Generalised Unitarity for One-Loop Amplitudes

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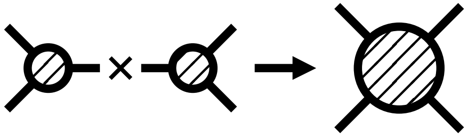
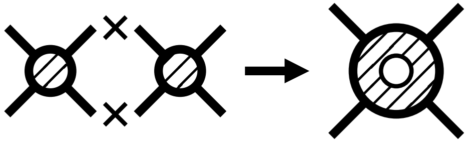
Overview

- On-shell methods for one-loop amplitudes
- Generalised unitarity approach
- Application to six gluon amplitudes

Motivations for On-Shell Constructions

- On-shell quantities are simpler than Feynman representations
- Construct S-matrix elements using analytic properties
- New insight from Twistor Space [Witten (2003)]
  - Use amplitudes as fundamental building blocks
  - Make use of complex momenta
  - Recursion:  [Britto,Cachazo,Feng,Witten (2004)]
  - Unitarity: 

[Bern,Dixon,Dunbar,Kosower (1994)]

[Bern,Dixon,Kosower (1997)]

[Britto,Cachazo,Feng (2004)]

Recent Advances with On-Shell Approaches

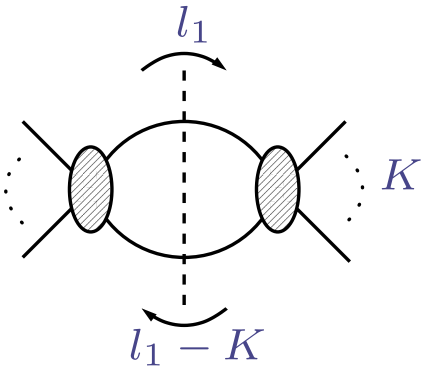
- Full numerical programs for NLO matrix elements:
  - Helac-1loop : QCD $2 \rightarrow 4$ Feynman+OPP [van Hameren,Ossola,Papadopoulos,Pittau]
  - Rocket ($n \sim 20$) : $n(g), q\bar{q} + n(g), t\bar{t} + 3g, W + 3j$ [Ellis,Giele,Kunszt,Melnikov,Zanderighi]
  - BlackHat : $8g, W + 3j$ full colour [Berger et al.]
  - Further C++ implementations [Giele,Winter;Lazopoulos]
- Generatation of compact analytic formulae @ NLO
  - One-Loop Recursion Relations [Berger,Bern,Forde,Dixon,Kosower]
  - Spinor Integration Reduction [Britto,Buchbinder,Cachazo,Feng,Mastrolia]
  - D-dimensional cuts [Anastasiou,Britto,Feng,Kunszt,Mastrolia,Yang;SB]
- Feasibility of complicated multi-loop amplitudes, e.g.
  - $\mathcal{N} = 4$ 4-loop 4-gluon [Bern,Czakon,Dixon,Kosower]
  - $\mathcal{N} = 4$ 2-loop 6-gluon [Bern,Dixon,Kosower,Roiban,Spradlin,Vergu,Volovich]
  - $\mathcal{N} = 8$ 4-loop 4-graviton [Bern,Carrasco,Dixon,Johansson,Kosower,Roiban]

Generalised unitarity for one-loop computations

Unitarity Cuts

- Cutkosky Rules to compute $\mathrm{Disc}_p(A^{1-\mathrm{loop}})$

$$\frac{1}{p^2 - m^2 + iO^+} \to i\delta^{(+)}(p^2 - m^2)$$

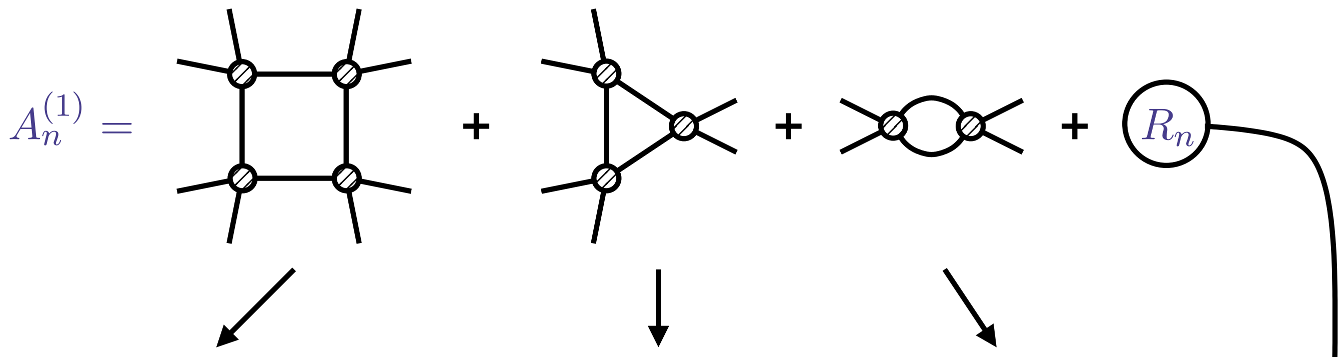


- Construction of amplitudes from double cuts [Bern,Dixon,Dunbar,Kosower (1994)]
- 4-dimensional cuts suitable for supersymmetric theories
  - $\mathcal{N} = 4$ all-n MHV [Bern,Dixon,Dunbar,Kosower (1994)]
  - $\mathcal{N} = 1$ all-n MHV, $\mathcal{N} = 4$ 6-point NMHV [Bern,Dixon,Dunbar,Kosower (1994)]

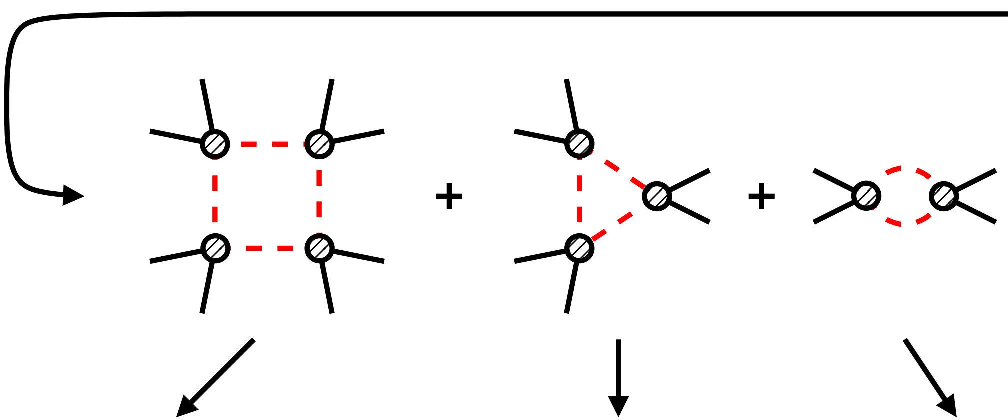
One-Loop Integral Basis

Generalised cuts: loops from trees:

[Bern,Dixon,Dunbar,Kosower;Britto,Cachazo,Feng,Mastrolia,Yang;Ossola,Papadopoulos,Pittau;Forde]



$$C_4 = \prod_{i=1}^{4} A_i(l) \qquad C_3 = \lim_{t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0} \qquad C_2 \sim \lim_{y/t\to\infty} \prod_{i=1}^{2} A_i(l(t))$$



D-dimensional cuts:

[Ossola,Papadopoulos,Pittau;
Giele,Kunszt,Melnikov;
Britto,Feng,Mastrolia;SB]

$$C_4^{[4]} = \lim_{\mu^2\to\infty} \prod_{i=1}^{4} A_i(l)|_{\mu^4} \qquad C_3^{[2]} = \lim_{\mu^2,t\to\infty} \prod_{i=1}^{3} A_i(l(t))|_{t^0,\mu^2} \quad C_2^{[2]} \sim \lim_{\mu^2,y/t\to\infty} \prod_{i=1}^{2} A_i(l(t))$$

Six-Gluon Helicity Amplitudes

- 29889 Feynman diagrams with 6-rank tensor hexagon integrals

$$\mathcal{A}_6^{(1)}(1,2,3,4,5,6) = \sum_{\sigma_6} \sum_{k=0}^{4} G_{n;k+1}\, \mathrm{tr}(T^{a_1} \ldots T^{a_k})\, \mathrm{tr}(T^{a_k+1} \ldots T^{a_5} T^{a_6}) \times A_{n;k+1}^{(1)}(1,2,3,4,5,6)$$

- $G_{6;1} = N_c$: sub-leading colours obtained from permutations
- $A_{6;1}$ only independent quanitity: evaluate in helicity basis
- Full analytic QCD corrections now available:

Mahlon(1993); Bern,Dixon,Dunbar,Kosower(1994,1995); Bidder,Bjerrum-Bohr,Dixon,Dunbar(2004); Bedford,Brandhuber,Spence,Travaglini(2004); Britto,Buchbinder,Cachazo,Feng(2005); Britto,Feng,Mastrolia(2005); Bern,Bjerrum-Bohr,Ita,Dunbar(2005); Bern,Dixon,Kosower(2005); Berger,Bern,Dixon,Forde,Kosower(2006); Xiao,Yang,Zhang(2006); Dunbar(2008)

Six-Gluon Integral Basis

Integrals

$I_4(s,t;\{M_i^2\})$

$I_3(\{M_i^2\})$

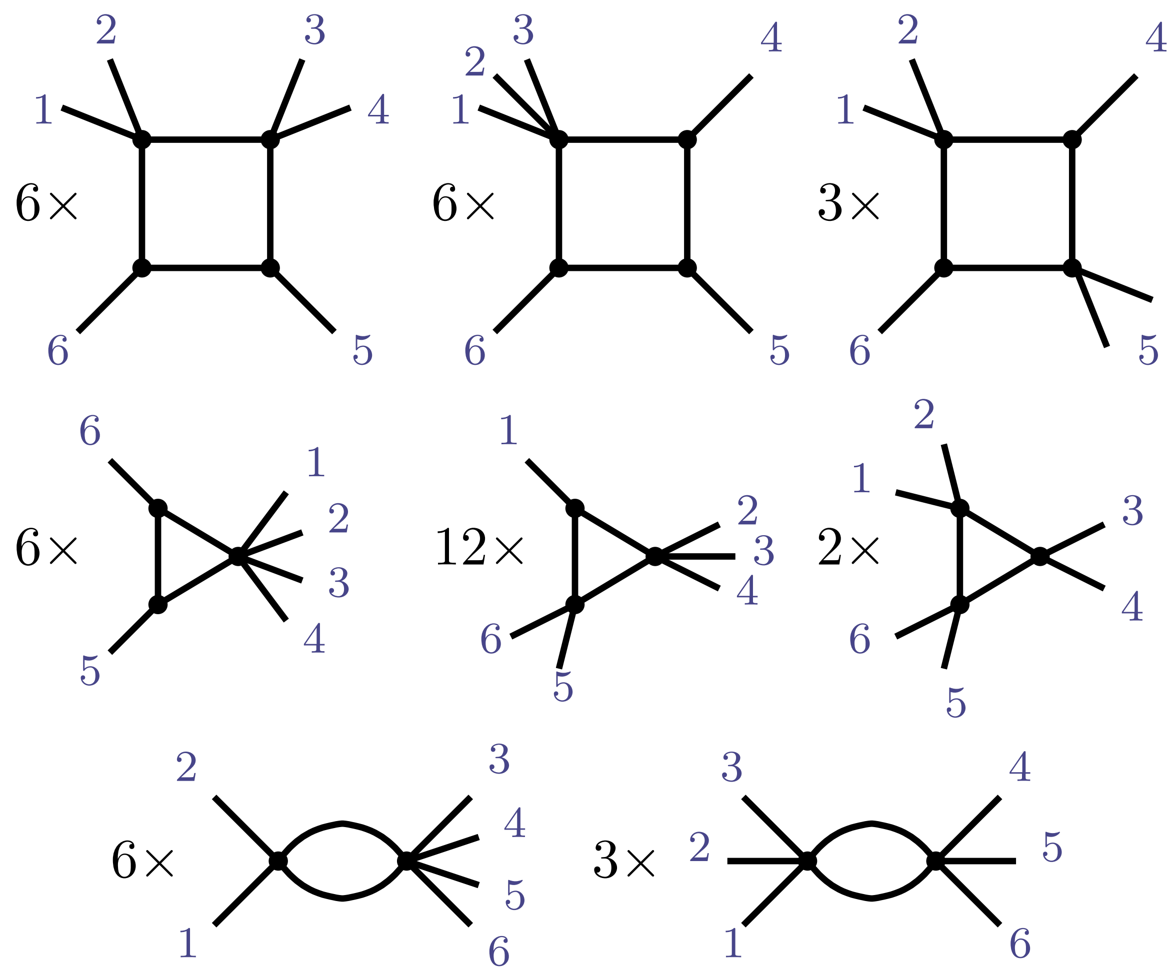
$I_2(s)$

Public codes:

`LoopTools`

`CutTools`

`QCDloop`



34 Integral coefficient (15 boxes, 20 triangles, 9 bubbles)

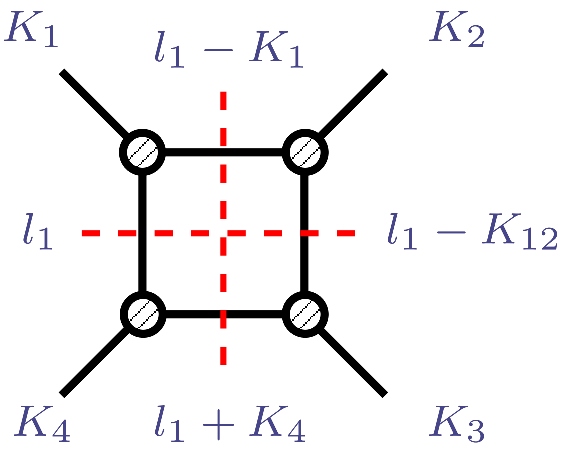
Generalised Unitarity

- Cutting more propagators to isolate the various integral functions

[Bern,Dixon,Kosower (1997)]

[Britto,Cachazo,Feng (2004)]

$$\{l^2 = 0, (l - K_1)^2 = 0, (l - K_{12})^2 = 0, (l + K_4)^2 = 0\}$$



- 2 complex solutions
- 4D loop momentum completely fixed

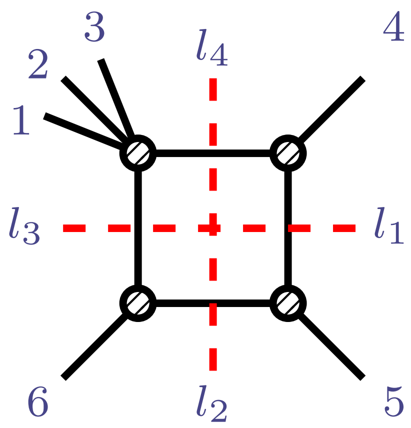
$$C_4(K_1, K_2, K_3, K_4) = \frac{1}{2} \sum_{l^\pm} A^{(0)}(l_1, K_1, -l_2) A^{(0)}(l_2, K_2, -l_3) A^{(0)}(l_3, K_3, -l_4) A^{(0)}(l_4, K_4, -l_1)$$

Loop momentum paramterisation

$$l^\nu = a(K_1^\flat)^\nu + b(K_2^\flat)^\nu + c\langle K_1^\flat|\gamma^\nu|K_2^\flat] + d\langle K_2^\flat|\gamma^\nu|K_1^\flat]$$

Box Contributions for A_6^g

- One-mass box: $C_{4;123|4|5|6}$



$$\{l_1^2 = 0, (l_1 - 5)^2 = 0, (l_1 - 5 - 6)^2 = 0, (l + 4)^2 = 0\}$$

$$l_1^{(1)} = \frac{[56]}{2[46]}\langle 4|\gamma^\mu|5], \qquad l_1^{(2)} = \frac{\langle 56\rangle}{2\langle 46\rangle}\langle 5|\gamma^\mu|4]$$

- Simple substitution into the products of tree amplitudes:

$$C_{4;123}(1^\pm, 2^+, 3^+, 4^+, 5^+, 6^+) = 0$$

$$C_{4;123}(1^-, 2^-, 3^+, 4^+, 5^+, 6^+) = 0$$

$$C_{4;123}(1^-, 2^-, 3^-, 4^+, 5^+, 6^+) = 0$$

$$C_{4;123}(1^-, 2^+, 3^-, 4^+, 5^+, 6^+) = s_{45}s_{56}A^{(0)}(1^-, 2^+, 3^-, 4^+, 5^+, 6^+)$$

Box Contributions for A_6^g

Maximal Helicity Violating (MHV): $2-, 4+$

$$C_4(1^-,2^-,3^+,4^+,5^+,6^+) = -\frac{\langle 12\rangle^3}{\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\Big($$

$$\mathrm{I}_4\left(s_{16},s_{12},s_{345}\right)s_{12}s_{16} + \mathrm{I}_4\left(s_{56},s_{16},s_{234}\right)s_{56}s_{16} + \mathrm{I}_4\left(s_{12},s_{23},s_{456}\right)s_{12}s_{23} +$$

$$\mathrm{I}_4\left(s_{23},s_{34},s_{156}\right)s_{23}s_{34} + \mathrm{I}_4\left(s_{34},s_{45},s_{126}\right)s_{34}s_{45} + \mathrm{I}_4\left(s_{45},s_{56},s_{123}\right)s_{45}s_{56} +$$

$$\mathrm{I}_4\left(s_{123},s_{234},s_{23},s_{56}\right)s_{123}s_{234} + \mathrm{I}_4\left(s_{234},s_{345},s_{34},s_{16}\right)s_{234}s_{345} + \mathrm{I}_4\left(s_{345},s_{456},s_{45},s_{12}\right)s_{345}s_{456}\Big)$$

$$C_4(1^-,2^+,3^-,4^+,5^+,6^+) = -\frac{\langle 13\rangle^4}{2\langle 16\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\Big($$

$$\mathrm{I}_4\left(s_{12},s_{23},s_{456}\right)s_{12}s_{23} + \mathrm{I}_4\left(s_{34},s_{45},s_{126}\right)s_{34}s_{45} + \mathrm{I}_4\left(s_{56},s_{16},s_{234}\right)s_{16}s_{56} + \mathrm{I}_4\left(s_{45},s_{56},s_{123}\right)s_{45}s_{56}$$

$$+\,\mathrm{I}_4\left(s_{23},s_{34},s_{156}\right)\left(\frac{s_{23}s_{34}\langle 14\rangle^4\langle 23\rangle^4}{\langle 13\rangle^4\langle 24\rangle^4} + \frac{s_{23}s_{34}\langle 12\rangle^4\langle 34\rangle^4}{\langle 13\rangle^4\langle 24\rangle^4}\right)$$

$$+\,\mathrm{I}_4\left(s_{16},s_{12},s_{345}\right)\left(\frac{s_{12}s_{16}\langle 16\rangle^4\langle 23\rangle^4}{\langle 13\rangle^4\langle 26\rangle^4} + \frac{s_{12}s_{16}\langle 12\rangle^4\langle 36\rangle^4}{\langle 13\rangle^4\langle 26\rangle^4}\right)$$

$$+\,\mathrm{I}_4\left(s_{345},s_{234},s_{34},s_{16}\right)\left(\frac{\langle 15\rangle^4\langle 2|K_{34}|5]\langle 5|K_{34}|2]\langle 23\rangle^4}{\langle 13\rangle^4\langle 25\rangle^4} + \frac{\langle 12\rangle^4\langle 35\rangle^4\langle 2|K_{34}|5]\langle 5|K_{34}|2]}{\langle 13\rangle^4\langle 25\rangle^4}\right)$$

$$+\,\mathrm{I}_4\left(s_{234},s_{123},s_{23},s_{56}\right)\langle 1|K_{23}|4]\langle 4|K_{23}|1] + \mathrm{I}_4\left(s_{123},s_{345},s_{12},s_{45}\right)\langle 3|K_{45}|6]\langle 6|K_{45}|3]\Big)$$

Box Contributions for A_6^g

Next-to-Maximal Helicity Violating (NMHV): $3^-, 3^+$

$$C_4(1^-,2^-,3^-,4^+,5^+,6^+) =$$

$$\mathrm{I}_4\left(s_{16},s_{123},s_{23},s_{45}\right)\frac{s_{16}s_{123}^4}{\langle 45\rangle\langle 56\rangle\langle 4|K_{123}|1]\langle 6|K_{123}|3][21][32]} + \mathrm{I}_4\left(s_{12},s_{23},s_{456}\right)\frac{s_{12}s_{23}s_{123}^3}{\langle 45\rangle\langle 56\rangle\langle 4|K_{123}|1]\langle 6|K_{123}|3][21][32]}$$

$$+\,\mathrm{I}_4\left(s_{45},s_{56},s_{123}\right)\frac{s_{45}s_{56}s_{123}^3}{\langle 45\rangle\langle 56\rangle\langle 4|K_{123}|1]\langle 6|K_{123}|3][21][32]} + \mathrm{I}_4\left(s_{34},s_{456},s_{56},s_{12}\right)\frac{s_{34}s_{456}s_{123}^3}{\langle 45\rangle\langle 56\rangle\langle 4|K_{123}|1]\langle 6|K_{123}|3][21][32]}$$

$$+\,\mathrm{I}_4\left(s_{23},s_{345},s_{45},s_{16}\right)\left(-\frac{s_{23}\langle 3|K_{12}|6]^4}{\langle 34\rangle\langle 45\rangle\langle 3|K_{612}|6]\langle 5|K_{612}|2][21][61]} - \frac{s_{23}\langle 12\rangle^3[54]^3}{\langle 16\rangle\langle 2|K_{612}|5]\langle 6|K_{612}|3][43]}\right)$$

$$+\,\mathrm{I}_4\left(s_{16},s_{12},s_{345}\right)\left(-\frac{s_{12}s_{16}\langle 3|K_{12}|6]^4}{s_{345}\langle 34\rangle\langle 45\rangle\langle 3|K_{612}|6]\langle 5|K_{612}|2][21][61]} - \frac{s_{12}s_{16}\langle 12\rangle^3[54]^3}{s_{345}\langle 16\rangle\langle 2|K_{612}|5]\langle 6|K_{612}|3][43]}\right)$$

$$+\,\mathrm{I}_4\left(s_{34},s_{45},s_{126}\right)\left(-\frac{s_{34}s_{45}\langle 3|K_{12}|6]^4}{s_{345}\langle 34\rangle\langle 45\rangle\langle 3|K_{612}|6]\langle 5|K_{612}|2][21][61]} - \frac{s_{34}s_{45}\langle 12\rangle^3[54]^3}{s_{345}\langle 16\rangle\langle 2|K_{612}|5]\langle 6|K_{612}|3][43]}\right)$$

$$+\,\mathrm{I}_4\left(s_{56},s_{126},s_{12},s_{34}\right)\left(-\frac{s_{56}s_{126}\langle 3|K_{12}|6]^4}{s_{345}\langle 34\rangle\langle 45\rangle\langle 3|K_{612}|6]\langle 5|K_{612}|2][21][61]} - \frac{s_{56}s_{126}\langle 12\rangle^3[54]^3}{s_{345}\langle 16\rangle\langle 2|K_{612}|5]\langle 6|K_{612}|3][43]}\right)$$

$$+\,\mathrm{I}_4\left(s_{12},s_{234},s_{34},s_{56}\right)\left(-\frac{s_{12}\langle 1|K_{23}|4]^4}{\langle 16\rangle\langle 56\rangle\langle 1|K_{234}|4]\langle 5|K_{234}|2][32][43]} - \frac{s_{12}\langle 23\rangle^3[65]^3}{\langle 34\rangle\langle 2|K_{234}|5]\langle 4|K_{234}|1][61]}\right)$$

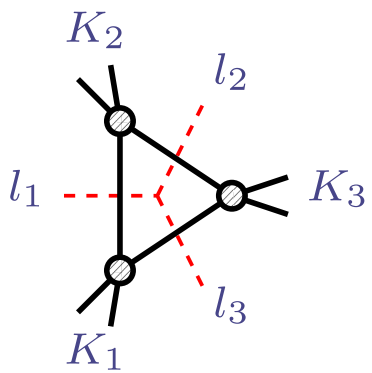
$$+\,\mathrm{I}_4\left(s_{23},s_{34},s_{156}\right)\left(-\frac{s_{23}s_{34}\langle 1|K_{23}|4]^4}{s_{234}\langle 16\rangle\langle 56\rangle\langle 1|K_{234}|4]\langle 5|K_{234}|2][32][43]} - \frac{s_{23}s_{34}\langle 23\rangle^3[65]^3}{s_{234}\langle 34\rangle\langle 2|K_{234}|5]\langle 4|K_{234}|1][61]}\right)$$

$$+\,\mathrm{I}_4\left(s_{56},s_{16},s_{234}\right)\left(-\frac{s_{16}s_{56}\langle 1|K_{23}|4]^4}{s_{234}\langle 16\rangle\langle 56\rangle\langle 1|K_{234}|4]\langle 5|K_{234}|2][32][43]} - \frac{s_{16}s_{56}\langle 23\rangle^3[65]^3}{s_{234}\langle 34\rangle\langle 2|K_{234}|5]\langle 4|K_{234}|1][61]}\right)$$

$$+\,\mathrm{I}_4\left(s_{45},s_{156},s_{16},s_{23}\right)\left(-\frac{s_{45}s_{156}\langle 1|K_{23}|4]^4}{s_{234}\langle 16\rangle\langle 56\rangle\langle 1|K_{234}|4]\langle 5|K_{234}|2][32][43]} - \frac{s_{45}s_{156}\langle 23\rangle^3[65]^3}{s_{234}\langle 34\rangle\langle 2|K_{234}|5]\langle 4|K_{234}|1][61]}\right)$$

Further Reduction: Triangle Coefficients

- Complex parameterisation for the loop momentum: [Ossola,Papadopoulos,Pittau] [Forde]



$$l^\nu = a_1 (K_1^\flat)^\nu + a_2 (K_2^\flat)^\nu + \frac{t}{2} \langle K_1^\flat | \gamma^\nu | K_2^\flat] + \frac{a_3}{2t} \langle K_2^\flat | \gamma^\nu | K_1^\flat]$$

a_1, a_2, a_3 fixed by on-shell constraints

- Consider t as a complex variable:

$$\int dt J_t A_1 A_2 A_3 = \int dt J_t \underbrace{\mathrm{Inf}_t}[A_1 A_2 A_3] + \sum_i \frac{\mathrm{Res}_{t=t_i}(A_1 A_2 A_3)}{(t - t_i)}$$

Laurent series around $t = \infty$
$\mathrm{Inf}_t[X(t)] = x_0 + x_1 t + x_2 t^2 + x_3 t^3$

- Integrals over non-zero powers of t vanish

$$C_3 = -\sum_\gamma \mathrm{Inf}_t[A_1 A_2 A_3]|_{t^0}$$

Triangle Contributions for A_6^g

$C_{3;2345}(1^-,2^+,3^-,4^+,5^+,6^+) =$

$$\frac{i\langle 12\rangle^3[61]\langle 36\rangle^4}{\langle 23\rangle\langle 26\rangle^4\langle 34\rangle\langle 45\rangle\langle 56\rangle} - \frac{2i\langle 12\rangle^2\langle 13\rangle[61]\langle 36\rangle^3}{\langle 23\rangle\langle 26\rangle^3\langle 34\rangle\langle 45\rangle\langle 56\rangle} + \frac{3i\langle 12\rangle\langle 13\rangle^2[61]\langle 36\rangle^2}{\langle 23\rangle\langle 26\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle} - \frac{2i\langle 13\rangle^3[61]\langle 36\rangle}{\langle 23\rangle\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}$$

$C_{3;12|345}(1^-,2^+,3^-,4^+,5^+,6^+) =$

$$\frac{i\langle 12\rangle^2\langle 36\rangle^3[61]^3\langle 16\rangle^3}{\langle 26\rangle^4\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6|K_{12}|6]^2} - \frac{i\langle 12\rangle\langle 36\rangle^2\langle 3|K_{12}|6][61]^2\langle 16\rangle^3}{\langle 26\rangle^3\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6|K_{12}|6]^2} + \frac{2i\langle 36\rangle\langle 3|K_{12}|6]^2[61]\langle 16\rangle^3}{\langle 26\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6|K_{12}|6]^2} +$$

$$\frac{2i\langle 12\rangle^2\langle 36\rangle^3[61]^2[62]\langle 16\rangle^2}{\langle 26\rangle^3\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6|K_{12}|6]^2} + \frac{2i\langle 12\rangle\langle 36\rangle^2\langle 3|K_{12}|6][61][62]\langle 16\rangle^2}{\langle 26\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6|K_{12}|6]^2} + \frac{2i\langle 36\rangle\langle 3|K_{12}|6]^2[62]\langle 16\rangle^2}{\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6|K_{12}|6]^2} +$$

$$\frac{3i\langle 12\rangle^2\langle 36\rangle^3[61][62]^2\langle 16\rangle}{\langle 26\rangle^2\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6|K_{12}|6]^2} + \frac{3i\langle 12\rangle\langle 36\rangle^2\langle 3|K_{12}|6][62]^2\langle 16\rangle}{\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6|K_{12}|6]^2} + \frac{2i\langle 12\rangle^2\langle 36\rangle^3[62]^3}{\langle 26\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 6|K_{12}|6]^2}$$

Three-mass coefficients much longer ($\sim$ 200 terms)

Matching to IR structure

- One and two mass triangles not unique, e.g.

$$I_3^{1m}(s) = -\frac{c_\Gamma}{s\epsilon^2}\left(\frac{\mu_R^2}{-s}\right)^\epsilon$$

$$I_4^{1m}(s,t,M^2) = -\frac{2c_\Gamma}{st\epsilon^2}\left(\left(\frac{\mu_R^2}{-s}\right)^\epsilon + \left(\frac{\mu_R^2}{-t}\right)^\epsilon - \left(\frac{\mu_R^2}{-M^2}\right)^\epsilon\right) + \mathcal{O}(\epsilon)$$

- Universal IR pole structure [Giele,Glover(1991)]

$$A_n^{(1)} = -\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu_R^2}{-s_{i,i+1}}\right)^\epsilon + \mathcal{O}(\epsilon^{-1})$$

- Eliminate triangles in favour of $D = 6 - 2\epsilon$ dimensional box integrals
- Three-mass triangles still remain (only 4 non-zero coefficients in A_6^g)

Matching to IR structure

Maximal Helicity Violating (MHV): $2-, 4+$

$$C_{3+4}(1^-,2^-,3^+,4^+,5^+,6^+) = -\frac{\langle 12\rangle^3}{\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\Bigg(\frac{1}{\epsilon^2}\sum_{i=1}^{6}\left(\frac{\mu_R^2}{-s_{i,i+1}}\right)^\epsilon +$$

$$\mathrm{I}_4^{D=6-2\epsilon}(s_{16},s_{12},s_{345}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{56},s_{16},s_{234}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{12},s_{23},s_{456}) +$$

$$\mathrm{I}_4^{D=6-2\epsilon}(s_{23},s_{34},s_{156}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{34},s_{45},s_{126}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{45},s_{56},s_{123}) +$$

$$\mathrm{I}_4^{D=6-2\epsilon}(s_{123},s_{234},s_{23},s_{56}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{234},s_{345},s_{34},s_{16}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{345},s_{456},s_{45},s_{12})\Bigg)$$

$$C_{3+4}(1^-,2^+,3^-,4^+,5^+,6^+) = -\frac{\langle 13\rangle^4}{\langle 16\rangle\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\Bigg(\frac{1}{\epsilon^2}\sum_{i=1}^{6}\left(\frac{\mu_R^2}{-s_{i,i+1}}\right)^\epsilon +$$

$$+\,\mathrm{I}_4^{D=6-2\epsilon}(s_{12},s_{23},s_{456}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{34},s_{45},s_{126}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{23},s_{34},s_{156})\left(\frac{\langle 14\rangle^4\langle 23\rangle^4}{\langle 13\rangle^4\langle 24\rangle^4} + \frac{\langle 12\rangle^4\langle 34\rangle^4}{\langle 13\rangle^4\langle 24\rangle^4}\right)$$

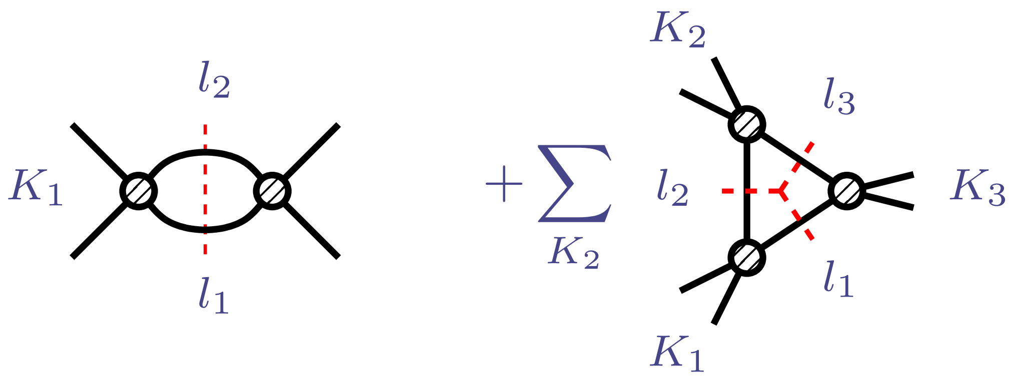
$$+\,\mathrm{I}_4^{D=6-2\epsilon}(s_{56},s_{16},s_{234}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{45},s_{56},s_{123}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{16},s_{12},s_{345})\left(\frac{\langle 16\rangle^4\langle 23\rangle^4}{\langle 13\rangle^4\langle 26\rangle^4} + \frac{\langle 12\rangle^4\langle 36\rangle^4}{\langle 13\rangle^4\langle 26\rangle^4}\right)$$

$$+\,\mathrm{I}_4^{D=6-2\epsilon}(s_{234},s_{123},s_{23},s_{56}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{123},s_{345},s_{12},s_{45}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{345},s_{234},s_{34},s_{16})\left(\frac{\langle 15\rangle^4\langle 23\rangle^4}{\langle 13\rangle^4\langle 25\rangle^4} + \frac{\langle 12\rangle^4\langle 35\rangle^4}{\langle 13\rangle^4\langle 25\rangle^4}\right)\Bigg)$$

Extracting Bubble Coefficients

- Two free integrations $\rightarrow$ more complicated residue structure
- Coefficient still written in terms of "Inf" expansions

$$C_{2;K} = -i\mathrm{Inf}_t[\mathrm{Inf}_y[A_1 A_2]]|_{t^0, y^k \rightarrow Y_k} - \frac{1}{2} \sum_{y_\pm} \sum_{K_2} \mathrm{Inf}_t[A_1 A_2' A_3'(K_2)]|_{t^k \rightarrow T_k}$$



- Alternative approaches:
  - Spinor Integration [Britto,Buchbinder,Cachazo,Feng,Mastrolia]
  - Generalising Cauchy's Theorem:
    Double cut from Stoke's Theorem [Mastrolia]

Bubble Contributions for A_6^g

$$C_2(1^-,2^-,3^+,4^+,5^+,6^+) =$$

$$\mathrm{I}_2\left(s_{156}\right)\Bigg(-\frac{\langle15\rangle^2\langle25\rangle\langle1|K_{561}|5]\langle2|K_{561}|5]^2[21]^3\langle12\rangle^3}{3s_{12}^3\left(s_{16}-s_{156}\right)^2\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}-\frac{\langle15\rangle\langle25\rangle^2\langle1|K_{561}|5]^2\langle2|K_{561}|5][21]^3\langle12\rangle^3}{3s_{12}^3\left(s_{16}-s_{156}\right)^2\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}$$

$$-\frac{11\langle25\rangle\langle1|K_{61}|5][21]\langle12\rangle^3}{6s_{12}\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}-\frac{11\langle15\rangle\langle2|K_{61}|5][21]\langle12\rangle^3}{6s_{12}\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}\Bigg)+\mathrm{I}_2\left(s_{16}\right)\Bigg(\frac{\langle15\rangle^2\langle25\rangle\langle1|K_{561}|5]\langle2|K_{561}|5]^2[21]^3\langle12\rangle^3}{3s_{12}^3\left(s_{16}-s_{156}\right)^2\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}$$

$$+\frac{\langle15\rangle\langle25\rangle^2\langle1|K_{561}|5]^2\langle2|K_{561}|5][21]^3\langle12\rangle^3}{3s_{12}^3\left(s_{16}-s_{156}\right)^2\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}+\frac{11\langle25\rangle\langle1|K_{61}|5][21]\langle12\rangle^3}{6s_{12}\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}+\frac{11\langle15\rangle\langle2|K_{61}|5][21]\langle12\rangle^3}{6s_{12}\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}$$

$$+\frac{11\langle12\rangle^3}{6\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}\Bigg)+\mathrm{I}_2\left(s_{234}\right)\Bigg(-\frac{\langle14\rangle^2\langle24\rangle\langle1|K_{4561}|4]\langle2|K_{4561}|4]^2[21]^3\langle12\rangle^3}{3s_{12}^3\left(s_{23}-s_{234}\right)^2\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}$$

$$-\frac{\langle14\rangle\langle24\rangle^2\langle1|K_{4561}|4]^2\langle2|K_{4561}|4][21]^3\langle12\rangle^3}{3s_{12}^3\left(s_{23}-s_{234}\right)^2\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}-\frac{11\langle24\rangle\langle1|K_{561}|4][21]\langle12\rangle^3}{6s_{12}\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}-\frac{11\langle14\rangle\langle2|K_{561}|4][21]\langle12\rangle^3}{6s_{12}\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}\Bigg)$$

$$+\mathrm{I}_2\left(s_{23}\right)\Bigg(\frac{\langle14\rangle^2\langle24\rangle\langle1|K_{4561}|4]\langle2|K_{4561}|4]^2[21]^3\langle12\rangle^3}{3s_{12}^3\left(s_{23}-s_{234}\right)^2\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}+\frac{\langle14\rangle\langle24\rangle^2\langle1|K_{4561}|4]^2\langle2|K_{4561}|4][21]^3\langle12\rangle^3}{3s_{12}^3\left(s_{23}-s_{234}\right)^2\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}$$

$$+\frac{11\langle24\rangle\langle1|K_{561}|4][21]\langle12\rangle^3}{6s_{12}\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}+\frac{11\langle14\rangle\langle2|K_{561}|4][21]\langle12\rangle^3}{6s_{12}\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}+\frac{11\langle12\rangle^3}{6\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle}\Bigg)$$

Bubble Contributions for A_6^g

More IR fitting and spurious singularity structure analysis:

$$I_2(s) \rightarrow \{K_0(s), L_k(s,t)\}$$

$$\begin{aligned}
&\widehat{C}_2(1^-,2^-,3^+,4^+,5^+,6^+) = \\
&\frac{11\langle 12\rangle^3}{6\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\left(I_2(s_{61})+I_2(s_{23})\right) \\
&+L_0\left(s_{16},s_{156}\right)\left(\frac{11\langle 25\rangle\langle 1|K_{61}|5][21]\langle 12\rangle^3}{6s_{12}\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}+\frac{11\langle 15\rangle\langle 2|K_{61}|5][21]\langle 12\rangle^3}{6s_{12}\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\right) \\
&+L_0\left(s_{23},s_{234}\right)\left(\frac{11\langle 24\rangle\langle 1|K_{561}|4][21]\langle 12\rangle^3}{6s_{12}\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}+\frac{11\langle 14\rangle\langle 2|K_{561}|4][21]\langle 12\rangle^3}{6s_{12}\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\right) \\
&+L_2\left(s_{16},s_{156}\right)\left(\frac{\langle 12\rangle^3\langle 15\rangle^2\langle 25\rangle\langle 1|K_{56}|5]\langle 2|K_{561}|5]^2[21]^3}{3s_{12}^3\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}+\frac{\langle 12\rangle^3\langle 15\rangle\langle 25\rangle^2\langle 1|K_{56}|5]^2\langle 2|K_{561}|5][21]^3}{3s_{12}^3\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\right) \\
&+L_2\left(s_{23},s_{234}\right)\left(\frac{\langle 12\rangle^3\langle 14\rangle^2\langle 24\rangle\langle 1|K_{56}|4]\langle 2|K_{561}|4]^2[21]^3}{3s_{12}^3\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}+\frac{\langle 12\rangle^3\langle 14\rangle\langle 24\rangle^2\langle 1|K_{56}|4]^2\langle 2|K_{561}|4][21]^3}{3s_{12}^3\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\right)
\end{aligned}$$

Direct Extraction of R_n

- D-dimensional cuts via massive cuts [Bern,Morgan (1995)] [Bern,Dixon,Dunbar,Kosower (1997)]

$$\int d^D l \prod_{i=1}^{C} \delta((l-K_i)^2) = \int d^{-2\epsilon}\mu \int d^4\bar{l} \prod_{i=1}^{C} \delta((\bar{l}-K_i)^2-\mu^2)$$

- Rational terms from boundary values of a contour intergral over μ [SB(2008)]

$$\int d^{-2\epsilon}\mu \int d^4\bar{l} \prod_{i=1}^{4} \delta(\bar{l}_i^2-\mu^2) A_1A_2A_3A_4$$

$$= \oint d^{-2\epsilon}\mu \sum_\sigma \left[\mathrm{Inf}_{\mu^2}[A_1A_2A_3A_4(\bar{l}^\sigma)] + \sum_i \frac{\mathrm{Res}_{\mu^2=\mu_i^2}(A_1A_2A_3A_4(\bar{l}^\sigma))}{\mu^2-\mu_i^2} \right]$$

$$\Rightarrow C_{4;K}^{[4]} = \sum_\sigma \mathrm{Inf}_{\mu^2}[A_1A_2A_3A_4(\bar{l}^\sigma)]\Big|_{\mu^4}$$

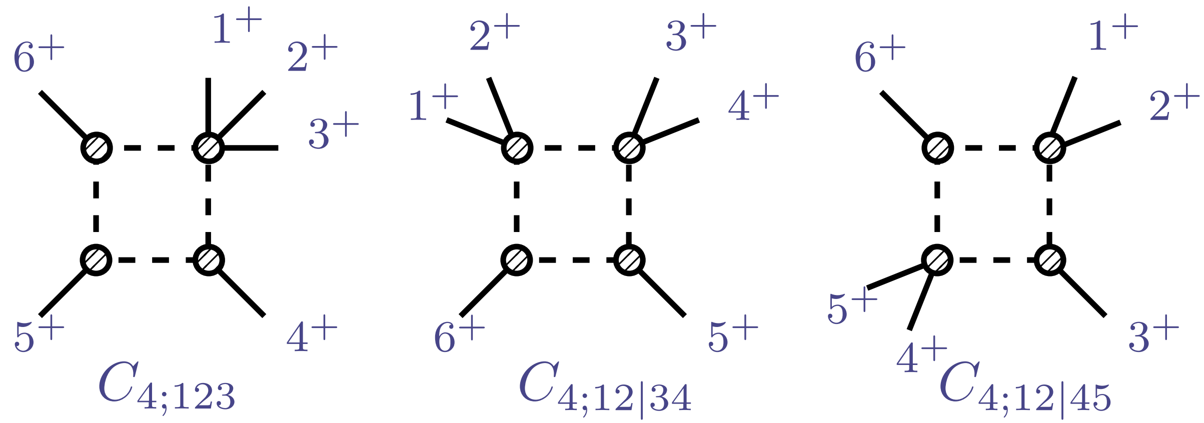
- Triangle and Bubble coefficients follow from similar analysis

Rational Contributions to A_6^g

- Only need to consider massive internal scalar $\boxed{R_n^g = \left(1 - \frac{N_f}{N_c}\right) R_n^s}$

$$R_6^g(1^+, 2^+, 3^+, 4^+, 5^+, 6^+) = -\frac{1}{6} \sum_{\sigma \in S_n} \Bigg($$

$$C^{[4]}_{4;\sigma(1)\sigma(2)\sigma(3)} + C^{[4]}_{4;\sigma(1)\sigma(2)|\sigma(3)\sigma(4)} + \frac{1}{2} C^{[4]}_{4;\sigma(1)\sigma(2)|\sigma(4)\sigma(5)} \Bigg)$$



Compact tree amplitudes from massive BCFW recursion

[SB,Glover,Khoze,Svrček (2005)]

Rational Contributions to A_6^g

- Solution for loop momentum around $\mu \to \infty$, e.g. "s_{123}" one-mass box:

$$l_1^{\pm} \overset{\mu\to\infty}{\to} \pm|\mu|\sqrt{\frac{\langle 5|6|4]}{\langle 4|6|5] s_{45}}}\left(|4\rangle[5| - \frac{\langle 4|6|5]}{\langle 5|6|4]}|5\rangle[6|\right)$$

$$A_6^{(1)}(1^+,2^+,3^+,4^+,5^+,6^+) = \sum_{\sigma\in S_n}\Bigg(\frac{2i\left(s_{45}\langle 6|1+2|3][51][64]^2 - s_{46}\langle 5|1+2|3][54]^2[61]\right)[56]}{\langle 12\rangle\langle 23\rangle \text{tr}_5(5,4,6,1)\text{tr}_5(5,4,6,3)}$$
$$+\frac{2i\langle 5|1+2|6]\langle 6|1+2|5][12][43][65]^2}{\langle 12\rangle\langle 34\rangle \text{tr}_5(5,2,6,1)\text{tr}_5(5,4,6,3)}$$
$$+\frac{2i\left(\langle 3|1+2|3]\langle 6|1+2|6] - s_{36}s_{12}\right)[12][54][63]^2}{\langle 12\rangle\langle 45\rangle \text{tr}_5(2,3,6,1)\text{tr}_5(5,3,6,4)}\Bigg)$$

D-dimensional cuts vs. On-shell recursion

- Alternative method for generating compact rational contributions:
  - Unitarity bootstrap method [Bern,Dixon,Kosower(2005)]
- Works efficiently for gluon amplitudes$\rightarrow$ not a universal approach
  - Gluon amplitudes, $n \le 8$ [Berger,Bern,Dixon,Forde,Kosower]
  - Higgs+4 partons [SB,Glover,Risager]

[Glover,Mastrolia,Williams]

[Berger,Del-Duca,Dixon;Dixon,Sofianatos]

$$
\begin{aligned}
R_6(1^+,2^+,3^+,4^+,5^+,6^+) = &-\frac{[32][41]}{3\langle16\rangle\langle23\rangle\langle45\rangle\langle56\rangle} - \frac{\langle35\rangle[32][51]}{3\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle} - \frac{[42][51]}{3\langle16\rangle\langle23\rangle\langle34\rangle\langle56\rangle} \\
&- \frac{\langle13\rangle[43][51]}{3\langle12\rangle\langle16\rangle\langle23\rangle\langle34\rangle\langle56\rangle} - \frac{[43][52]}{3\langle12\rangle\langle16\rangle\langle34\rangle\langle56\rangle} - \frac{\langle36\rangle[32][61]}{3\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle} - \frac{\langle46\rangle[42][61]}{3\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle} \\
&- \frac{\langle13\rangle\langle46\rangle[43][61]}{3\langle12\rangle\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle\langle56\rangle} - \frac{[52][61]}{3\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle} - \frac{\langle13\rangle[53][61]}{3\langle12\rangle\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle} - \frac{\langle14\rangle[54][61]}{3\langle12\rangle\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle} \\
&- \frac{\langle46\rangle[43][62]}{3\langle12\rangle\langle16\rangle\langle34\rangle\langle45\rangle\langle56\rangle} - \frac{[53][62]}{3\langle12\rangle\langle16\rangle\langle34\rangle\langle45\rangle} - \frac{\langle24\rangle[54][62]}{3\langle12\rangle\langle16\rangle\langle23\rangle\langle34\rangle\langle45\rangle} - \frac{[54][63]}{3\langle12\rangle\langle16\rangle\langle23\rangle\langle45\rangle}
\end{aligned}
$$

- D-dimensional cuts easier to implement numerically [Ellis et al.,Berger et al.]

Full MHV amplitude

$$
\begin{aligned}
&A_{6;1}^{(1)}(1^-,2^-,3^+,4^+,5^+,6^+) = -\frac{\langle 12\rangle^3}{\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\Bigg(\frac{1}{\epsilon^2}\sum_{i=1}^{6}\left(\frac{\mu_R^2}{-s_{i,i+1}}\right)^{\epsilon} + \\
&\mathrm{I}_4^{D=6-2\epsilon}(s_{16},s_{12},s_{345}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{56},s_{16},s_{234}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{12},s_{23},s_{456}) + \\
&\mathrm{I}_4^{D=6-2\epsilon}(s_{23},s_{34},s_{156}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{34},s_{45},s_{126}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{45},s_{56},s_{123}) + \\
&\mathrm{I}_4^{D=6-2\epsilon}(s_{123},s_{234},s_{23},s_{56}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{234},s_{345},s_{34},s_{16}) + \mathrm{I}_4^{D=6-2\epsilon}(s_{345},s_{456},s_{45},s_{12})\Bigg) \\
&+ \frac{11\langle 12\rangle^3}{6\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\left(I_2(s_{61}) + I_2(s_{23})\right) \\
&+ L_0(s_{16},s_{156})\left(\frac{11\langle 25\rangle\langle 1|K_{61}|5][21]\langle 12\rangle^3}{6s_{12}\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle} + \frac{11\langle 15\rangle\langle 2|K_{61}|5][21]\langle 12\rangle^3}{6s_{12}\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\right) \\
&+ L_0(s_{23},s_{234})\left(\frac{11\langle 24\rangle\langle 1|K_{561}|4][21]\langle 12\rangle^3}{6s_{12}\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle} + \frac{11\langle 14\rangle\langle 2|K_{561}|4][21]\langle 12\rangle^3}{6s_{12}\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\right) \\
&+ L_2(s_{16},s_{156})\left(\frac{\langle 12\rangle^3\langle 15\rangle^2\langle 25\rangle\langle 1|K_{56}|5]\langle 2|K_{561}|5]^2[21]^3}{3s_{12}^3\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle} + \frac{\langle 12\rangle^3\langle 15\rangle\langle 25\rangle^2\langle 1|K_{56}|5]^2\langle 2|K_{561}|5][21]^3}{3s_{12}^3\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\right) \\
&+ L_2(s_{23},s_{234})\left(\frac{\langle 12\rangle^3\langle 14\rangle^2\langle 24\rangle\langle 1|K_{56}|4]\langle 2|K_{561}|4]^2[21]^3}{3s_{12}^3\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle} + \frac{\langle 12\rangle^3\langle 14\rangle\langle 24\rangle^2\langle 1|K_{56}|4]^2\langle 2|K_{561}|4][21]^3}{3s_{12}^3\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle}\right) \\
&- \frac{\langle 4|K_{35}|4][53]^2\langle 12\rangle^3}{3s_{126}\langle 16\rangle\langle 34\rangle\langle 45\rangle\langle 2|K_{16}|5]\langle 6|K_{12}|3]} + \frac{\langle 5|K_{46}|5][64]^2\langle 12\rangle^3}{3s_{123}\langle 23\rangle\langle 45\rangle\langle 56\rangle\langle 1|K_{23}|4]\langle 3|K_{12}|6]} \\
&- \frac{\langle 35\rangle[64][65]\langle 12\rangle^3}{6\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 1|K_{23}|4]\langle 3|K_{12}|6]} + \frac{\langle 12|K_{34}|5\rangle[53]^2\langle 12\rangle^2}{3\langle 16\rangle\langle 34\rangle\langle 45\rangle\langle 2|K_{16}|5]\langle 5|K_{34}|2]\langle 6|K_{12}|3]} \\
&- \frac{\langle 1|4|3]\langle 12\rangle^2}{6\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle[32]} + \frac{\langle 2|5|6]\langle 12\rangle^2}{6\langle 16\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle[61]} - \frac{\langle 1|K_{24}|3]\langle 1|4|3]\langle 12\rangle}{6s_{234}\langle 16\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle[32]} \\
&- \frac{[63]^3}{3\langle 45\rangle^2[21][32][61]} + \frac{\langle 1|53|5\rangle[53]^2\langle 12\rangle^2}{3\langle 16\rangle\langle 34\rangle\langle 45\rangle\langle 2|K_{16}|5]\langle 5|K_{34}|2]\langle 6|K_{12}|3]} - \frac{\langle 1|K_{24}|3]^2\langle 12\rangle}{3s_{234}\langle 16\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle[32]} \\
&- \frac{\langle 15\rangle\langle 1|K_{56}||K_{34}|5\rangle[43][65]\langle 12\rangle}{6s_{234}\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 1|K_{23}|4]\langle 5|K_{34}|2]} - \frac{\langle 24\rangle\langle 35\rangle\langle 5|K_{12}|6][53]^2[65]\langle 12\rangle}{3\langle 34\rangle^2\langle 45\rangle\langle 2|K_{16}|5]\langle 5|K_{34}|2]\langle 6|K_{12}|3][61]} \\
&+ \frac{\langle 5|K_{12}|6]^2[65]\langle 12\rangle}{6\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 3|K_{12}|6]\langle 5|K_{34}|2][61]} - \frac{\langle 35\rangle\langle 4|K_{12}|3]\langle 4|K_{12}|6]\langle 5|K_{12}|6][53]}{3\langle 34\rangle^2\langle 45\rangle^2\langle 5|K_{34}|2]\langle 6|K_{12}|3][21][61]} \\
&- \frac{\langle 35\rangle\langle 4|K_{12}|6]^2[63]}{3\langle 34\rangle^2\langle 45\rangle^2\langle 5|K_{34}|2][21][61]} - \frac{\langle 15\rangle^2\langle 16\rangle[43]^3[65]}{3s_{234}\langle 56\rangle^2\langle 1|K_{23}|4]\langle 5|K_{34}|2][32]} - \frac{\langle 15\rangle^2\langle 1|K_{24}|3][43]^2[65]}{3s_{234}\langle 45\rangle\langle 56\rangle\langle 1|K_{23}|4]\langle 5|K_{34}|2][32]} \\
&- \frac{\langle 4|K_{35}||K_{12}|5\rangle\langle 5|K_{12}|6]^2[65]}{3\langle 34\rangle\langle 45\rangle^2\langle 56\rangle\langle 3|K_{12}|6]\langle 5|K_{34}|2][21][61]} + \frac{\langle 35\rangle\langle 1|K_{24}|3]^3}{3s_{234}\langle 16\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 5|K_{34}|2][32]}
\end{aligned}
$$

Progress for LHC processes

- Easy extensions for massive internal propagators [Kilgore]
  - Loop momentum parameterisation unchanged
  - Re-solve on-shell constraints
- Application for massive fermion amplitudes [Ellis,Giele,Kunszt,Melnkov] [SB]
- Combination with real radiation:
  - `MadDipole` [Frederix,Gehrmann,Greiner]
  - `Sherpa` [Gleisberg,Krauss]
  - Automated Dipole Subtraction [Hasegawa,Moch,Uwer]
- Efficient scaling with increasing number of external legs
  - 20 gluon amplitudes! [Giele,Zanderighi]

Conclusions

- Efficient techniques for one-loop amplitude calculations
  - Succesful numerical implementations [`ROCKET,BlackHat,Helac-1loop`]
  - Full cross-section for $W + 3j$ [Berger et al.]
- Automation of analytic extraction: `FORM,Maple`
  - Compact(ish) formulae
  - Internal masses
  - Arbitrary helicity configurations
- Open questions
  - Systematic treatment of IR consistency conditions?
  - Generalisations to higher loops in QCD?

Additional Slides

Spinor-Helicity Method

Two-component Weyl-spinor representation: [Good Review: Mangano,Parke (1991)]

$$\langle pq\rangle = e^{i\theta_{pq}}\sqrt{|p\cdot q|} \qquad [pq] = e^{-i\theta_{pq}}\sqrt{|p\cdot q|}$$

Momenta translated to matrix representation with Pauli matrices:

$$p^\mu = \frac{1}{2}\langle p|\bar\sigma^\mu|p]$$

Fermion wave-functions

$$u_+(p) = |p\rangle \qquad u_-(p) = |p]$$
$$\bar u_+(p) = \rangle p| \qquad \bar u_-(p) = [p|$$

Polarisation vectors in light-like axial gauge

$$\varepsilon_+^\mu(p,\xi) = \frac{\langle\xi|\bar\sigma^\mu|p]}{\sqrt{2}\langle\xi p\rangle} \qquad \varepsilon_-^\mu(p,\xi) = \frac{\langle p|\bar\sigma^\mu|\xi]}{\sqrt{2}[p\xi]}$$

BCFW Recursion

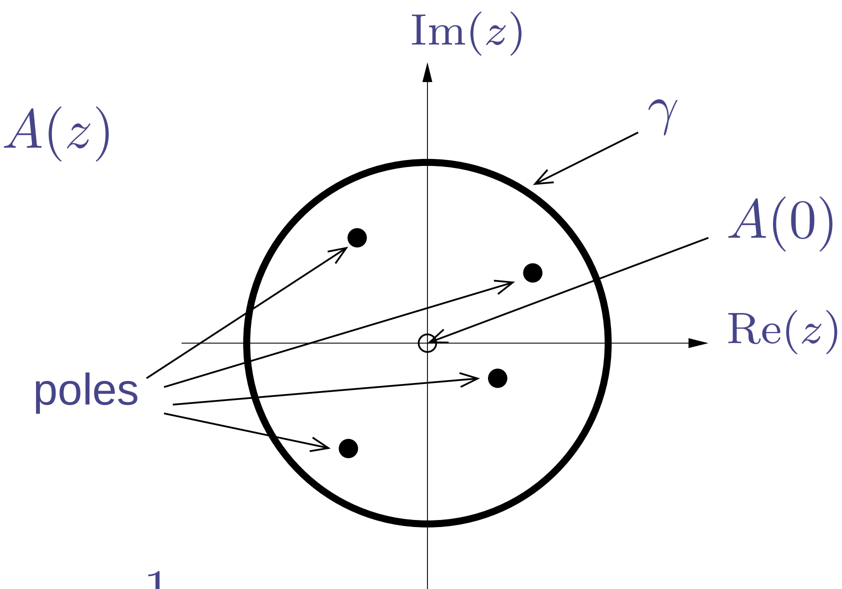
[Britto,Cachazo,Feng,Witten]

n-particle tree amplitude with complex deformation $\langle p_1, p_2] \to \langle p_1(z), p_2(z)]$,

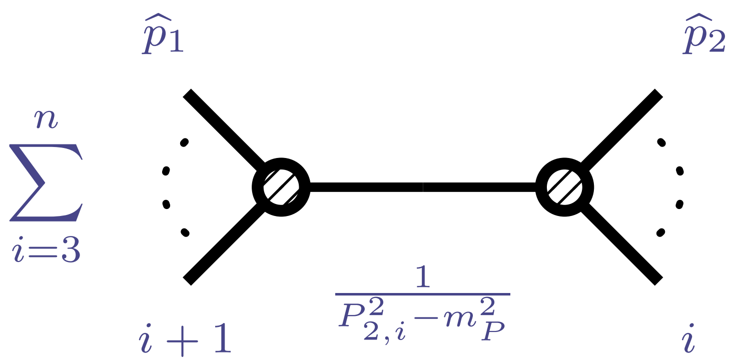
$$A(p_1, p_2, \ldots, p_n) \to A(\widehat{p_1}(z), \widehat{p_2}(z), \ldots, p_n) = A(z)$$

Momentum conservation: $\widehat{p_1}(z) + \widehat{p_2}(z) = p_1 + p_2$

On-shell conditions: $\widehat{p_1}(z)^2 = 0, \widehat{p_2}(z)^2 = 0$



$$0 = \frac{1}{2\pi i} \oint_\gamma dz \frac{A(z)}{z} = A(0) + \sum_{i=3}^{n-1} A_L(z_i) \frac{1}{P_i^2 - m_i^2} A_R(z_i) + A_\infty$$



Massive particles [SB,Glover,Khoze,Svrček]

One-loop generalisations

[Bern,Dixon,Kosower]

D-Dimensional Cuts and Rational Terms

- Full loop amplitudes from D-dimensional cuts [Bern,Morgan (1995)]
- Expanded integral basis: $l_D = l_4 + l_{-2\epsilon} \Rightarrow l_4^2 = \mu^2$

[Bern,Dixon,Dunbar,Kosower (1997)]

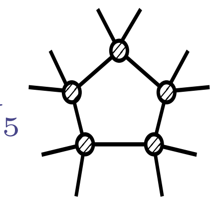
[Ossola,Papadopoulos,Pittau (2006)]

[Anastasiou,Britto,Feng,Kunszt,Mastrolia (2006)]

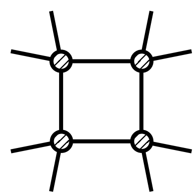
[Britto,Feng,Mastrolia;Yang (2008)]

[Giele,Kunszt,Melnikov (2008)]

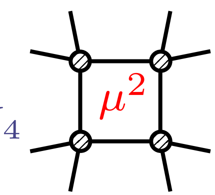
$$A_n^{(1),D}(\{p_i\}) = \sum_{K_5} C_{5;K_5}$$



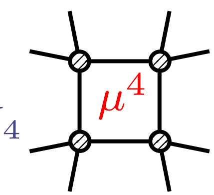
$$+ \sum_{K_4} C_{4;K_4}^{[0]}$$



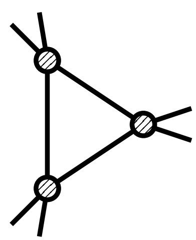
$$+ \sum_{K_4} C_{4;K_4}^{[2]}$$



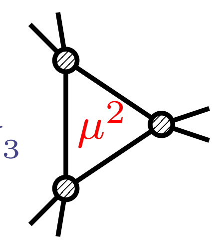
$$+ \sum_{K_4} C_{4;K_4}^{[4]}$$



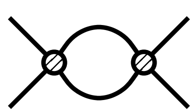
$$+ \sum_{K_3} C_{3;K_3}$$



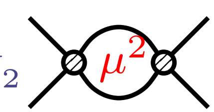
$$+ \sum_{K_3} C_{3;K_3}^{[2]}$$



$$+ \sum_{K_2} C_{2;K_2}$$



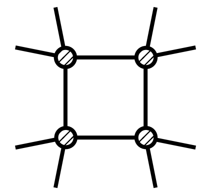
$$+ \sum_{K_2} C_{2;K_2}^{[2]}$$



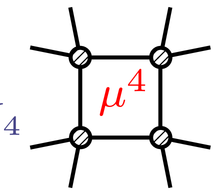
D-Dimensional Cuts and Rational Terms

- Take the $D \to 4-2\epsilon$ limit

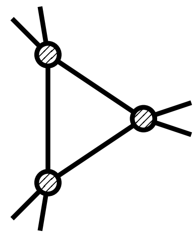
$$A_n^{(1),4-2\epsilon}(\{p_i\}) = \sum_{K_4} C_{4;K_4}$$



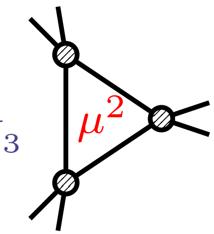
$$+ \sum_{K_4} C^{[4]}_{4;K_4}$$



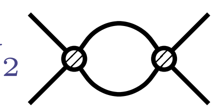
$$+ \sum_{K_3} C_{3;K_3}$$



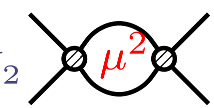
$$+ \sum_{K_3} C^{[2]}_{3;K_3}$$



$$+ \sum_{K_2} C_{2;K_2}$$



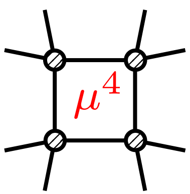
$$+ \sum_{K_2} C^{[2]}_{2;K_2}$$



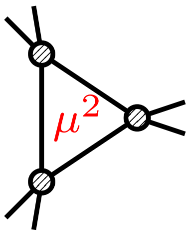
$$+ \mathcal{O}(\epsilon)$$

D-Dimensional Cuts and Rational Terms

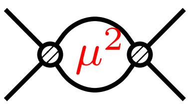
- Evaluate three non-zero integrals [Bern,Morgan (1995)]



$$= -\frac{1}{6} + \mathcal{O}(\epsilon)$$



$$= -\frac{1}{2} + \mathcal{O}(\epsilon)$$



$$= -\frac{K^2 - 3(m_1^2 + m_2^2)}{6} + \mathcal{O}(\epsilon)$$

- Identify rational contribution

[Giele,Kunszt,Melnikov]

[Ossola,Papdopoulos,Pittau]

[Britto,Feng,Mastrolia;Yang]

$$R_n(\{p_i\}) = -\frac{1}{6}\sum_{K_4} C^{[4]}_{4;K_4} - \frac{1}{2}\sum_{K_3} C^{[2]}_{3;K_3} - \frac{1}{6}\sum_{K_2} K_2^2 - 3(m_1^2 + m_2^2)C^{[2]}_{2;K_2}$$