NNLO INFRARED SUBTRACTION SCHEME

1. IR subtraction schemes
2. The TS subtraction scheme-motivations
3. Phase space integrals: structure, computation, results
4. Conclusions and outlooks

This work has been done in collaboration with
S. Moch, G. Somogyi, Z.Trocsanyi

Paolo Bolzoni – Calculations for modern and future colliders, Dubna, 10 July 2009

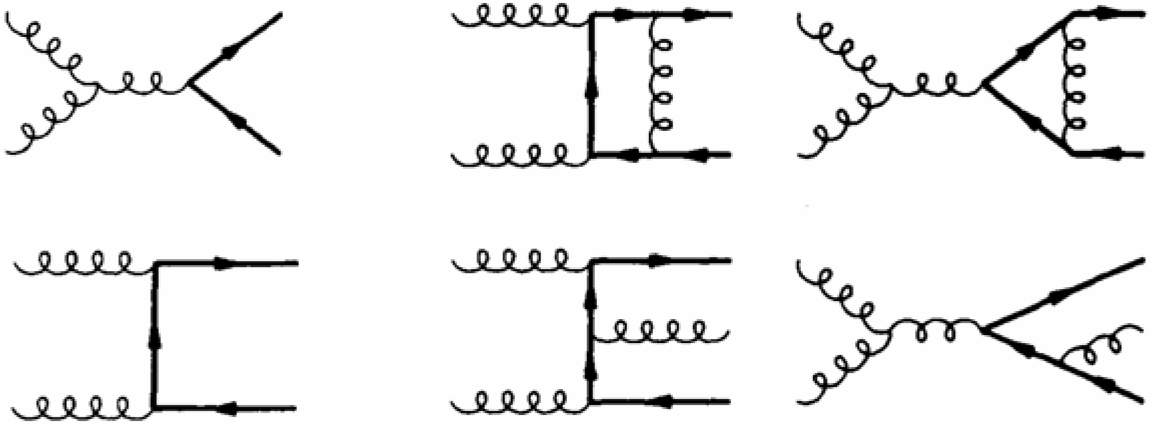
1. IR SUBTRACTION SCHEMES

THE KINOSHITA-LEE-NAUENBERG THEOREM

[T.Kinoshita (1962); T.D. Lee, M. Nauenberg (1963)]

$$\sigma = \sigma^{LO} + \sigma^{NLO} + \ldots$$

Consider for example some NLO contributions to the partonic QCD process: $gg \to QQ$



We know that mass singularities cancel out according to the KLN theorem between real and virtual quantum corrections at the same order in perturbation theory

$$\sigma^{NLO} = \sigma^{V} + \int d\sigma^{R}$$

Even if the two contributions are separately divergent, their sum is finite

WHAT IS A SUBTRACTION SCHEME I

Consider the following toy example where the PS of the unresolved parton is parametrized by $x \in (0,1)$:

$$d\sigma^R(x) = x^{-1-\epsilon} S(x)\, dx$$

$$\sigma^V = \frac{S(0)}{\epsilon} + S^V$$

$$\begin{aligned}
\sigma &= \sigma^V + \int_0^1 d\sigma^R(x) = \sigma^V + \int_0^1 x^{-1-\epsilon} S(x)\, dx \\
&= \sigma^V + \int_0^1 \left[\frac{S(x) - S(0)}{x}\right] dx - \frac{S(0)}{\epsilon} + O(\epsilon) \\
&= S^V + \int_0^1 \left[\frac{S(x) - S(0)}{x}\right] dx + O(\epsilon),
\end{aligned}$$

where we have subtracted and readded $x^{-1-\varepsilon} S(0)$ in the integrand and expanded in ε

In this form the first integral is computable with standard numerical methods like MC

WHAT IS A SUBTRACTION SCHEME II

What we did in the previous example was:

1. The subtraction from the real emission contribution of a COUNTERTERM:

$$d\sigma^A(x) = x^{-1-\epsilon}\, S(0)\, dx$$

2. The readdition of the same counterterm integrated over the phase space of the emitted parton:

$$\int d\sigma^A(x) = \int_0^1 x^{-1-\epsilon} S(0)\, dx = -\frac{S(0)}{\epsilon} + O(\epsilon)$$

In general the choice of the counterterms is arbitrary under these constaints:

(i) The singular soft and/or collinear behavior of the real cross section is reproduced
(i) Their integration over the PS of the unresolved partons is factorized
(ii) It respects the delicate mechanism of cancellation between real and virtual contributions

Each choice of the counterterms defines an IR subtraction scheme and choosen one their integration can be done once and for all…

2.THE TROCSANYI-SOMOGYI (TS) SUBTRACTION SCHEME

[Z. Trocsanyi, G. Somogyi, V. Del Duca, Z. Nagy]

WHY THIS NEW SUBTRACTION SCHEME?

Existing subtraction methods can not straightforwardly be generalized to NNLO

[S. Weinzierl; M. Grazzini, S. Frixione];
[A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, G. Heinrich]

This new subtraction scheme is worked out for colorless incoming particle at NNLO and for hadronic initiated processes at NLO in an NNLO compatible way

[Z. Trocsanyi, G. Somogyi, V. Del Duca, Z. Nagy]

The proposed scheme can be generalized to any order in perturbation theory and is based on simple separaition of soft and collinear singularities due to new phase space mappings

[Z. Trocsanyi, G. Somogyi, Z. Nagy]

NNLO computations are very long and complicated and we wish to have an independent method of evaluation of these challenging observables (e.g. for the NNLO $e^+e^- \to$ 3 jets)

[S. Weinzierl; A. Gehrmann-De Ridder, T. Gehrmann, E.W.N. Glover, G. Heinrich]

THE NLO COUNTERTERMS IN THE TS SCHEME

$$\sigma^{NLO} = \int_{m+1} d\sigma^{R}_{m+1} J_{m+1} + \int_{m} d\sigma^{V}_{m} J_{m}$$

$$\sigma^{LO} = \int_{m} d\sigma^{B}_{m} J_{m}$$

J_m is in general a function that defines the properties of the observed m-jets

$$-d\sigma^{R,A}_{m+1}$$

Regularizes the real emission cross section in Its unresolved region of the phase space

$$\left(\int_1 d\sigma^{R,A}_{m+1}\right)$$

Regularizes the virtual emission cross section

THE NNLO COUNTERTERMS IN THE TS SCHEME

$$\sigma^{NNLO} = \int_{m+2} d\sigma^{RR}_{m+2} \, J_{m+2} + \int_{m+1} d\sigma^{RV}_{m+1} \, J_{m+1} + \int_{m} d\sigma^{VV}_{m} \, J_{m}$$

$$-d\sigma^{RR,A_1}_{m+2}$$

Regularizes the doubly-real cross section in the singly-unresolved region of the phase space

$$-d\sigma^{RR,A_2}_{m+2} + d\sigma^{RR,A_{12}}_{m+2}$$

Regularizes the doubly-real cross section in the doubly-unresolved region of the phase space and avoid the double subtraction

$$-d\sigma^{RV,A_1}_{m+1}$$

Regularizes the real-virtual cross section in the singly-unresolved region of the phase space

$$-\left(\int_1 d\sigma^{RR,A_1}_{m+2}\right)^{A_1}$$

Regularizes the first counterterm integrated over one-unresolved parton when the other one becomes also unresolved

WHAT IS NEEDED TO DEFINE SUCH A SCHEME

To complete such a scheme three problems must be solved:

1. **The disentanglement of overlapping of soft and/or collinear configurations to avaoid multiple subtractions at any order**

   [Z. Nagy ,Z. Trocsanyi, G. Somogyi]

2. **From the strict unresolved limits to the whole phase space a proper mapping of momenta in needed to respect QCD factorization and the IR divergences cancellations**

   [Z. Trocsanyi, G. Somogyi, V. Del Duca]

3. **The computation of the integrated subtraction terms over the phase space of the unresolved partons**

The first two problems have been solved. Here we discuss the third one

ANALYTIC AND NUMERICAL EVALUATION OF THE INTEGRATED COUNTERTERMS

1.The cancellations of all IR poles in this scheme is more convincingly once the structure of the integrated subtraction terms is exhibited analytically

2. The analytic results are very fast and very accurate compared to numerical ones

HOWEVER

we will see that the analytic results also show that the integrated counterterms consist of very smooth functions

HENCE

The final results for the integrated real-virtual counterterms can be conveniently given e.g. in the form of interpolating tables computed once and for all including efficiently also those cases (like the finite parts) for which the analytic results can not be carried out

THE INTEGRATION OF THE COUNTERTERMS: the state of art of analytic computation

$$\int_1 d\sigma_{m+2}^{RR,A_1}$$

Collinear, soft and collinear-soft integrals already computed analytically

[Z. Trocsanyi, G. Somogyi, V. Del Duca]

$$\int_1 d\sigma_{m+1}^{RV,A_1}$$

Collinear, soft and collinear-soft integrals: computed

[U. Aglietti, V. Del Duca, C. Duhr, G. Somogyi, Z. Trocsanyi; P.Bolzoni]

$$\int_1 \left(\int_1 d\sigma_{m+2}^{RR,A_1}\right)^{A_1}$$

Nested integrals: analytic computation finished

[P. Bolzoni, S. Moch, G. Somogyi, Z. Trocsany]

Numerical evaluation of all the singly-unresolved integrals computed numerically using sector decomposition (T. Binoth, G. Heinrich)

[Z. Trocsanyi, G. Somogyi,]

$$\int_2 \left[d\sigma_{m+2}^{RR,A_2} - d\sigma_{m+2}^{RR,A_{12}}\right]$$

To be done

3. PHASE SPACE INTEGRALS:
Structure, computation and results
(We show only the collinear case as an illustrative example)

THE COLLINEAR and NESTED COLLINEAR INTEGRALS

$$x = \frac{2\tilde{p}_{ir}\cdot Q}{Q^2}$$

Kinematic variable that describes the splitting couple of partons

$k = -1, 0, 1, 2$ $\qquad$ $\kappa = 0, 1$

δ	Function	$g_I^{(\pm)}(z)$
0	g_A	1
∓ 1	$g_B^{(\pm)}$	$(1-z)^{\pm\epsilon}$
0	$g_C^{(\pm)}$	$(1-z)^{\pm\epsilon}{}_2F_1(\pm\epsilon,\pm\epsilon,1\pm\epsilon,z)$
± 1	$g_D^{(\pm)}$	${}_2F_1(\pm\epsilon,\pm\epsilon,1\pm\epsilon,1-z)$

$$\begin{aligned}\mathcal{I}(x;\epsilon,d_0,\kappa,k,\delta,g_I^{(\pm)}) &= x\int_0^1 d\alpha\,(1-\alpha)^{2d_0-1}\,[\alpha(\alpha+(1-\alpha)x)]^{-1-(1+\kappa)\epsilon} \\ &\times\int_0^1 dv\,[v(1-v)]^{-\epsilon}\left(\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x}\right)^{k+\delta\epsilon} g_I^{(\pm)}\left(\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x}\right)\end{aligned}$$

$$\begin{aligned}\mathcal{I}{*}\mathcal{I}_r(x;\epsilon,\alpha_0,d_0;k,l) &= x\int_0^{\alpha_0}\mathrm{d}\alpha\int_0^1\mathrm{d}v\,\alpha^{-1-\epsilon}\,(1-\alpha)^{2d_0-1}\,[\alpha+(1-\alpha)x]^{-1-\epsilon} \\ &\times[v(1-v)]^{-\epsilon}\left[\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x}\right]^k \mathcal{I}\left(x\frac{\alpha+(1-\alpha)xv}{2\alpha+(1-\alpha)x};\epsilon,\alpha_0,d_0;0,l,0,1\right),\end{aligned}$$

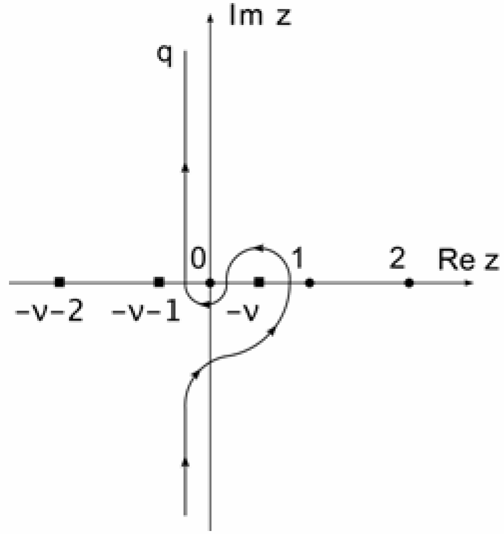
The integration variable v accounts for different fractions of momentmum that the parton p_r carries away from the splitting (ir)

The collinear counterterms to the cross section are consructed from these 'master' integrals

THE METHOD OF MB REPRESENTATION

1. Convert sums into products in the integrand

$$(a+b)^{-\nu} = \frac{1}{\Gamma(\nu)} \int_{q-i\infty}^{q+i\infty} \frac{dz}{2\pi i}\, a^{-\nu-z}\, b^{z}\, \Gamma(\nu+z)\Gamma(-z)$$



2. Integrate the over the real variables to obtain MB integrals

[V. A. Smirnov , J. B. Tausk]

3. Compute the MB integrals converting them into sums over residua

[MB.m: M. Czakon; AMBRE.m: J. Gluza, F. Haas, K. Kajda, T. Riemann]

4. Perform the sums

[XSummer: S. Moch, P. Uwer]

AN EXAMPLE

$$\mathcal{E}(x;\epsilon,d_0) = x^2 \int_0^1 d\alpha \, \frac{\alpha^{-1-\epsilon}\,(1-\alpha)^{2d_0}}{[\alpha + (1-\alpha)]^{-1-\epsilon}\,[2\alpha + (1-\alpha)x]^{-1}} =$$

$$= \int_{q_1 - i\infty}^{q_1 + \infty} \frac{dz_1}{2\pi i} \int_{q_2 - i\infty}^{q_2 + i\infty} \frac{dz_2}{2\pi i}\, 2^{z_2}\, x^{-\epsilon - z_1 - z_2}$$

$$\times \Gamma\left(\begin{matrix} -z_1,\ -z_2,\ 2d_0 - 1 - \epsilon - z_1 - z_2,\ 1 + \epsilon + z_1,\ 1 + z_2,\ -\epsilon + z_1 + z_2 \\ 2d_0 - 1 - 2\epsilon,\ 1 + \epsilon \end{matrix}\right)$$

In this case for $d_0 \geq 3$ a good choise is q_1=-1/4, q_2=-1/8,ε=-1/2

Computing residua to perform the analytic continuation to ε=0, expanding in ε and converting MB integrals into sums, one gets:

$$\mathcal{E}(x;\epsilon,d_0) = -\frac{1}{\epsilon} - \log(2)\Sigma_0(x,d_0) + \log(x)\Sigma_1(x,d_0) - \Sigma_2(x,d_0)$$

$\Sigma_0(x,d_0)$,$\Sigma_1(x,d_0)$and $\Sigma_2(x,d_0)$ are double sums

The extraction of poles comes out in a convenient way

SUMS

To have an ides of the sums involved let's look at one sum of the example

$$\Sigma_2(x,d_0) = \sum_{m,n=1}^{\infty} \left(\frac{x}{2}\right)^m x^n \binom{2d_0-2+m+n}{m+n} \left[S_1(2d_0-2+m+n) - S_1(m+n)\right]$$

If d_0 is chosen as a positive integer it happens that

$$\binom{2d_0-2+m+n}{m+n} \left[S_1(2d_0-2+m+n) - S_1(m+n)\right]$$

Is a polynomial in m and n, thus the sum can be expressed in terms of these functions

$$\sum_{n=1}^{\infty} \frac{x^n}{n^k} = \begin{cases} \mathrm{Li}_k(x) & \text{if } k \geq 0 \\ \frac{1}{(1-x)^{1-k}} \sum_{i=0}^{-k-1} \left\langle \begin{matrix} -k \\ i \end{matrix} \right\rangle x^{-k-i} & \text{if } k < 0 \end{cases}$$

This is a simple case, in general the sums are more complicated and we used XSummer

[XSummer: S. Moch, P. Uwer]

COLLINEAR INTEGRALS:
New analytic results

$$\mathcal{I}(x;\epsilon,d_0;1,k,\delta,g_I^{(\pm)}) = \frac{\delta_{k,-1}}{2(2-\delta)}\frac{1}{\epsilon^2} - \left[\frac{2\delta_{k,-1}\log(x)}{3-\delta} + \frac{1-\delta_{k,-1}}{2[1+k(1-\delta_{k,-1})]}\frac{1}{\epsilon}\right] + \mathcal{G}_{I,k}^{(\pm)}(x) + \mathcal{F}(x;\epsilon,d_0,k) + O(\epsilon)$$

$I = C, D$

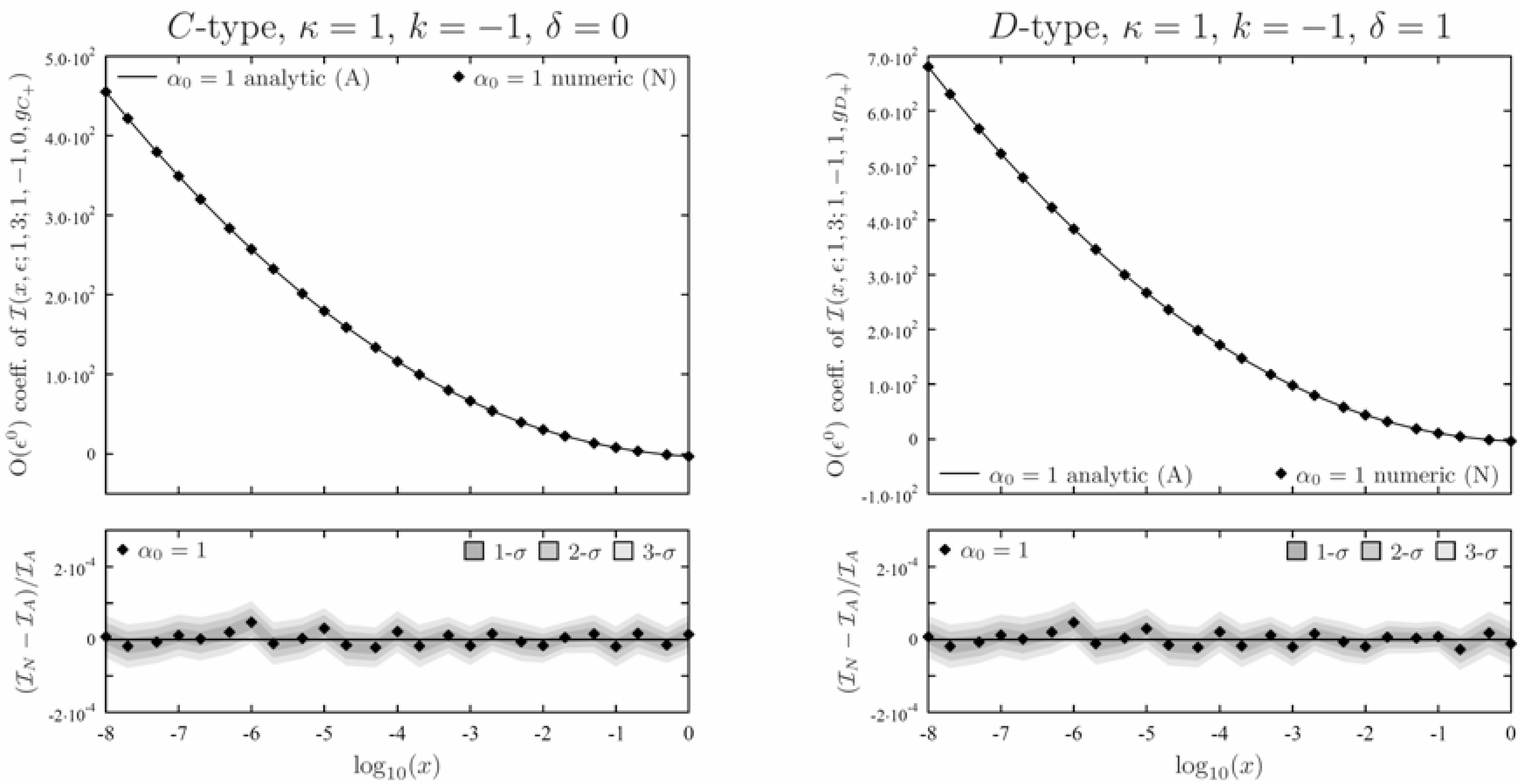
$k = -1, 0, 1, 2$

$$\mathcal{G}_{I,k}^{(\pm)}(x) = \begin{pmatrix} \left(\frac{2}{3}\pm\frac{1}{2}\right)\zeta_2 + \frac{1}{3}\log^2(x) & 1\pm\frac{1}{2} & \frac{1}{2}\pm\frac{3}{8} & \frac{13}{36}\pm\frac{11}{36} \\ \left(\frac{13}{36}\mp\frac{1}{16}\right)\zeta_2 + \left(\frac{1}{2}\pm\frac{1}{2}\right)\log^2(x) & 1\pm\frac{1}{2} & \frac{1}{2}\pm\frac{1}{8} & \frac{13}{36}\pm\frac{1}{18} \end{pmatrix}$$

$$\begin{aligned}\mathcal{F}(x;\epsilon,d_0=3,-1) &= -\frac{3}{2}\zeta_2 + \log^2(x) - \frac{x(1-x)(35x^3-133x^2+188x-116)}{24(1-x)^5} \\ &+ \frac{x(25x^4-116x^3+212x^2-192x+96)}{12(1-x)^5}\log(x) \\ &+ \frac{(2-x)(x^4-3x^3+4x^2-2x+1)}{(1-x)^5}\mathrm{Li}_2(1-x)\end{aligned}$$

$$\lim_{x\to1}\mathcal{F}(x;\epsilon,d_0=3,-1) = -\frac{8731}{3600} - \frac{3}{2}\zeta_2$$

COLLINEAR INTEGRALS: Numeric and analytic results



Comparison between numeric and analytic results. The numeric results have been obtained using standard residuum subtraction and a Monte Carlo integration program

NESTED COLLINEAR INTEGRALS:
New analytic results

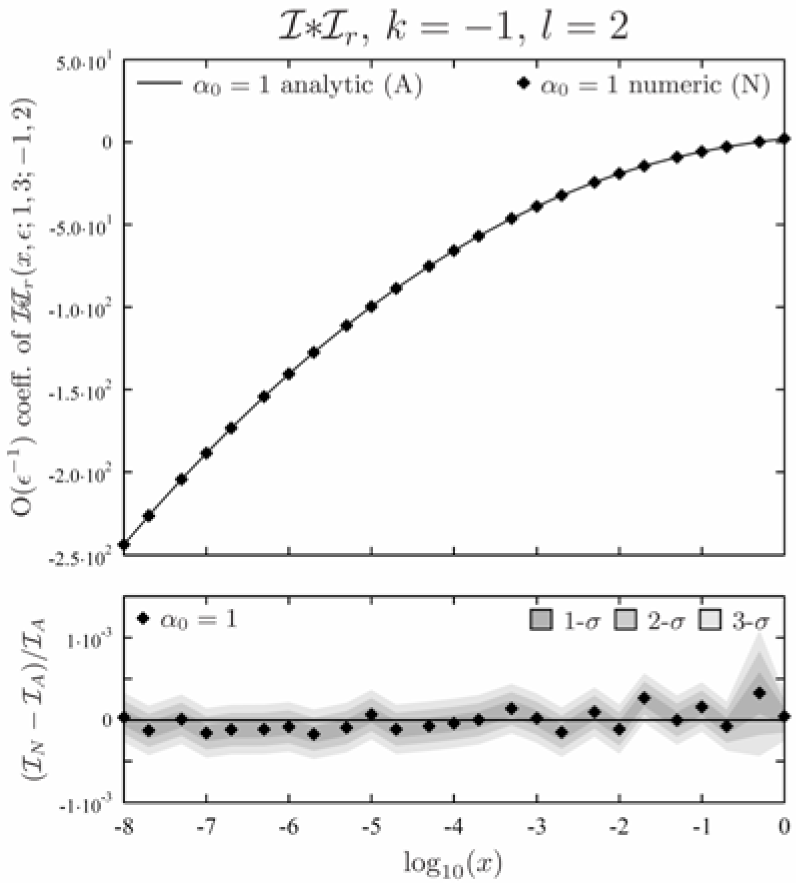
$$\mathcal{I}{*}\mathcal{I}_r(x,\epsilon;1,3;-1,2) = -\frac{1}{12}\frac{1}{\epsilon^3} + \left(-\frac{2}{9} + \frac{1}{3}\log(x)\right)\frac{1}{\epsilon^2} + \left[\frac{1}{(1-x)^5}\left(-\frac{1}{3}\zeta_2 - \frac{25}{36}\log(x)\right.\right.$$
$$\left. + \frac{1}{3}\log(1-x)\log(x) + \frac{1}{3}\mathrm{Li}_2(x)\right) + \frac{1}{(1-x/2)^5}\left(\frac{1}{6}\log\left(\frac{x}{2}\right)\right) + \frac{1}{(1-x)^4}\left(-\frac{13}{36} + \frac{1}{6}\log(x)\right)$$
$$+\frac{1/6}{(1-x/2)^4} + \frac{1}{(1-x)^3}\left(-\frac{7}{72} - \frac{1}{18}\log(x)\right) + \frac{1/12}{(1-x/2)^3}$$
$$+\frac{1}{(1-x)^2}\left(-\frac{1}{6} - \frac{2}{9}\log(x)\right) + \frac{1/18}{(1-x/2)^2} + \frac{1}{(1-x)}\left(-\frac{25}{72} - \frac{7}{12}\log(x)\right)$$
$$+\frac{1/24}{(1-x/2)} + \frac{31}{216} + \frac{1}{6}\log(2) + \frac{19}{9}\log(x) + \frac{2}{3}\log(1-x)\log(x) - \frac{2}{3}\log^2(x)$$
$$\left. + \frac{2}{3}\mathrm{Li}_2(x)\right]\frac{1}{\epsilon} + \mathrm{O}(\epsilon^0).$$

$$\lim_{x\to 1}\mathcal{I}{*}\mathcal{I}_r(x,\epsilon;1,3;-1,2) = -\frac{1}{12}\frac{1}{\epsilon^3} - \frac{2}{9}\frac{1}{\epsilon^2} + \left(\frac{3091}{675} + \frac{2}{3}\zeta_2 - \frac{31}{6}\log(2)\right)\frac{1}{\epsilon} + \mathrm{O}(\epsilon^0)$$

NESTED COLLINEAR INTEGRALS: Numerical and analytic results

Comparison between numeric and analytic results. The agreement is excellent. This analytic examples shows that the expansion coefficients of the integrals building the counterterms are are smooth functions

For practical purposes the integrals can be used in terms of interpolating tables



An available package SUMI.m evaluates all the singly-unresolved integrals from both the analytic results and from evaluation of MB integrals representations in the complex plane up to finite parts of the Laurent expansion

4.CONCLUSIONS AND OUTLOOKS

CONCLUSIONS AND OUTLOOKS

1. We have established a systematic method for the integration of counterterms in the TS scheme

   - MB representation and subsequent summation (up to triple sums)
   - Real-virtual and iterated integrated counterterms computed
   - Doubly-real integrated counterterms are feasible

2. The counterterms are all very smooth functions

3. We have studied couterterms in the universal process-independent TS subtraction scheme

   - Worked out for NNLO QCD with colorless incoming particles
   - Hopefully also more efficient

4. A Mathematica file SUMI.m contains all the analytic results and MB representations which can be used to efficiently evaluate the interals [arXiv: 0905.4390]

5. A NNLO subtraction scheme for hadron-initiated QCD processes and for heavy quark production are the future natural steps