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Geometrical methods in loop calculations

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partly based on work with **R. Delbourgo** and **M. Yu. Kalmykov**

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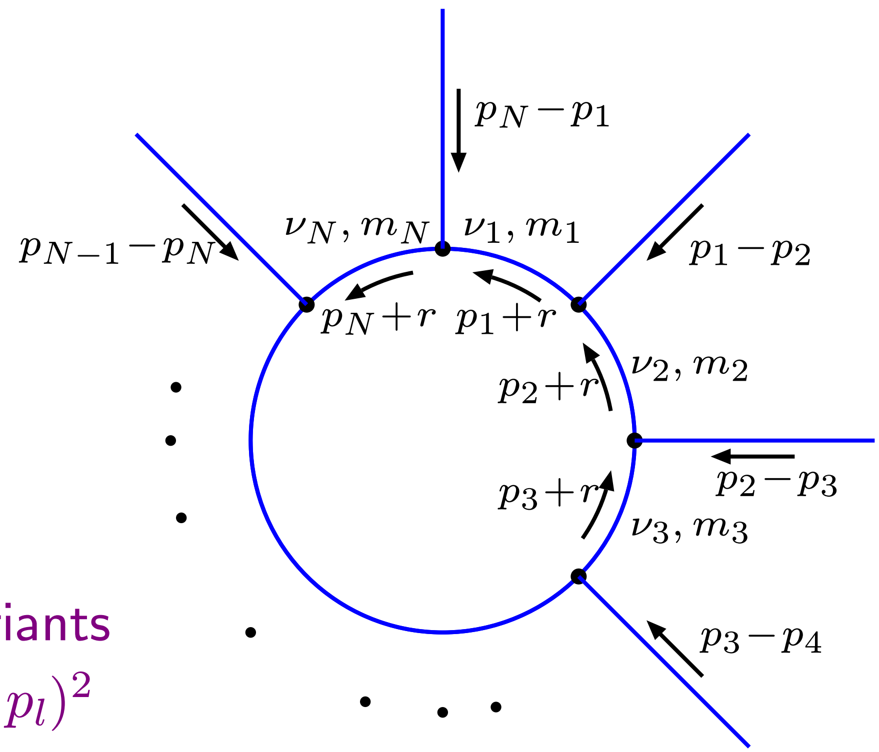
One-loop N-point function $J^{(N)}(n;\nu_1,\ldots,\nu_N)$



Depends on
$\frac{1}{2}N(N-1)$ invariants
$k_{jl}^2 = (p_j - p_l)^2$
and N masses m_i

$$J^{(N)}(n;\nu_1,\ldots,\nu_N) \equiv \int \frac{\mathrm{d}^n k}{\left[(p_1+k)^2 - m_1^2\right]^{\nu_1} \cdots \left[(p_N+k)^2 - m_N^2\right]^{\nu_N}}$$

Geometrical Approach

The idea is to use geometrical description not only when analyzing the singularities (thresholds, etc.), but also when *calculating* dimensionally-regulated Feynman integrals. In particular, it may be used to predict types of functions (and their arguments) appearing in higher orders of ε-expansion.
Such geometrical approach was developed and summarized in

A.I.D. and R. Delbourgo, J. Math. Phys. **39** (1998) 4299.

Examples include results for *all* terms of the ε-expansion for the one-loop two-point function with arbitrary masses, one-loop three-point integrals with massless internal lines and arbitary (off-shell) external momenta and two-loop vacuum diagrams with arbitrary masses:

A.I.D., Phys. Rev. **D61** (2000) 087701;

A.I.D. and M.Yu. Kalmykov, Nucl. Phys. B (PS) **89** (2000) 283; Nucl. Phys. **B605** (2001) 266

as well as the three-point function with arbitrary momenta and masses:

A.I.D., AIHENP-99 Proceedings (hep-th/9908032); Nucl.Instr.Meth. **A559** (2006) 293

Feynman parameters

Parametric representation for the one-loop N-point function:

$$J^{(N)}(n;1,\ldots,1) = \mathrm{i}^{1-n}\pi^{n/2}\,\Gamma\left(N-\frac{n}{2}\right)\int\limits_0^1\ldots\int\limits_0^1 \frac{\left(\prod \mathrm{d}\alpha_i\right)\cdot\delta\left(\sum\alpha_i-1\right)}{\left[\sum\sum\limits_{j<l}\alpha_j\alpha_l k_{jl}^2-\sum\alpha_i m_i^2\right]^{N-n/2}}$$

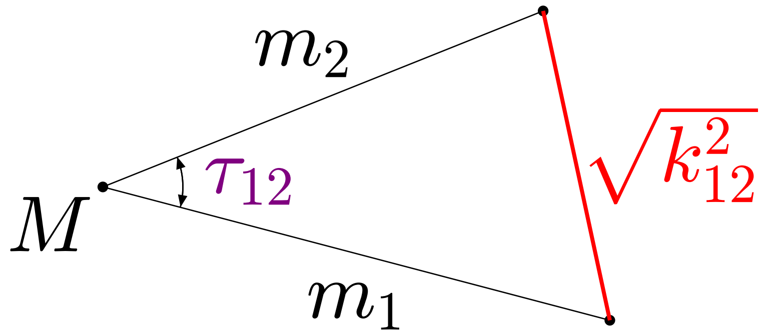
By using $\sum\alpha_i = 1$ we can make the quadratic form homogeneous in α_i:

$$\left[\sum\sum\limits_{j<l}\alpha_j\alpha_l k_{jl}^2-\left(\sum\alpha_i\right)\left(\sum\alpha_i m_i^2\right)\right] \Rightarrow -\left[\sum\alpha_i^2 m_i^2+2\sum\sum\limits_{j<l}\alpha_j\alpha_l m_j m_l c_{jl}\right],$$

$$c_{jl}\equiv\frac{m_j^2+m_l^2-k_{jl}^2}{2m_j m_l}, \qquad c_{jl}=\cos\tau_{jl}=\begin{cases}1, & k_{jl}^2=(m_j-m_l)^2 \quad \text{pseudothreshold}\\ -1, & k_{jl}^2=(m_j+m_l)^2 \quad \text{threshold}\end{cases}$$

Direct geometrical interpretation: when $-1\le c_{jl}\le 1$ (i.e., angles τ_{jl} are real)

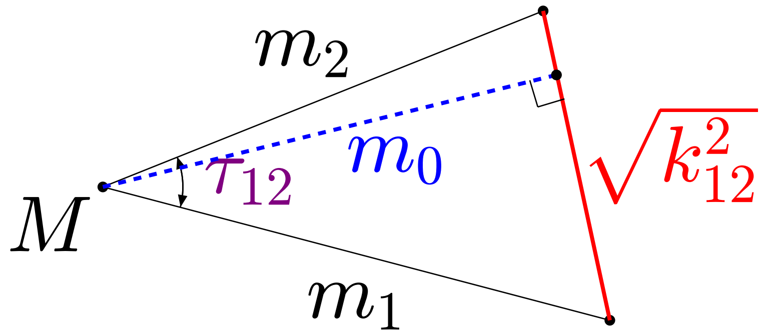
Two-point function: the basic triangle



$$\cos\tau_{12} = c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}$$

$$c_{12} = \cos\tau_{12} = \begin{cases} 1, & k_{12}^2 = (m_1 - m_2)^2 \quad \text{pseudothreshold} \quad (\tau_{12} = 0) \\ -1, & k_{12}^2 = (m_1 + m_2)^2 \quad \text{threshold} \quad (\tau_{12} = \pi) \end{cases}$$

Two-point function: the basic triangle



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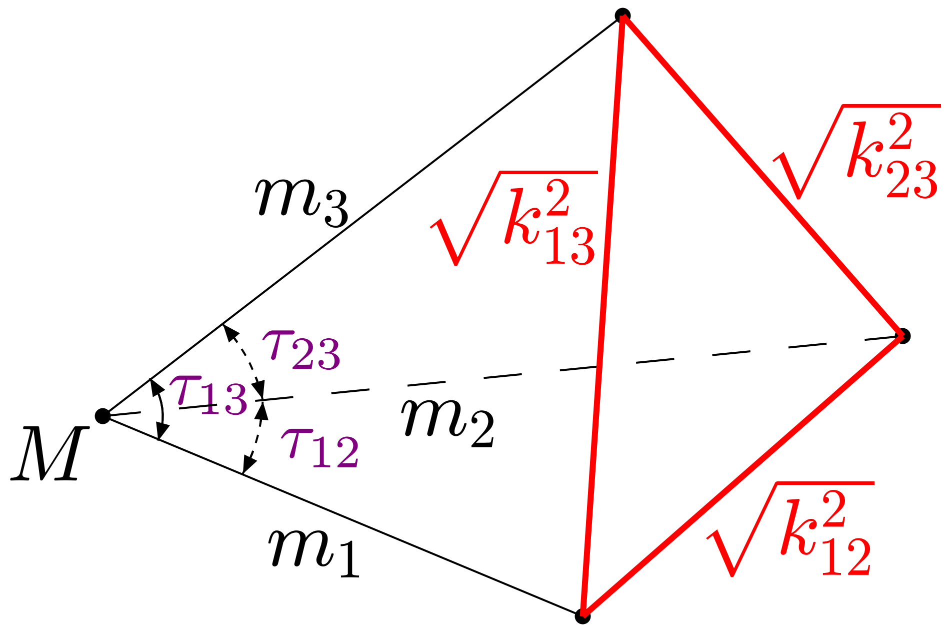
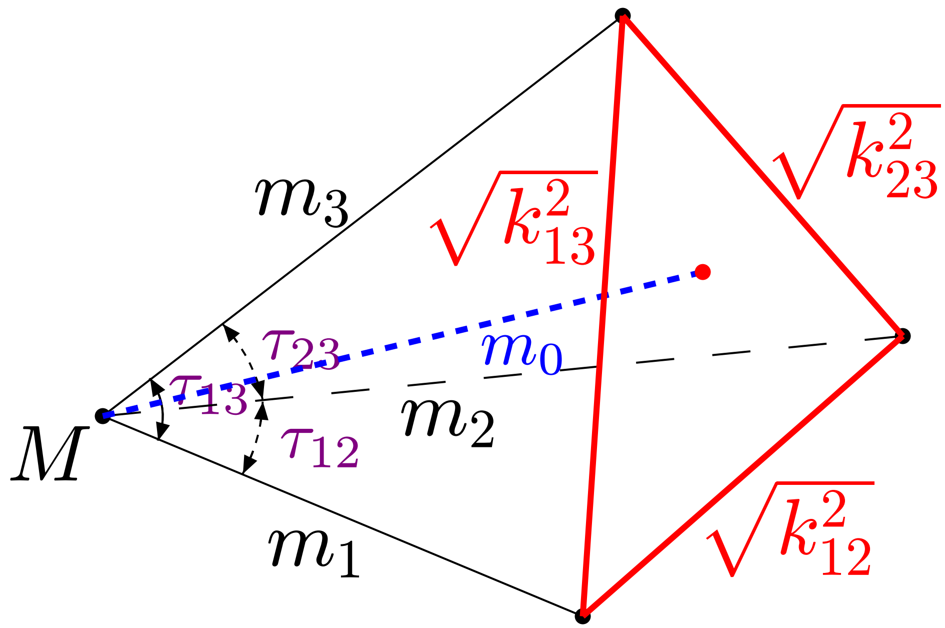
Three-point function: the basic tetrahedron

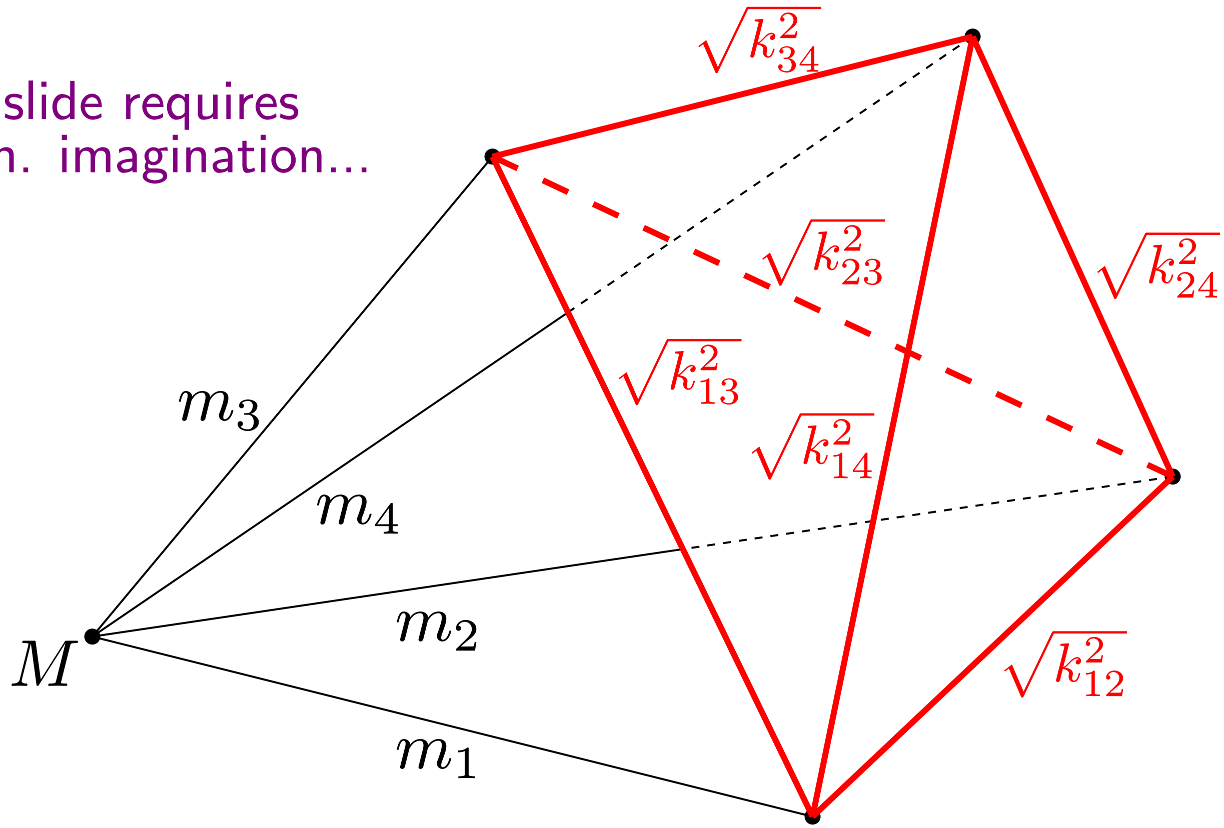


Three-point function: the basic tetrahedron



The basic simplex for $N = 4$

This slide requires 4-dim. imagination...

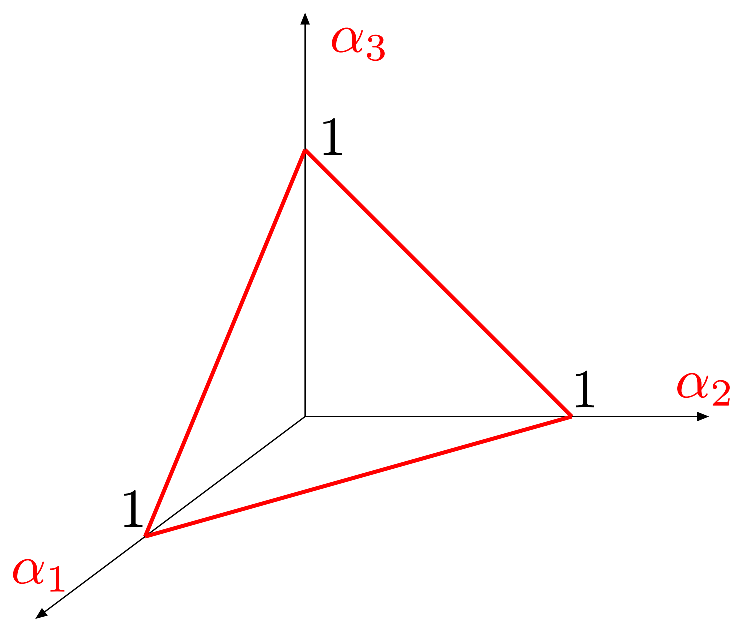


$$D^{(N)} = \det \|c_{jl}\| \ , \qquad \Lambda^{(N)} = \det \|(k_{jN} \cdot k_{lN})\| \ ,$$

$$V^{(N)} = \frac{(\Pi m_i)}{N!}\sqrt{D^{(N)}}, \qquad \overline{V}_0^{(N-1)} = \frac{1}{(N-1)!}\sqrt{\Lambda^{(N)}}, \qquad m_0 = (\Pi m_i)\sqrt{\frac{D^{(N)}}{\Lambda^{(N)}}}$$

Feynman parameters: limits of integration

$$\int_0^1 \ldots \int_0^1 \left(\prod \mathsf{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right) \{\ldots\} = \int_0^\infty \ldots \int_0^\infty \left(\prod \mathsf{d}\alpha_i\right) \cdot \delta\left(\sum \alpha_i - 1\right) \{\ldots\}$$



Feynman parameters: substitutions

Using linear and *quadratic* substitutions of α variables, we arrive at

$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N}\pi^{n/2}\Gamma\left(N-\frac{n}{2}\right)(\Pi f_i)\int\limits_0^\infty\ldots\int\limits_0^\infty \frac{\left(\prod \mathsf{d}\alpha_i\right)\cdot\delta\left(\alpha^T\|C\|\alpha-1\right)}{\left(\sum\alpha_i f_i\right)^{n-N}}$$

Modified matrix: $C_{jl} = \left(\sqrt{F_j^{(N)}}c_{jl}\sqrt{F_l^{(N)}}\right)$, with $F_i^{(N)} = \frac{\partial}{\partial m_i^2}\left(m_i^2 D^{(N)}\right)$

obeying $\sum\limits_{l=1}^{N} c_{jl}F_l^{(N)}\frac{1}{m_l} = D^{(N)}\frac{1}{m_j} \Rightarrow \sum\limits_{l=1}^{N} C_{jl}\frac{\sqrt{F_l^{(N)}}}{m_l} = D^{(N)}\frac{\sqrt{F_j^{(N)}}}{m_j} \Rightarrow$

Eigenvector: $f_i = \frac{\sqrt{F_i^{(N)}}}{m_i}$, Eigenvalue: $D^{(N)} = \det\|c_{jl}\|$ (Gram determinant)

Feynman parameters: diagonalization

Whenever a quadratic form occurs, an obvious idea is to *diagonalize* it: "rotate" variables $\alpha_i \to \beta_i$ so that $\alpha^T \|C\| \alpha = \sum_{i=1}^{N} \lambda_i \beta_i^2$
One of the β's (say β_N) is directed along f_i, so that $\lambda_N = D^{(N)}$ and denominator $(\sum \alpha_i f_i)$ is proportional to β_N.

Assume (for a moment) that all $\lambda_i > 0$ and rescale $\beta_i = \frac{\gamma_i}{\sqrt{\lambda_i}} \Rightarrow$

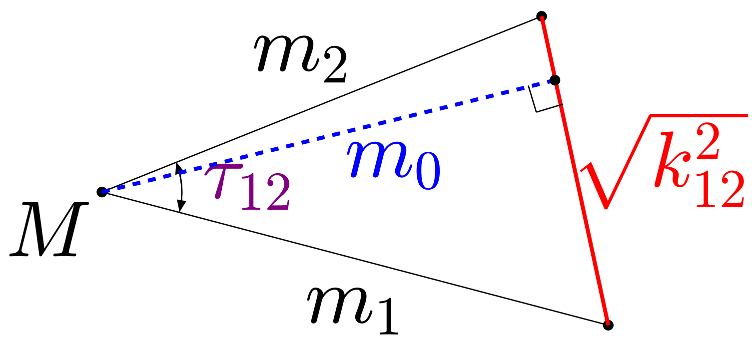
$$J^{(N)}(n;1,\ldots,1) = 2\mathrm{i}^{1-2N} \pi^{n/2}\, \Gamma\left(N - \tfrac{n}{2}\right) \frac{{m_0}^{n-N-1}}{\sqrt{\Lambda^{(N)}}} \int\limits_{\Omega^{(N)}} \!\!\ldots \int \frac{\prod \mathrm{d}\gamma_i}{\gamma_N^{\,n-N}}\, \delta\left(\sum \gamma_i^2 - 1\right)$$

Remarkably: the same N-dim. solid angle $\Omega^{(N)}$ as in the *basic simplex*!

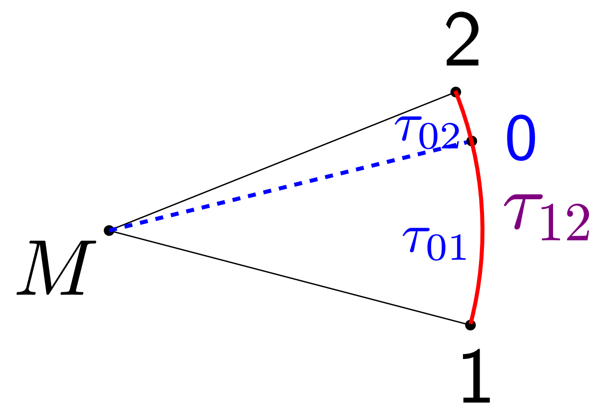
Special case: $N = n$ ($N = 2$ in 2d, $N = 3$ in 3d, $N = 4$ in 4d, etc.)

If some of λ_i are negative – *hyperbolic* surface (instead of *spherical*)
$\leftrightarrow$ analytical continuation!

Two-point function, geometrical approach



the basic triangle

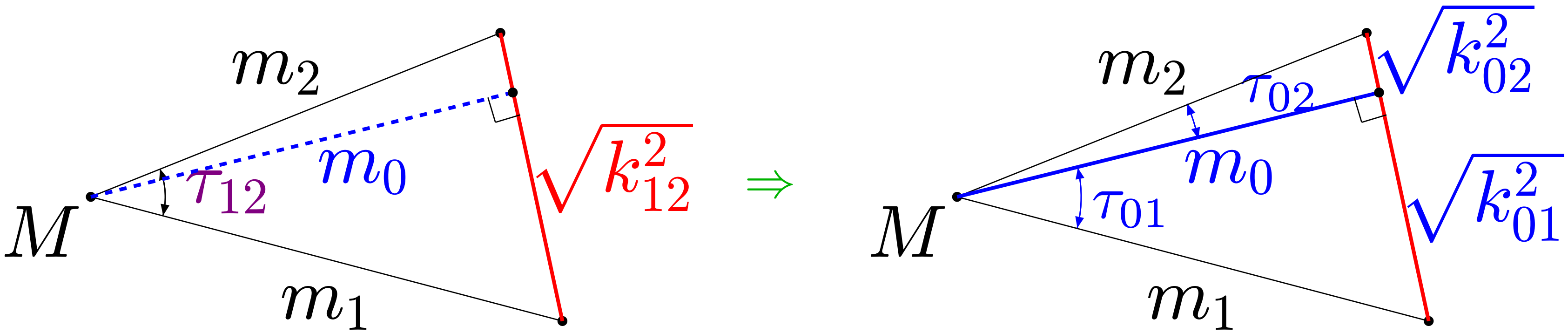


the arc τ_{12}

$$\cos\tau_{12} \equiv c_{12} = \frac{m_1^2 + m_2^2 - k_{12}^2}{2m_1 m_2}, \quad D^{(2)} = 1 - c_{12}^2 = \sin^2\tau_{12}, \quad \Lambda^{(2)} = k_{12}^2,$$

$$m_0 = m_1 m_2 \sqrt{\frac{D^{(2)}}{\Lambda^{(2)}}} = \frac{m_1 m_2 \sin\tau_{12}}{\sqrt{k_{12}^2}}, \qquad \cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01} + \tau_{02} = \tau_{12}.$$

Two-point function, splitting the basic triangle



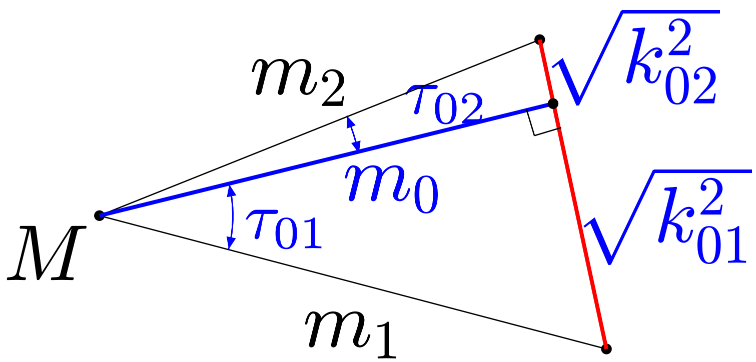
$$k_{01}^2 = \frac{(k_{12}^2 + m_1^2 - m_2^2)^2}{4k_{12}^2}, \qquad k_{02}^2 = \frac{(k_{12}^2 - m_1^2 + m_2^2)^2}{4k_{12}^2}$$

$$J^{(2)}\left(n; 1, 1 | k_{12}^2; m_1, m_2\right) = \frac{1}{2k_{12}^2}\Big\{(k_{12}^2 + m_1^2 - m_2^2) J^{(2)}\left(n; 1, 1 | k_{01}^2; m_1, m_0\right)$$
$$+ (k_{12}^2 - m_1^2 + m_2^2) J^{(2)}\left(n; 1, 1 | k_{02}^2; m_2, m_0\right)\Big\}$$

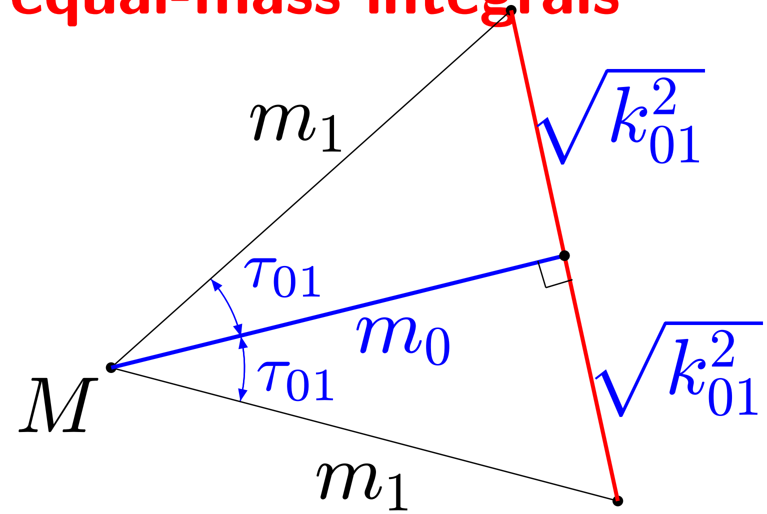
This is an example of a functional relation between integrals with different momenta and massses, similar to those described in

O.V. Tarasov, Phys.Lett. **B670** (2008) 67

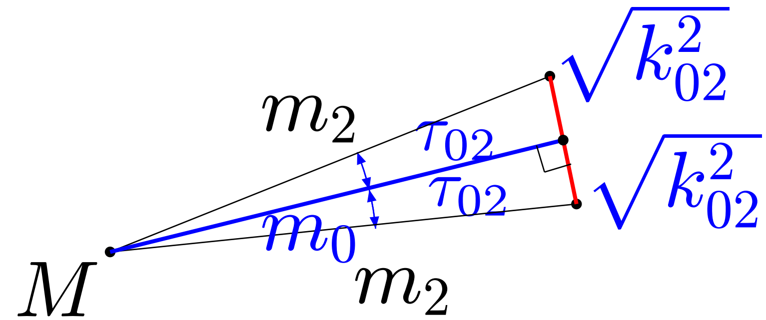
Two-point function, reduction to equal-mass integrals



$\Rightarrow \quad \frac{1}{2}$



$+ \quad \frac{1}{2}$



$\Rightarrow$ equal-mass functions

$$
\begin{aligned}
J^{(2)}\left(n;1,1|k_{12}^2;m_1,m_2\right) \;=\; & \frac{1}{4k_{12}^2}\Big\{(k_{12}^2+m_1^2-m_2^2)J^{(2)}\left(n;1,1|4k_{01}^2;m_1,m_1\right) \\
& +(k_{12}^2-m_1^2+m_2^2)J^{(2)}\left(n;1,1|4k_{02}^2;m_2,m_2\right)\Big\}
\end{aligned}
$$

with $\quad k_{01}^2=\dfrac{(k_{12}^2+m_1^2-m_2^2)^2}{4k_{12}^2}, \qquad k_{02}^2=\dfrac{(k_{12}^2-m_1^2+m_2^2)^2}{4k_{12}^2}$

Two-point function, hypergeometric representation

$$J^{(2)}(n;1,1|k_{12}^2;m_1,m_2) = \mathrm{i}\pi^{n/2}\Gamma\left(\frac{4-n}{2}\right)\,\frac{m_0^{n-3}}{\sqrt{k_{12}^2}}\left\{\Omega_1^{(2;n)}+\Omega_2^{(2;n)}\right\}$$

with

$$\Omega_i^{(2;n)} = \int\limits_0^{\tau_{0i}} \frac{\mathrm{d}\theta}{\cos^{n-2}\theta} = \tan\tau_{0i}\;\;(\cos\tau_{0i})^{4-n}\;\;{}_2F_1\left(\begin{array}{c}1,(4-n)/2\\ 3/2\end{array}\Big|\sin^2\tau_{0i}\right)$$

$$c_{12} = \frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \quad D^{(2)} = 1-c_{12}^2 = \sin^2\tau_{12}, \quad m_0 = m_1m_2\sqrt{\frac{D^{(2)}}{k_{12}^2}},$$

$$\cos\tau_{0i} = \frac{m_0}{m_i}, \quad \tau_{01}+\tau_{02} = \tau_{12}.$$

Two-point function in $n = 4 - 2\varepsilon$ dimensions, ε-expansion

$$J^{(2)}(4-2\varepsilon;1,1) = i\pi^{2-\varepsilon}\frac{\Gamma(1+\varepsilon)}{2(1-2\varepsilon)}\left\{\frac{m_1^{-2\varepsilon}+m_2^{-2\varepsilon}}{\varepsilon}+\frac{m_1^2-m_2^2}{\varepsilon k_{12}^2}(m_1^{-2\varepsilon}-m_2^{-2\varepsilon})\right.$$
$$\left. -\frac{\left[\Delta(m_1^2,m_2^2,k_{12}^2)\right]^{1/2-\varepsilon}}{(k_{12}^2)^{1-\varepsilon}}\sum_{j=0}^{\infty}\frac{(2\varepsilon)^j}{j!}\left[\mathrm{Ls}_{j+1}(\pi-2\tau_{01})+\mathrm{Ls}_{j+1}(\pi-2\tau_{02})-2\mathrm{Ls}_{j+1}(\pi)\right]\right\}$$

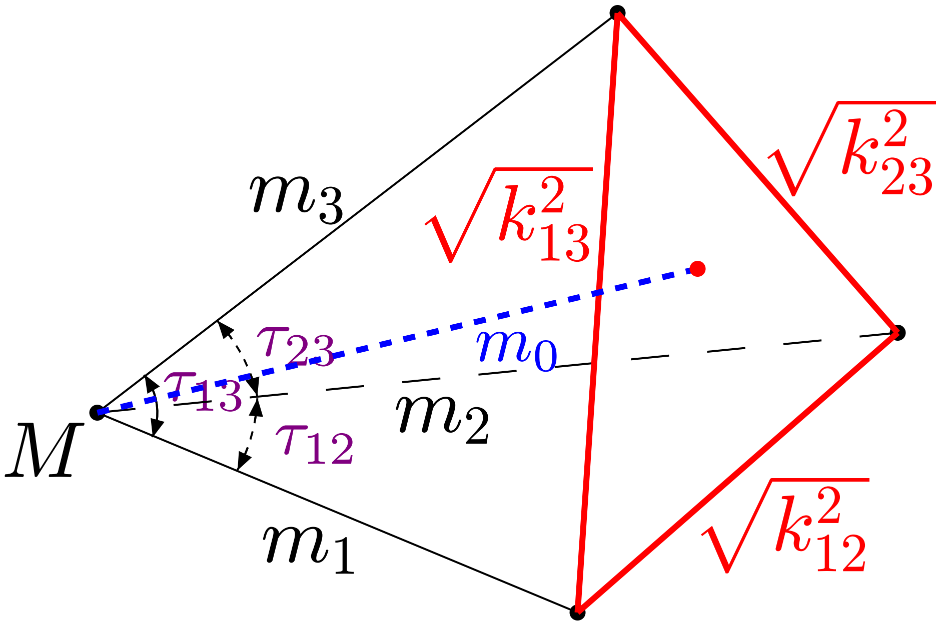
where $\Delta(m_1^2, m_2^2, k_{12}^2) = 4m_1^2m_2^2D^{(2)} = 4m_1^2m_2^2\sin^2\tau_{12}$,
so that $\frac{1}{4}\sqrt{\Delta}$ is the triangle area.

These results are represented in terms of the log-sine integrals,

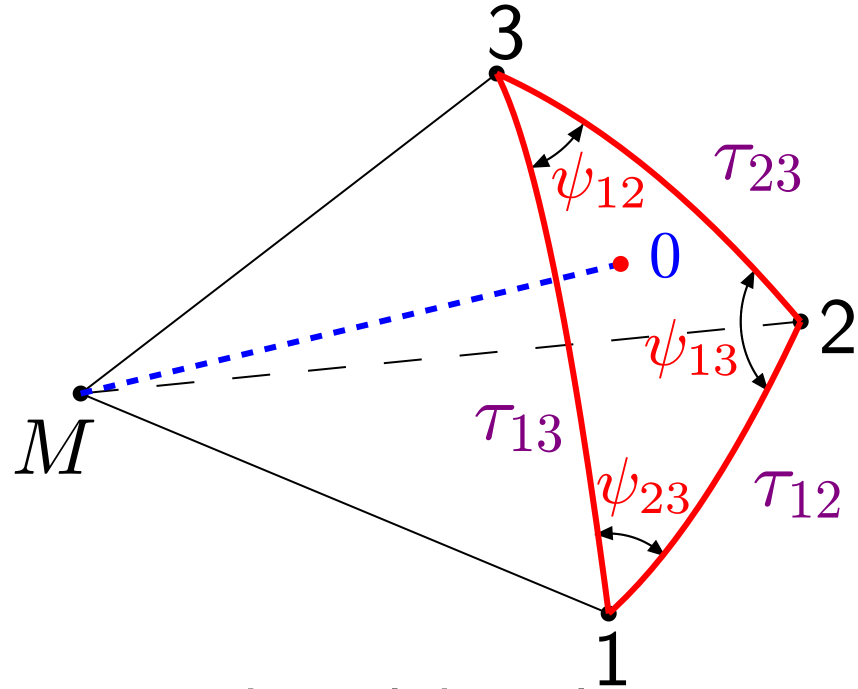
$$\mathsf{Ls}_j(\theta) = -\int_0^\theta \mathrm{d}\phi\ \ln^{j-1}\left|2\sin\frac{\phi}{2}\right| .$$

Analytic continuation $\Rightarrow$ Nielsen polylogarithms (to all orders)

Three-point function: geometrical approach



the basic tetrahedron



the solid angle

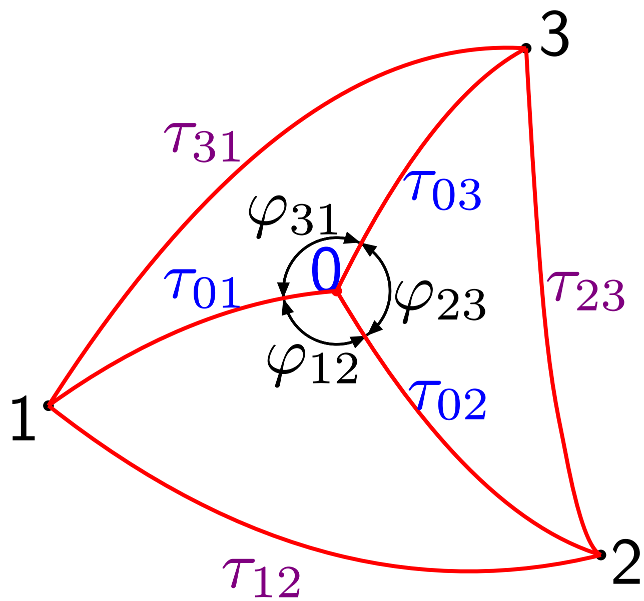
Special case $n = 3 \Rightarrow$ the area of spherical triangle ("spherical excess"):

$$\Omega^{(3;3)} = \psi_{12} + \psi_{23} + \psi_{31} - \pi \ .$$

Compare with: B. G. Nickel, J. Math. Phys. **19** (1978) 542

Three-point function: splitting the solid angle

Relation to the angles accociated with a spherical (or hyperbolic) triangle:



$$\varphi_{12} + \varphi_{23} + \varphi_{31} = 2\pi$$

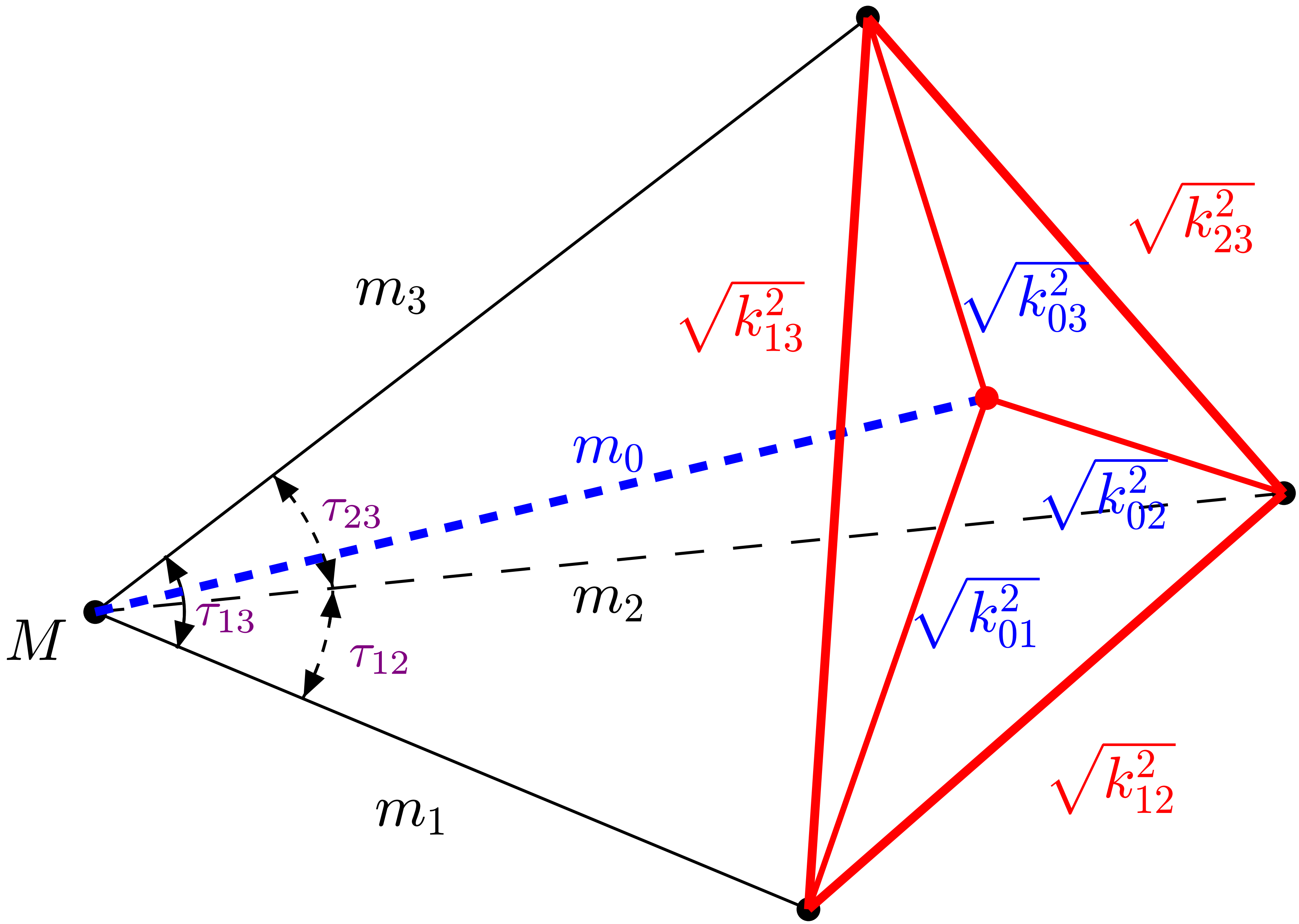
$$\cos\tau_{12} = \frac{m_1^2+m_2^2-k_{12}^2}{2m_1m_2}, \text{ etc.}$$

$$\cos\tau_{0i} = \frac{m_0}{m_i} \quad (i=1,2,3)$$

$$m_0 = m_1 m_2 m_3 \sqrt{\frac{D^{(3)}}{\Lambda^{(3)}}}$$

$$D^{(3)} = \begin{vmatrix} 1 & c_{12} & c_{13} \\ c_{12} & 1 & c_{23} \\ c_{13} & c_{23} & 1 \end{vmatrix}, \quad \Lambda^{(3)} = \tfrac{1}{4}\left[2k_{12}^2k_{13}^2+2k_{13}^2k_{23}^2+2k_{23}^2k_{12}^2-(k_{12}^2)^2-(k_{13}^2)^2-(k_{23}^2)^2\right]$$

Three-point function: the basic tetrahedron



Three-point function: splitting the basic tetrahedron

$$J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$$

$$= \frac{m_1^2m_2^2m_3^2}{\Lambda^{(3)}}\left\{\frac{F_1^{(3)}}{m_1^2}J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{03}^2,k_{02}^2;m_0,m_2,m_3\right)\right.$$

$$+\frac{F_2^{(3)}}{m_2^2}J^{(3)}\left(n;1,1,1\middle|k_{03}^2,k_{13}^2,k_{01}^2;m_1,m_0,m_3\right)$$

$$\left.+\frac{F_3^{(3)}}{m_3^2}J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)\right\}$$

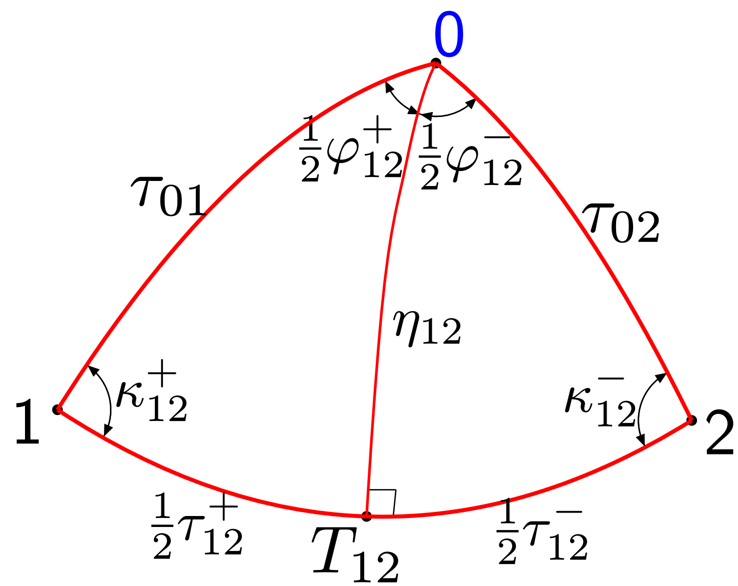
with $k_{01}^2 = m_1^2 - m_0^2, \qquad k_{02}^2 = m_2^2 - m_0^2, \qquad k_{03}^2 = m_3^2 - m_0^2,$

$$F_3^{(3)} = \frac{1}{4m_1^2m_2^2}\left[k_{12}^2\left(k_{13}^2+k_{23}^2-k_{12}^2+m_1^2+m_2^2-2m_3^2\right)-(m_1^2-m_2^2)\left(k_{13}^2-k_{23}^2\right)\right],$$ etc.

$$\frac{F_1^{(3)}}{m_1^2}+\frac{F_2^{(3)}}{m_2^2}+\frac{F_3^{(3)}}{m_3^2}=\frac{\Lambda^{(3)}}{m_1^2m_2^2m_3^2}$$

Three-point function: further splitting

One of the three triangles ($\frac{1}{2}(\varphi_{12}^{+} + \varphi_{12}^{-}) = \varphi_{12}$):



Three-point function: reduction to integrals with two equal masses

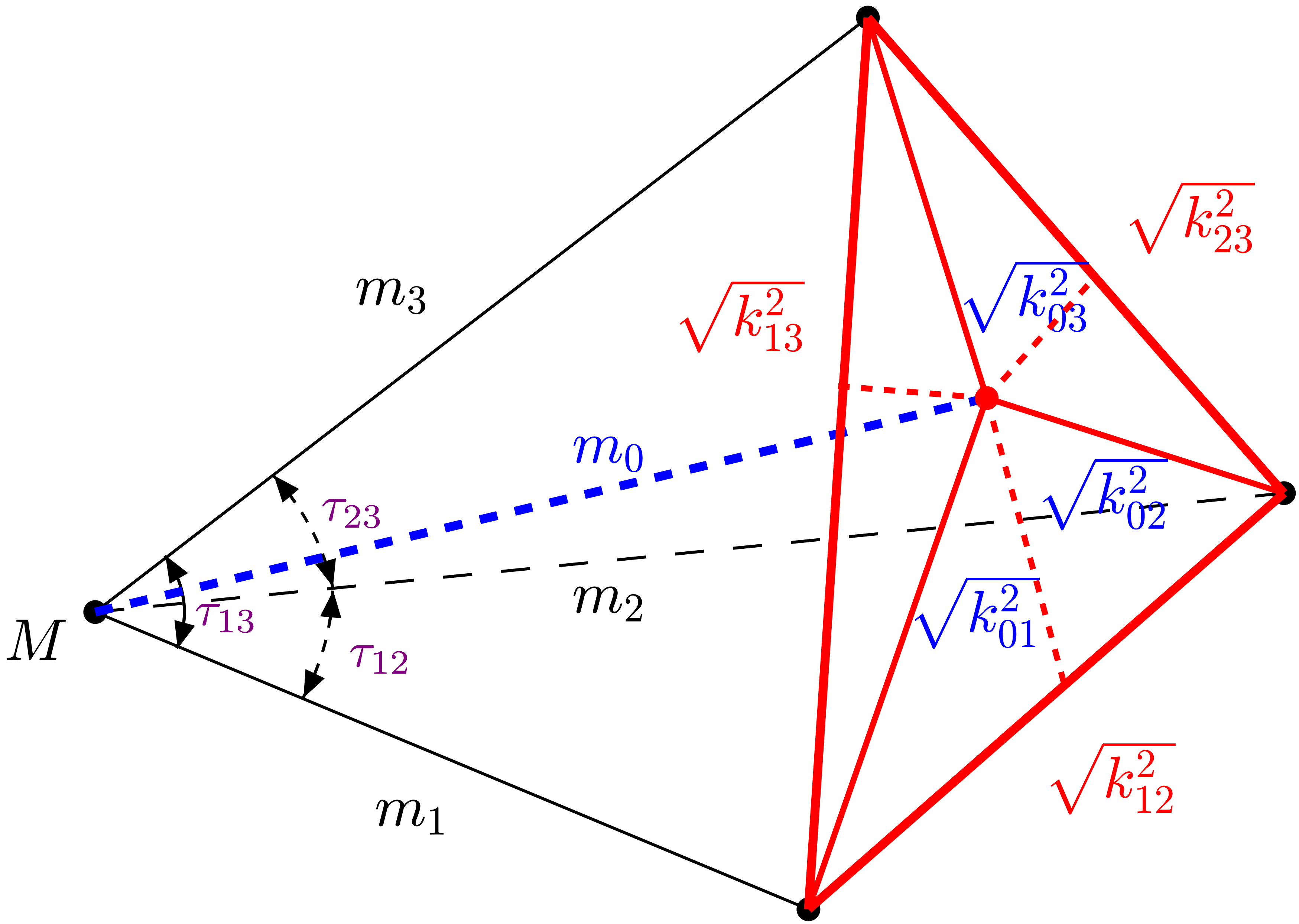
$$J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$$

$$= \frac{1}{2k_{12}^2}\left\{(k_{12}^2+m_1^2-m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{01}^2,k_{01}^2,\frac{(k_{12}^2+m_1^2-m_2^2)^2}{k_{12}^2};m_1,m_1,m_0\right)\right.$$

$$\left.+(k_{12}^2-m_1^2+m_2^2)J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{02}^2,\frac{(k_{12}^2-m_1^2+m_2^2)^2}{k_{12}^2};m_2,m_2,m_0\right)\right\}$$

— similarly to the reduction of the two-point function

Three-point function: the basic tetrahedron



Number of dimensionless variables

in $J^{(3)}\left(n;1,1,1\middle|k_{23}^2,k_{13}^2,k_{12}^2;m_1,m_2,m_3\right)$:

$$6-1(\text{dimension})=5$$

in $J^{(3)}\left(n;1,1,1\middle|k_{02}^2,k_{01}^2,k_{12}^2;m_1,m_2,m_0\right)$:

$$6-2(\text{relations})-1(\text{dimension})=3$$

in $J^{(3)}\left(n;1,1,1\middle|k_{01}^2,k_{01}^2,\frac{(k_{12}^2+m_1^2-m_2^2)^2}{k_{12}^2};m_1,m_1,m_0\right)$

$$6-3(\text{relations})-1(\text{dimension})=2$$

Three-point function in $n = 4 - 2\varepsilon$ dimensions

$$J^{(3)}(n;1,1,1) = -\frac{\mathrm{i}\pi^{n/2}}{\sqrt{\Lambda^{(3)}}}\,\Gamma\left(3-\frac{n}{2}\right)\; m_0^{n-4}\,\Omega^{(3;n)}\;,$$

$$\Omega^{(3;n)} = \int\!\!\int_{\Omega^{(3)}} \frac{\sin^{n-2}\theta\;\mathrm{d}\theta\;\mathrm{d}\phi}{\cos^{n-3}\theta} \quad = \quad \omega\left(\tfrac{1}{2}\varphi_{12}^{+},\eta_{12}\right)+\omega\left(\tfrac{1}{2}\varphi_{12}^{-},\eta_{12}\right) +\omega\left(\tfrac{1}{2}\varphi_{23}^{+},\eta_{23}\right)+\omega\left(\tfrac{1}{2}\varphi_{23}^{-},\eta_{23}\right) +\omega\left(\tfrac{1}{2}\varphi_{31}^{+},\eta_{31}\right)+\omega\left(\tfrac{1}{2}\varphi_{31}^{-},\eta_{31}\right)\,,$$

with

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right) = \frac{1}{2\varepsilon}\int\limits_0^{\varphi/2}\mathrm{d}\phi\left[1-\left(1+\frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon}\right] = \frac{1}{2}\sum_{j=0}^{\infty}\frac{(-\varepsilon)^j}{(j+1)!}\int\limits_0^{\varphi/2}\mathrm{d}\phi\;\ln^{j+1}\!\left(1+\frac{\tan^2\eta_{12}}{\cos^2\phi}\right)$$

A.I.D., hep-th/9908032, Nucl.Instr.Meth. A559 (2006) 293

The result for arbitrary ε can be presented in terms of Appell's hypergeometric function F_1,

$$\omega\left(\tfrac{1}{2}\varphi,\eta\right) = \frac{1}{2\varepsilon}\left[\frac{\varphi}{2} - \sin\frac{\varphi}{2}\cos\frac{\varphi}{2}\cos^{2\varepsilon}\tau_0\; F_1\left(1,1,\varepsilon;\tfrac{3}{2}\middle|\; \sin^2\frac{\varphi}{2},\sin^2\frac{\tau}{2}\right)\right],$$

with $\cos\tau_0 = \cos\eta\;\cos\frac{\tau}{2}$,

$$F_1(a,b,b',c|x,y) = \sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(a)_{j+l}\;(b)_j\;(b')_l}{(c)_{j+l}}\;\frac{x^j\;y^l}{j!\;l!}$$

Similar functions occurred in

O.V. Tarasov, Nucl. Phys. B (PS) **89** (2000) 237

J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

Some special cases: L.G. Cabral-Rosetti, M.A. Sanchis-Lozano, hep-ph/0206081

Special value of n: $n = 4$ $(\varepsilon \to 0)$:

$$\int\limits_0^{\varphi/2} \mathrm{d}\phi \; \ln\left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right) = \tfrac{1}{2}\tau \; \ln\left(\frac{1+\sin\eta}{1-\sin\eta}\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi+\tau\right) + \tfrac{1}{2}\mathrm{Cl}_2\left(\varphi-\tau\right) - \mathrm{Cl}_2\left(\varphi\right)$$

Compare with: P. Wagner, Indag. Math. 7 (1996) 527

After analytical continuation, corresponds to

G. 'tHooft and M. Veltman, Nucl. Phys. B153 (1979) 365

Analytic Continuation: Arbitrary Dimension

Consider $$\int\limits_0^{\varphi_0} \mathrm{d}\phi \left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} .$$

Substitute $z \Rightarrow e^{2\mathrm{i}\phi}$, so that $\cos^2\phi \Rightarrow \dfrac{(1+z)^2}{4z}$,

$$1 + \frac{\tan^2\eta}{\cos^2\phi} \Rightarrow \frac{(z+\rho)(z+1/\rho)}{(z+1)^2}, \quad \text{with} \quad \rho \equiv \frac{1-\sin\eta}{1+\sin\eta}$$

In this way, $$\int\limits_0^{\varphi_0} \mathrm{d}\phi \left(1 + \frac{\tan^2\eta}{\cos^2\phi}\right)^{-\varepsilon} \Rightarrow \frac{\mathrm{i}}{2} \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right]^{-\varepsilon},$$

with $z_0 \leftrightarrow e^{2\mathrm{i}\varphi_0}$.

Analytic Continuation: Expansion in ε

Expanding in ε, we get

$$Q_j \equiv \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln^j \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] .$$

The first term, $\mathcal{O}(1)$:

$$\begin{aligned} Q_1 \;\; &\equiv \;\; \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right] \\ &= \;\; \mathrm{Li}_2\left(-z_0\rho\right) + \mathrm{Li}_2\left(-z_0/\rho\right) - 2\mathrm{Li}_2\left(-z_0\right) + \tfrac{1}{2}\ln^2\rho \end{aligned}$$

Analytic Continuation: Expansion in ε (continued)

$$
\begin{aligned}
Q_2 \;&\equiv\; \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z}\; \ln^2\left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right] \\
&=\; \ln\rho \left[2\mathrm{Li}_2\left(\frac{1-\rho}{1+z_0\rho}\right) + 2\mathrm{Li}_2\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) - 2\mathrm{Li}_2\left(\frac{\rho-1}{z_0+\rho}\right) - 2\mathrm{Li}_2\left(\frac{z_0(1-\rho)}{z_0+\rho}\right) \right. \\
&\quad \left. -\mathrm{Li}_2\left(\frac{1-\rho^2}{1+z_0\rho}\right) - \mathrm{Li}_2\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) + \mathrm{Li}_2\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{Li}_2\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right) \right] \\
&\quad +4\mathrm{S}_{1,2}\left(\frac{1-\rho}{1+z_0\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(\rho-1)}{1+z_0\rho}\right) + 4\mathrm{S}_{1,2}\left(\frac{\rho-1}{z_0+\rho}\right) - 4\mathrm{S}_{1,2}\left(\frac{z_0(1-\rho)}{z_0+\rho}\right) \\
&\quad -\mathrm{S}_{1,2}\left(\frac{1-\rho^2}{1+z_0\rho}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(\rho^2-1)}{\rho(1+z_0\rho)}\right) - \mathrm{S}_{1,2}\left(\frac{\rho^2-1}{\rho(z_0+\rho)}\right) + \mathrm{S}_{1,2}\left(\frac{z_0(1-\rho^2)}{z_0+\rho}\right)
\end{aligned}
$$

Compare with: J. Fleischer, F. Jegerlehner, O.V. Tarasov, Nucl. Phys. **B672** (2003) 303

ε-expansion: higher terms

ε^2-term

$$Q_3 = \int\limits_{z_0}^{1} \frac{\mathrm{d}z}{z} \; \ln^3 \left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2} \right],$$

etc.

Some special cases considered in J.G. Körner, Z. Merebashvili, M. Rogal, Phys. Rev. D71 (2005) 054028

ε-expansion: recursive calculation

$$
\begin{aligned}
Q_0(z_0,\rho) &= -\ln z_0, \\
Q_j(z_0,\rho) &= Q_{j-1}(z_0,\rho)\ln\left[\frac{(z+\rho)(z+1/\rho)}{(z+1)^2}\right] \\
&\quad + \int\limits_{z_0}^{1} \mathrm{d}z\, Q_{j-1}(z,\rho)\left[\frac{1}{z+\rho}+\frac{1}{z+1/\rho}-\frac{2}{z+1}\right]
\end{aligned}
$$

$\Rightarrow$ higher terms can be expressed in terms of *multiple polylogarithms*

$$
\mathrm{Li}_{n_1,\ldots,n_m}(z_1,\ldots,z_m) = \sum_{0<k_1<k_2<\ldots<k_m} \frac{z_1^{k_1} z_2^{k_2}\ldots z_m^{k_m}}{k_1^{n_1} k_2^{n_2}\ldots k_m^{n_m}}
$$

A.B. Goncharov, Math. Res. Lett. 5 (1998) 497,

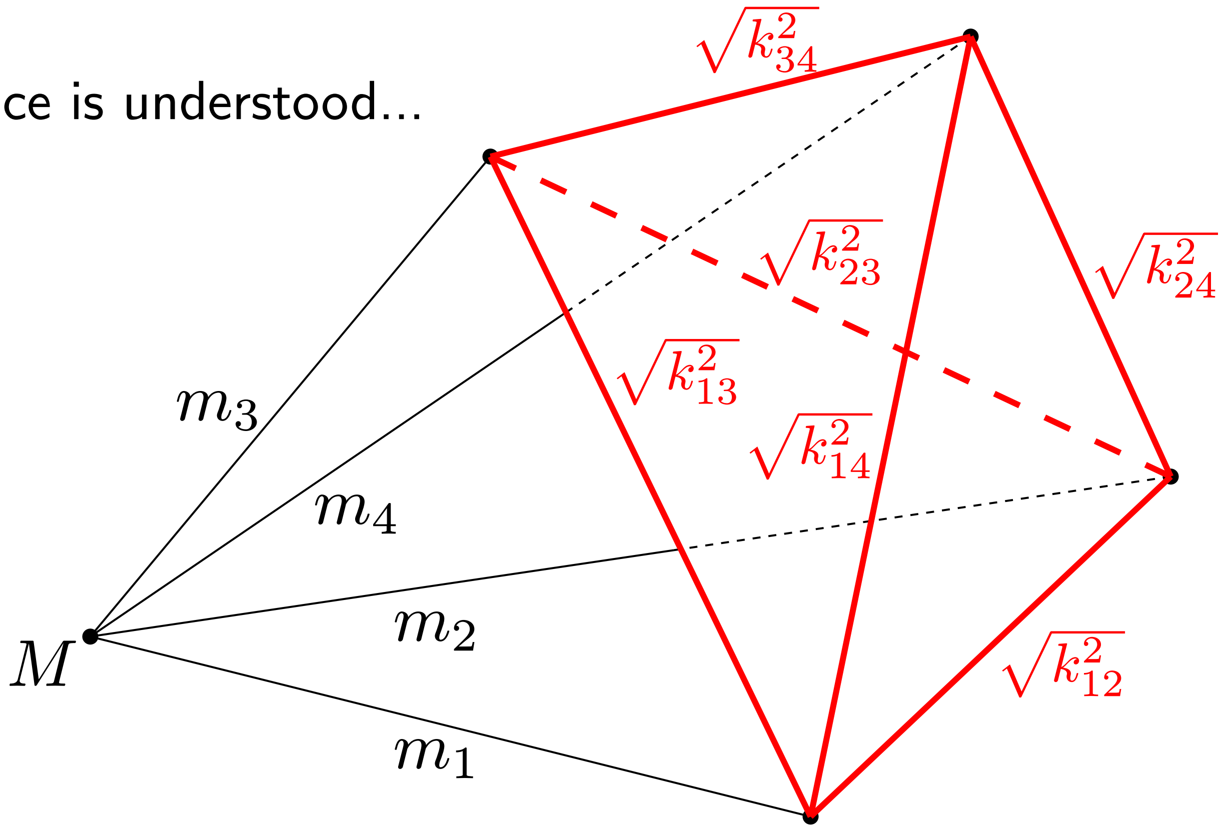
J. Vollinga and S. Weinzierl, Comp. Phys. Commun. **167** (2005) 177

or *two-dimensional harmonic polylogarithms*,

T. Gehrmann and E. Remiddi, Nucl. Phys. **B601** (2001) 248

Four-point function: the basic simplex for $N = 4$

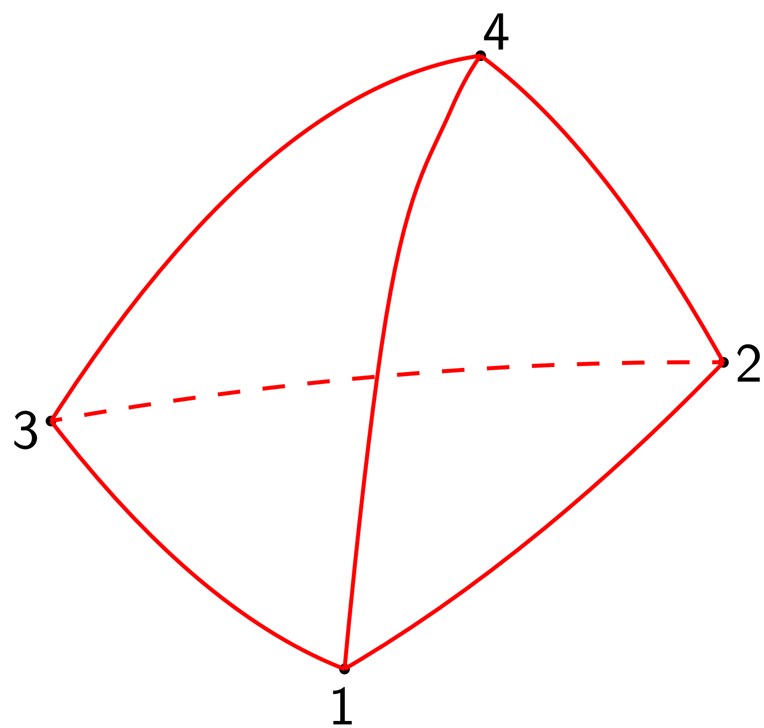
4-dim. space is understood...



$$D^{(N)} = \det \|c_{jl}\| \; , \qquad \Lambda^{(N)} = \det \|(k_{jN} \cdot k_{lN})\| \; ,$$

$$V^{(N)} = \frac{(\Pi m_i)}{N!} \sqrt{D^{(N)}}, \qquad \overline{V}_0^{(N-1)} = \frac{1}{(N-1)!} \sqrt{\Lambda^{(N)}}, \qquad m_0 = (\Pi m_i) \sqrt{\frac{D^{(N)}}{\Lambda^{(N)}}}$$

Geometrical approach: 4-point function



The spherical tetrahedon

Summary

- A geometrical way to calculate dimensionally-regulated Feynman diagrams is reviewed.

- In the one-loop N-point case, results can be related to certain volume integrals in non-Euclidean geometry. For example, the result for the four-point function can be associated with the content of a spherical or hyperbolic tetrahedron in three-dimensional spherical or hyperbolic space (Lobachevsky, Schläfli, ...)

- Geometrical approach provides straightforward way of reducing general integrals to those with lesser number of independent variables, and allows to derive functional relations between integrals with different momenta and masses.

- Analytical continuation of the results to other regions of kinematical variables (momenta and masses of the particles) is discussed. In a number of cases, analytic results can be presented in terms of the (generalized) polylogarithms and associated functions. In more complicated cases, multiple polylogarithms may appear.