_{12} = -3,\ s_{23} = -6,\ s_{34} = -5,\ s_{45} = -7,\ s_{15} = -2$

In: RedE3[$p_1^2, \ldots, p_5^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{15}, m_1^2, \ldots, m_5^2$]/.{D4->D0,C3->C0,B2->B0}

Out: 0.218741

2. Comparison with Sector Decomposition : F_0

Point:

$p_1^2 = p_2^2 = p_3^2 = p_4^2 = p_5^2 = p_6^2 = -1,\ m_1^2 = m_2^2 = m_3^2 = m_4^2 = m_5^2 = m_6^2 = 1,$

$s_{12} = s_{23} = s_{34} = s_{45} = s_{56} = s_{16} = s_{123} = s_{234} = -1,\ s_{345} = -5/2$

In: RedF0[$p_1^2, \ldots, p_6^2, s_{12}, s_{23}, s_{34}, s_{45}, s_{56}, s_{16}, s_{123}, s_{234}, s_{345}, m_1^2, \ldots, m_6^2$]/.{D4->D0}

Out: 0.013526

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Numerical cross checks

3. Comparison with LoopTools : E_0, E_1, E_2, E_3, E_4, E_{34}, E_{123}, E_{002}

Point:

$p_1^2 = p_2^2 = 0$, $p_3^2 = p_5^2 = 49/256$, $p_4^2 = 9/100$, $m_1^2 = m_2^2 = m_3^2 = 49/256$, $m_4^2 = m_5^2 = 81/1600$,
$s_{12} = 4$, $s_{23} = -1/5$, $s_{34} = 1/5$, $s_{45} = 3/10$, $s_{15} = -1/2$

```
In: RedE0[ p1^2,...,p5^2,s12,s23,s34,s45,s15,m1^2,...,m5^2 ]/.D4->D0
Out: 41.3403 - 45.9721*I
In: RedEget[rank1 ,p1^2,...,p5^2,s12,s23,s34,s45,s15,m1^2,...,m5^2 ]/.D4->D0
Out: ee1 =-2.38605 + 5.27599*I, ee2 =-5.80875 + 0.597891*I,
     ee3 =-14.4931 + 20.8149*I, ee4 =-11.3362 + 18.1593*I
In: RedEcoef[ee34 ,p1^2,...,p5^2,s12,s23,s34,s45,s15,m1^2,...,m5^2 ]/.{D4->D0,C3->C0}
Out: 7.1964 + 3.10115*I
In: RedEcoef[ee123 ,p1^2,...,p5^2,s12,s23,s34,s45,s15,m1^2,...,m5^2 ]/.{D4->D0,C3->C0,B2->B0}
Out:-0.149527 - 0.31059*I
In: RedEcoef[ee002 ,p1^2,...,p5^2,s12,s23,s34,s45,s15,m1^2,...,m5^2 ]/.{D4->D0,C3->C0,B2->B0}
Out: 0.154517 - 0.387727*I
```


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Conclusions

- The full numerical implementation of the analytical reduction of one-loop tensor integrals down to scalar master integrals has been described
- Reduction formulas have been implemented in a Mathematica and a Fortran program
- Results and cross checks with other programs were shown
- Mathematica program publicly available at:

http://www-zeuthen.desy.de/theory/research/CAS.html