Precision Studies of Radiative Corrections to Bhabha Scattering and Fermion Pair Production

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Outline

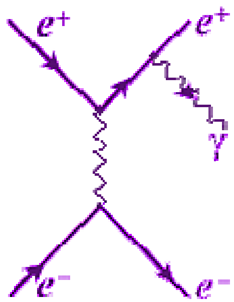
We will describe radiative corrections to Bremsstrahlung and related processes – focusing on applications to luminosity, fermion pair production, and radiative return.

- BHLUMI and the Bhabha Luminosity Process
- The KK MC and fermion pair production
- Radiative Return Applications
- Comparisons to other related results

The e^+e^- Luminosity Process

- Calculating normalized cross sections requires knowing the beam luminosity $\mathcal{L}$. If *N* events are observed, the normalized cross section is $\sigma = N/\mathcal{L}$.
- In e^+e^- scattering (SLC, LEP), the luminosity is calibrated using small angle Bhabha scattering

$$e^+e^- \longrightarrow e^+e^- + n\gamma$$



- This process has both experimental and theoretical advantages:
  - A large, clean signal
  - Almost pure QED (3% *Z* exchange)
- The angle cuts were 1-3 degrees at LEP1, 3-6 degrees at LEP2.

The BHLUMI Monte Carlo Program

BHLUMI was developed into an extremely precise tools for computing the Bhabha luminosity process in $e^+ e^-$ colliders.

The project was begun by S. Jadach, B.F.L. Ward, E. Richter-Was, and Z. Was and continued with contributions by S. Yost, M. Melles, M. Skrzypek, W. Placzek and others.

Historical Progress in Bhabha Scattering

Year	Expt.	Theory
1982	2%	2%
1990	0.8%	1%
1992	0.6%	0.25%
1997	0.15%	<0.11%
1999	0.05%	<0.06%

BHLUMI (1992–1999)

for LEP 1 parameters

ILC requirement: 0.01% is already within reach.

YFS Monte Carlo Structure

BHLUMI uses Yennie-Frautschi-Suura exponentiation [YFS, Ann. Phys. 13 (1961) 379] as the framework for its MC generation.

This has the advantage of exactly extracting and exponentiating the leading soft photon contributions, eliminating soft photon singularities from the numerical calculations.

YFS exponentiation is based on an exact rearrangement of the differential cross section, explicitly canceling all infrared divergences. It was developed into a MC framework by S. Jadach and has been implemented in the programs YFS2, YFS3, BHLUMI, BHWIDE, KORALZ, KKMC, YFSWW3, YFSZZ, KoralW.

Overview of YFS Theory

In QED, all IR divergences cancel between real and virtual photon graphs in the integrated cross section.

YFS theory gives way of rearranging the cross section so that all real and virtual IR divergences are explicitly canceled, and the universal soft photon dependence is extracted and exponentiated.

The remaining terms – called YFS residuals – are free of IR singularities and may be calculated up to the desired order in perturbation theory.

Overview of YFS Theory

For Bhabha scattering with *n* radiated photons,

$$e^+(p_1)\, e^-(q_1) \longrightarrow e^+(p_2)\, e^-(q_2) + n(\gamma)(k_1, \ldots, k_n)$$

we will consider the construction of the residuals in the special case that all emission is from the e^+ line. This simplifies the equations and suffices to show the strategy.

The key fact that makes this work is that the IR behavior of all virtual photon corrections can be represented by a universal function B, while the IR behavior of all real photon corrections can be represented by a universal function $\tilde{B}$.

Example: e$^+$ line emission

1. Extract and exponentiate the virtual YFS function $B(p_1, p_2)$.
   A. Represent the amplitude for n real photon emission as the sum over terms with ℓ virtual photons: $\mathcal{M}_n = \sum_{l=0}^{\infty} \mathcal{M}_n^{(l)}$
   B. Extract the IR divergence, which is contained in the universal factors B: $\mathcal{M}_n^{(l)} = \sum_{m=0}^{l} M_n^{(l-m)} \frac{(\alpha B)^m}{m!}$

   where

$$B = \frac{i}{(2\pi)^3} \int \frac{d^4 k}{k^2 - m_\gamma^2 + i\epsilon} \left[\frac{-2p_{2\mu} - k_\mu}{-2p_2 k - k^2 - i\epsilon} - \frac{2p_{1\mu} - k_\mu}{2p_1 k - k^2 - i\epsilon} \right]^2$$

IR regulator m_γ

$$\mathrm{Re}\, B = \frac{-1}{2\pi} \left[\ln \frac{-t}{m_e^2} \left(\ln \frac{m_e^2}{m_\gamma^2} + \frac{1}{2} \ln \frac{-t}{m_e^2} - \frac{1}{2} \right) - \ln \frac{m_e^2}{m_\gamma^2} + 1 - \frac{\pi^2}{6} \right]$$

Example: e^+ line emission

C. Rearrange the sum over virtual loops to exponentiate the YFS soft virtual factor:

$$\mathcal{M}_n = \sum_{m=0}^{\infty} \frac{(\alpha B)^m}{m!} \sum_{l=m}^{\infty} M_{l-m} = e^{\alpha B} \sum_{l=0}^{\infty} M_n^{(l)}$$

2. Square the amplitude and construct the cross section with n real photons emitted from the e^+ line:

$$d\sigma = e^{2\alpha \mathrm{Re}\, B} \frac{1}{n\,!} \int dE_{n\gamma} \prod_{j=1}^{n} \frac{d^3 k_j}{\sqrt{k_j^2 + m_\gamma^2}} \delta\left(\sqrt{s} - E_{n\gamma} - \sum_{i=1}^{n} k_i^0\right) \left|\sum_{l=0}^{\infty} M_n^{(l)}\right|^2$$

Note that the amplitudes $M_n^{(l)}$ are now free of *virtual* IR divergences. $E_{n\gamma}$ is the total energy of the n photons.

Example: e^+ line emission

3. Extract and exponentiate the real YFS function $\tilde{B}(p_1, p_2)$.
   A. The squared amplitude can be rearranged in terms of soft photon factors S and YFS residuals $\tilde{\beta}_n$ as

$$\left|\sum_{l=0}^{\infty} M_n^{(l)}\right|^2 = \tilde{S}(k_1)\cdots\tilde{S}(k_n)\tilde{\beta}_0 + \sum_{i=1}^{n}\tilde{S}(k_1)\cdots\tilde{S}(k_{i-1})\tilde{S}(k_{i+1})\cdots\tilde{S}(k_n)\tilde{\beta}_1(k_i) + \ldots$$

$$+\sum_{i=1}^{n}\tilde{S}(k_i)\tilde{\beta}_{n-1}(k_1,\ldots,k_{i-1},k_{i+1},\ldots,k_n) + \tilde{\beta}_n(k_1,\ldots,k_n)$$

where the soft photon factor for the e^+ line is

$$\tilde{S}(k) = -\frac{\alpha}{4\pi^2}\left[\frac{p_{1\mu}}{p_1 k} - \frac{p_{2\mu}}{p_2 k}\right]^2$$

Example: e^+ line emission

B. The soft photon contribution is universal and now can be extracted in a cutoff-dependent YFS real IR function

IR regulator m_γ

Energy cut K_{max}

$$\tilde{B}(K_{\max}) = \int^{k^0 \le K_{\max}} \frac{d^3k}{\sqrt{|k|^2 + m_\gamma^2}} \frac{\tilde{S}(k)}{2\alpha}$$

$$= \frac{1}{2\pi}\left[\ln\frac{-t}{m_e^2}\left(\ln\frac{m_e^2}{m_\gamma^2} + \frac{1}{2}\ln\frac{-t}{m_e^2} - \ln\frac{E_1E_2}{K_{\max}^2}\right) - \ln\frac{m_e^2}{m_\gamma^2} + \ln\frac{E_1E_2}{K_{\max}^2} - \frac{\pi^2}{6}\right]$$

giving a cutoff-independent cross section

$$d\sigma = e^{2\alpha(\mathrm{Re}\,B + \tilde{B})}\int_{-\infty}^{\infty}\frac{dy}{2\pi}\, e^{iy(\sqrt{s} - E_{n\gamma}) + D}\left[\tilde{\beta}_0 + \sum_{n=1}^{\infty}\frac{1}{n\,!}\int dE_{n\gamma}\prod_{j=1}^{n}\frac{d^3k_j}{k_j^0}e^{-iyk_j^0}\tilde{\beta}_n\right]$$

if we define

$$D = \int^{k^0 \le K_{\max}} \frac{d^3k}{k^0}(e^{-iyk^0} - 1)\tilde{S}$$

Example: e^+ line emission

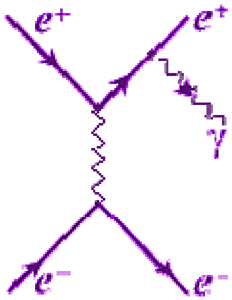
$$d\sigma = e^{2\alpha(\mathrm{Re}\,B+\widetilde{B})}\int_{-\infty}^{\infty}\frac{dy}{2\pi}\,e^{iy(\sqrt{s}-E_{n\gamma})+D}\left[\widetilde{\beta}_0+\sum_{n=1}^{\infty}\frac{1}{n\,!}\int dE_{n\gamma}\prod_{j=1}^{n}\frac{d^3k_j}{k_j^0}e^{-iyk_j^0}\widetilde{\beta}_n\right]$$

Essential features:

- The photon mass regulator cancels from the exponentiated YFS form factor $\mathrm{Re}\,B + \widetilde{B}$.
- The YFS residuals $\widetilde{\beta}_n$ are free of soft photon singularities, and can be evaluated in a stable manner for *arbitrarily* soft or collinear photons.
- The integrals of the soft factors $\widetilde{S}$ are singular, so a phase space cut is needed. However, a change of variables exists which flattens the integral, which is very convenient for MC generation.

YFS Soft Photon Techniques

For example, the cross section for one real photon can be expressed in terms of YFS residuals $\tilde{\beta}_0$, $\tilde{\beta}_1$ as



soft photon contribution

$$d\sigma_1(k) = \tilde{\beta}_0\, \tilde{S}(k) + \tilde{\beta}_1(k)$$

hard photon residual – well behaved for $k \longrightarrow 0$.

Here, $\tilde{\beta}_0 = \sigma_0$ is the Born cross section. Add as many residuals as needed to reach the desired precision level.

YFS Soft Photon Techniques

At the next order, the cross section for two real photons can be expressed in terms of YFS residuals $\tilde{\beta}_0$, $\tilde{\beta}_1$ and $\tilde{\beta}_2$ as

$$d\sigma_2(k_1, k_2) = \tilde{\beta}_0\, \tilde{S}(k_1)\, \tilde{S}(k_2)$$

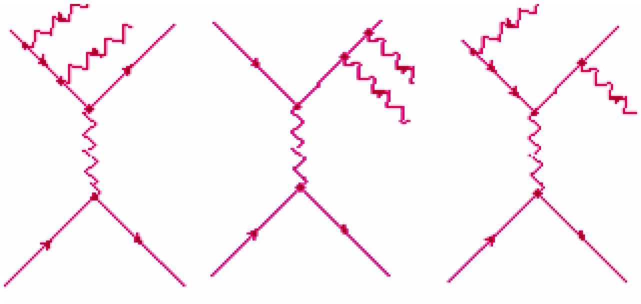
2 soft photons

$$+ \tilde{\beta}_1(k_1)\, \tilde{S}(k_2) + \tilde{\beta}_1(k_2)\, \tilde{S}(k_1)$$

1 soft, 1 hard photon

$$+ \tilde{\beta}_2(k_1, k_2).$$

2 hard photons



Leading Theoretical Uncertainties

The leading theoretical uncertainties in BHLUMI 4.04 are shown in the table for LEP1 and LEP2 parameters.

Source of Uncertainty	LEP 1	LEP 2
Missing Photonic $\mathcal{O}(\alpha^2 L)$	0.027%	0.04%
Missing Photonic $\mathcal{O}(\alpha^3 L^3)$	0.8%	1%
Vacuum Polarization	0.6%	0.25%
Light Pairs	0.15%	<0.11%
Z exchange	0.05%	<0.06%
TOTAL	0.061%	0.122%

"big logarithm"
$L = \ln(-t / m_e^2)$

LEP 1: $E_{cms} = 92$ GeV, $1^o < \theta < 3^o$ **LEP 2:** $E_{cms} = 176$ GeV, $3^o < \theta < 6^o$

The Missing Photonic Contribution

The biggest unimplemented part – the α^2 photonic contribution for e^+ and e^- line emission – is already known. The error budget was obtained by actually calculating these "missing" terms in a test version of BHLUMI.

Source of Uncertainty	LEP 1	LEP 2
Missing Photonic $\mathcal{O}(\alpha^2 L)$	0.027%	0.04%
Missing Photonic $\mathcal{O}(\alpha^3 L^3)$	0.8%	1%

Including known calculations in BHLUMI would remove *almost* the entire order α^2 contribution and the leading log contribution at order α^3.

Available exact $\mathcal{O}(\alpha^2)$ results

To be specific, the available exact O(α^2) results, presently used only to verify the BHLUMI precision, are…

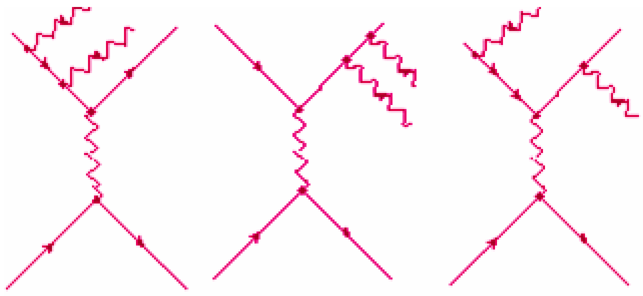
- Exact calculation of two-photon Bremsstrahlung: all graphs radiating two photons from any fermion lines have been calculated.
  [Jadach, Ward, Yost, Phys. Rev. D47 (1993) 2682]
- Order α correction to one-photon Bremsstrahlung: include all graphs where one real plus one virtual photon are emitted from either the e^+ or e^- line.
  [Jadach, Melles, Ward, Yost, Phys. Lett. B377 (1996) 168]
- Order α^2 correction to Bhabha scattering: two virtual photons on the e^+ or e^- line can be adapted from known* results.
  [Jadach, Melles, Ward, Yost, Phys. Lett. B450 (1999) 262]

*[Berends, Van Neerven, Burgers, Nucl. Phys. B297 (1988) 429]

Exact 2 Photon Bremsstrahlung

Historically, we started with the two-photon Bremsstrahlung process. The exact two photon Bremsstrahlung cross section:

$$e^+(p_1)\, e^-(q_1) \longrightarrow e^+(p_2)\, e^-(q_2) + \gamma(k_1) + \gamma(k_2)$$

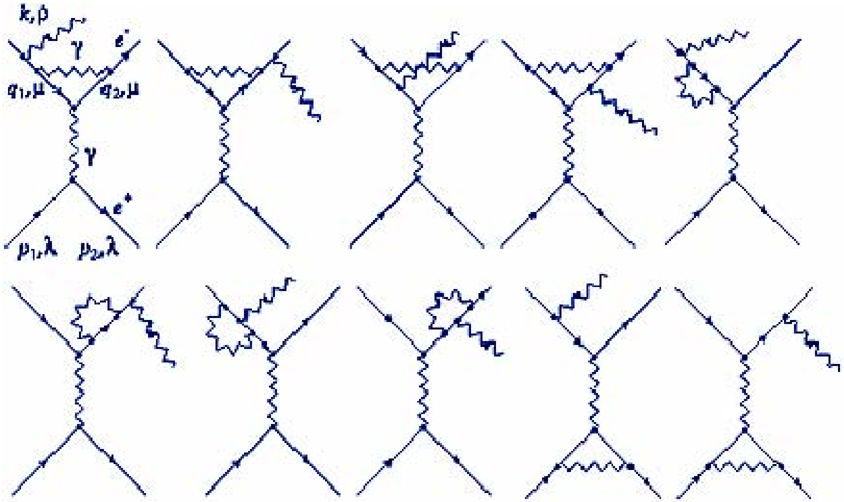


These diagrams were calculated exactly using helicity spinor techniques (which treat the electron as massless), adding electron masses perturbatively.

Real + Virtual Photon Emission

The second contribution completed was the case where one real photon is emitted, but another is virtual, present only in an internal "loop" in the diagram.

$$e^+(p_1)\, e^-(q_1) \longrightarrow e^+(p_2)\, e^-(q_2) + \gamma(k_1)$$



Computational Method

The graphs shown were calculated [Jadach, Melles, Ward, Yost, Phys. Lett. B377, 168 (1996)] using

- Helicity spinor methods
- Vermaseren’s algebraic manipulation program FORM
- Oldenborgh’s FF package of scalar one-loop Feynman integrals (later replaced by analytic expressions)

Finite e^- Mass Corrections

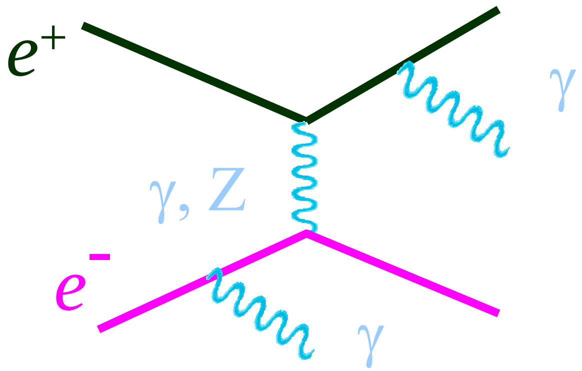
If the electron mass is 500 keV, and the ILC energy scale is 500 GeV, are mass effects still relevant?

Yes! Photons may be omitted collinearly with a fermion, leading to a large enhancement in the cross section. Integrating terms of the form $m_e^2/(pk)^2$ over k gives contributions of order 1. Such contributions do not appear in LL result, but begin to appear at NLL.

- Mass corrections are added via methods of Berends *et al* (CALCUL Collaboration), which were checked to show that all significant collinear mass corrections were included.

What's left at $\mathcal{O}(\alpha^2)$?

An *exact* calculation of "up-down" interference effects involving simultaneous emission from both lines has not yet been included.



These contributions go to zero at small angles, and are of order $m_e^2/|t|$ *without* cuts . But they become more important at larger angles.

Typical sizes for a test with BHLUMI are shown at right.

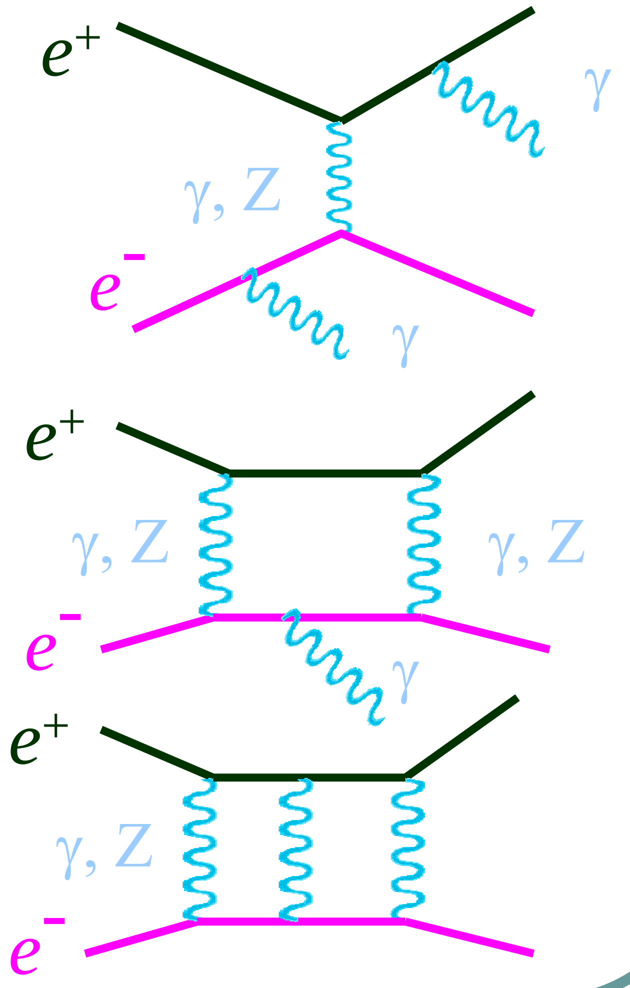
[Jadach, Richter-Was, Ward, Was, Phys. Lett. B253 (1991) 469]

Angle cut	size
$< 1^o$	< 0.001%
$3^o – 5^o$	0.01%
$9^o – 13^o$	0.09%

What's left at $\mathcal{O}(\alpha^2)$?

Up-down interference could be safely neglected for LEP1 and LEP2, but should be included for ILC physics at 0.01% accuracy. Examples of the $\mathcal{O}(\alpha^2)$ up-down interference contributions are shown, including virtual contributions.

There are several recent results on $\mathcal{O}(\alpha^2)$ Bhabha scattering. [*eg*, Bern, Dixon, Ghinculov, Phys. Rev. D63 (2001) 053007, A.A. Penin, Phys. Rev. Lett. 95 (2005) 056004, …] We are conducting tests to determine the effect of these contributions.

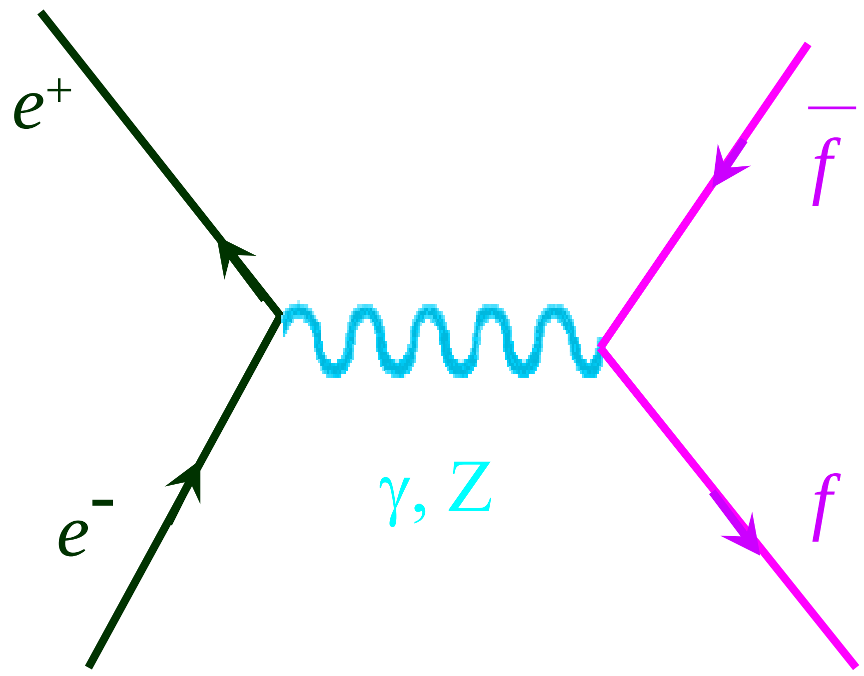


Fermion Pairs

Fermion pair production

$$e^+ e^- \rightarrow f \bar{f}$$

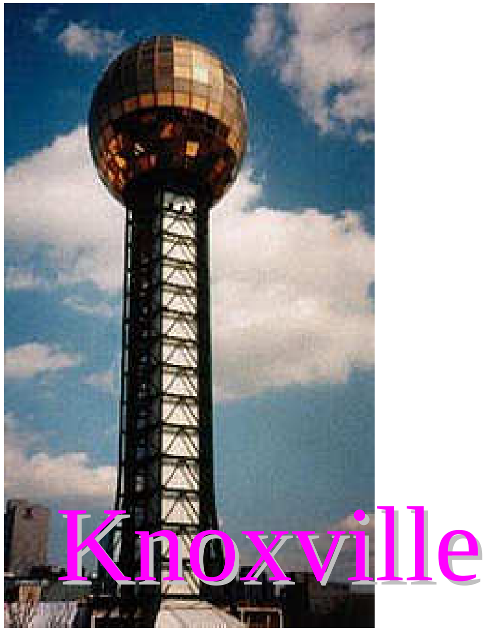
plays a critical role in extracting precision electroweak physics from e^+e^- colliders.



The KK Monte Carlo

The **KK MC** includes initial state photon radiation, and all other two-photon (real or virtual) to fermion pair processes.

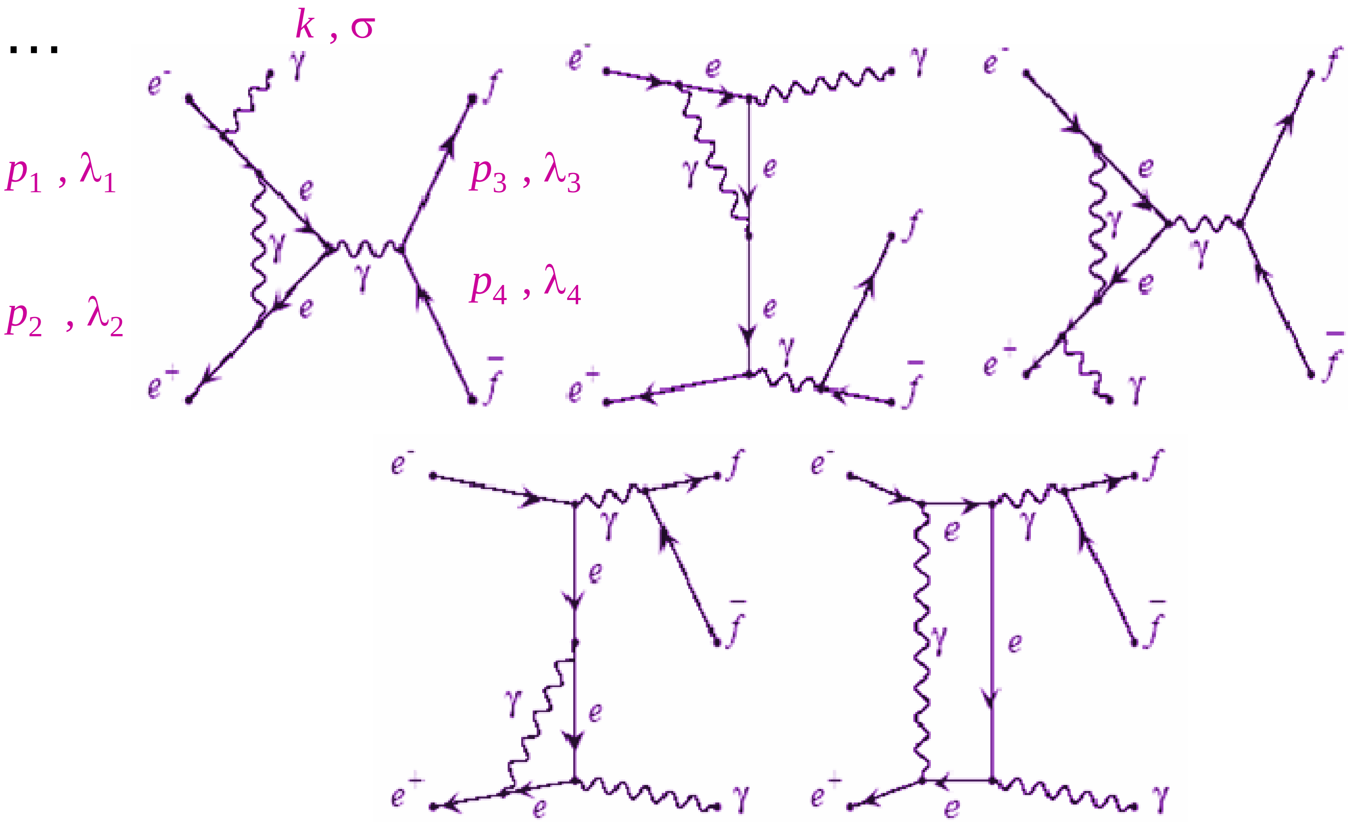
It also includes YFS exponentiation, and the effects of Z boson exchange.



[Jadach, Ward, Was, Phys. Rev D63 (2001) 113009, Comp. Phys Comm. 130 (2000) 260]

One Loop Radiative Corrections

For ISR, relevant one-loop Bremsstrahlung corrections include…



The Complete Result

The computational method is the same as for Bhabha scattering. Let's look at the result… [Jadach, Melles, Ward, Yost, Phys. Rev. D65 (2002) 073030]

$r_i = 2\, p_i \cdot k$

$$\frac{d\sigma_1^{\mathrm{ISR}(1)}}{d^2\Omega\, dr_1\, dr_2} = \frac{1}{(4\pi)^4 s'} \sum_{\lambda_i,\sigma} \mathrm{Re}[(\mathcal{M}_1^{\mathrm{ISR}(0)})^* \mathcal{M}_1^{\mathrm{ISR}(1)}].$$

and the differential cross section with virtual corrections can be expressed as

$$\frac{d\sigma_1^{\mathrm{ISR}(0)}}{d^2\Omega\, dr_1\, dr_2} = \frac{1}{2(4\pi)^4 s'} \sum_{\mathrm{helicities}} \left|\mathcal{M}_1^{\mathrm{ISR}(0)}\right|^2$$

$$\left|\mathcal{M}_1^{\mathrm{ISR}(0)}\right|^2 = \frac{e^6}{s^2 r_1 r_2}(t_1^2 + t_2^2 + u_1^2 + u_2^2) + \text{mass corrections}$$

The Complete Result

$$\mathcal{M}_1^{\rm ISR(1)} = \frac{\alpha}{4\pi}(f_0 + f_1 I_1 + f_2 I_2)\mathcal{M}_1^{\rm ISR(0)}$$

where I_1 and I_2 are spinor factors which vanish in the collinear limits $p_i{\cdot}k = 0$:

Spinor product: $|s_\lambda(p,k)|^2 = 2p\cdot k$

$$\begin{aligned} I_1 &= \sigma\lambda_3 s_{\lambda_1}(p_1,k)\boxed{s_{-\lambda_1}(p_2,k)} \\ &\quad\times \frac{s_{\lambda_3}(p_4,p_2)s_{-\lambda_3}(p_2,p_3) - s_{\lambda_3}(p_4,p_1)s_{-\lambda_3}(p_1,p_3)}{s_{-\sigma}(p_1,p_2)s_{-\sigma}(p_3,p_4)s_\sigma^2(p_{21},p_{34})}, \\ I_2 &= \lambda_1\lambda_3 \frac{s_{\lambda_1}(p_1,k)s_{-\lambda_1}(p_2,k)s_{\lambda_3}(p_4,k)s_{-\lambda_3}(p_3,k)}{s_{-\sigma}(p_1,p_2)s_{-\sigma}(p_3,p_4)s_\sigma^2(p_{21},p_{34})}, \end{aligned}$$

with $p_{ij} = p_i$ or p_j when $\sigma = \lambda_i$ or λ_j.

Form Factors

For helicities $\sigma = \lambda_1$. Otherwise interchange r_1 and r_2: (mass terms added later)

$$
\begin{aligned}
f_0 &= 2\left\{\ln\left(\frac{s}{m_e^2}\right) - 1 - i\pi\right\} + \frac{r_2}{1-r_2} + \frac{r_2(2+r_1)}{(1-r_1)(1-r_2)}\left\{\ln\left(\frac{r_2}{z}\right) + i\pi\right\} \\
&- \left\{3v + \frac{2r_2}{1-r_2}\right\}\mathrm{Lf}_1(-v) + \frac{v}{(1-r_2)}\, R_1(r_1,r_2) + r_2\, R_1(r_2,r_1),
\end{aligned}
$$

$$
\begin{aligned}
f_1 &= \frac{r_1 - r_2}{2(1-r_1)(1-r_2)} + \frac{z(1+z)}{2(1-r_1)^2(1-r_2)}\left\{\ln\left(\frac{r_2}{z}\right) + i\pi\right\} \\
&+ \frac{z}{1-r_2}\left\{\frac{1}{2}R_1(r_1,r_2) + r_2 R_2(r_1,r_2)\right\} + \frac{v}{4}\left\{R_1(r_1,r_2)\delta_{\sigma,1} + R_1(r_2,r_1)\delta_{\sigma,-1}\right\}
\end{aligned}
$$

$$
\begin{aligned}
f_2 &= 2 - \frac{1+z}{2(1-r_1)(1-r_2)} + \frac{z(r_2-r_1)}{2(1-r_1)^2(1-r_2)}\left\{\ln\left(\frac{r_2}{z}\right) + i\pi\right\} \\
&+ 2z\,\mathrm{Lf}_2(-v) + \frac{z}{1-r_2}\left\{\frac{1}{2}R_1(r_1,r_2) + (2-r_2)R_2(r_1,r_2)\right\} \\
&+ \frac{r_1 - r_2}{4}\left\{R_1(r_1,r_2)\delta_{\sigma,1} + R_1(r_2,r_1)\delta_{\sigma,-1}\right\}
\end{aligned}
$$

$r_i = 2\, p_i \cdot k$

$v = r_1 + r_2$

$z = 1 - v$

Special Functions

The form factors depend on the following special functions, formed from an IR-finite combination of a box integral and triangle integrals, which are designed to be stable in the collinear limits (small r_i).

$$\begin{aligned}
R_1(x,y) &= \mathrm{Lf}_1(-x)\left\{\ln\left(\frac{1-x}{y^2}\right) - 2\pi i\right\} \\
&+ \frac{2(1-x-y)}{1-x}\,\mathrm{Sf}_1\left(\frac{y}{1-x}, \frac{x(1-x-y)}{1-x}\right), \\
R_2(x,y) &= z + \frac{1}{1-x}\left\{\ln\left(\frac{y}{1-x-y}\right) + i\pi\right\} \\
&+ \mathrm{Lf}_2(-x)(\ln y + i\pi) - (1-x-y)\,\mathrm{Lf}_1(-x) - \frac{1}{2}\,\mathrm{Lf}_1^2(-x) \\
&+ \frac{1-x-y}{(x+y)(1-x)}\left\{x\,\mathrm{Lf}_1\left(\frac{-y}{1-x}\right) - y\,\mathrm{Lf}_2\left(\frac{-y}{1-x}\right)\right\} \\
&+ \left(\frac{1-x-y}{1-x}\right)^2 \mathrm{Sf}_2\left(\frac{y}{1-x}, \frac{x(1-x-y)}{1-x}\right),
\end{aligned}$$

Stabilized Functions

In the above expressions, $\mathrm{Sp}(x)$ is the Spence dilogarithm, also denoted $\mathrm{Li}_2(x)$. We also introduce a set of logarithmic and dilogarithmic difference functions, which are useful for preventing numerical instabilities in the calculation. They are defined recursively by

$$\mathrm{Lf}_0(x) = \ln(1-x), \qquad \mathrm{Lf}_{n+1}(x) = \frac{1}{x}\left(\mathrm{Lf}_n(x) - \mathrm{Lf}_n(0)\right),$$

$$\mathrm{Sf}_0(x,y) = \mathrm{Sp}(x+y), \qquad \mathrm{Sf}_{n+1}(x,y) = \frac{1}{y}\left(\mathrm{Sf}_n(x,y) - \mathrm{Sf}_n(x,0)\right)$$

These functions are very useful to handle differences of the form $(\ln(1+x) - \ln(1+x+y))/y$ or $(\mathrm{Sp}(x+y) - \mathrm{Sp}(x))/y$, and similar terms which are very common in Feynman diagram expansions.

Logarithmic Difference Functions

The logarithmic difference functions can be evaluated using

$$\mathrm{Lf}_n(x) = -\sum_{k=n}^{\infty} \frac{(-)^k}{k} x^{k-n}$$

if x is small.

Differences between these functions can be found using the identity

$$\frac{1}{y}\left\{\mathrm{Lf}_n(x+y) - \mathrm{Lf}_n(x)\right\} = \frac{1}{(x+y)^n}\left\{\frac{1}{1+x}\mathrm{Lf}_1\left(\frac{y}{1+x}\right) - \sum_{k=1}^{n}(x+y)^{k-1}\ \mathrm{Lf}_k(x)\right\}$$

Dilogarithmic Difference Functions

For y small, we can use an expansion

$$\mathrm{Sf}_n(x,y) = \sum_{k=n}^{\infty} \frac{y^{k-n}}{k!} \mathrm{Sp}^{(k)}(x)$$

in derivatives $\mathrm{Sp}^{(k)}(x)$ of the dilogarithm, which can be calculated recursively using

$$\begin{aligned} \mathrm{Sp}^{(1)}(x) &= \mathrm{Lf}_1(-x), \\ \mathrm{Sp}^{(n+1)}(x) &= \frac{1}{x}\left\{ \frac{(n-1)!}{(1-x)^n} - n\mathrm{Sp}^{(n)}(x) \right\} \quad \text{for} \quad n \geq 1 \end{aligned}$$

If x and y are both small, we can use a double expansion

$$\mathrm{Sf}_n(x,y) = \sum_{k=n}^{\infty} \sum_{l=n}^{k} \frac{1}{k^2} \binom{k}{l} x^{k-l} y^{l-n}$$

Finite Mass Corrections

Mass corrections were added following Berends, *et al* (CALCUL collaboration). We checked that all significant mass corrections are obtained in this manner.

The most important corrections for a photon with momentum k radiated collinearly with each incoming fermion line p_1 and p_2 are added via the prescription

$$\left|\mathcal{M}_{1\gamma}^{(m)}\right|^2 = -\sum_i \frac{e^2 m_e^2}{p\cdot k} \left|\mathcal{M}_{\rm Born}(p_i - k)\right|^2$$

At the cross-section level, the net effect is that the spin-averaged form factor f_0 receives an additional mass term

$$\begin{aligned}\langle f_0\rangle^m &= \frac{2m_e^2}{s}\left(\frac{r_1}{r_2}+\frac{r_2}{r_1}\right)\frac{z}{(1-r_1)^2+(1-r_2)^2} \\ &\times \left\{\langle f_0\rangle + \ln\left(\frac{s}{m_e^2}\right)(\ln z - 1) - \frac{3}{2}\ln z + \frac{1}{2}\ln^2 z + 1\right\}\end{aligned}$$

Next to Leading Log Approximation

For Monte Carlo use, it is desirable to have the shortest possible equation with sufficient accuracy. For most of the range of hard photon energies, the leading log (LL) and next to leading log (NLL) contributions suffice. These include all terms important in the collinear limits (r_i small).

- To NLL order, the spinor terms I_1 and I_2 can be dropped, since f_1 and f_2 are at most logarithmically divergent for small r_i.
- The spin-averaged collinear limit of f_0 is ...

$$\begin{aligned}\langle f_0\rangle^{\rm NLL} &= 2\{L-1\} + \frac{r_1(1-r_1)}{1+(1-r_1)^2} + \frac{r_2(1-r_2)}{1+(1-r_2)^2} + 2\ln r_1 \ln(1-r_2) \\ &+ 2\ln r_2 \ln(1-r_1) - \ln^2(1-r_1) - \ln^2(1-r_2) + 3\ln(1-r_1) \\ &+ 3\ln(1-r_2) + 2\,{\rm Sp}(r_1) + 2\,{\rm Sp}(r_2) + \langle f_0\rangle_m^{\rm NLL}\end{aligned}$$

big log $L = \ln\left(\frac{s}{m_e^2}\right)$

mass corrections

YFS Residuals

The Monte Carlo program will calculate YFS residuals, which are obtained by subtracting the YFS factors containing the infrared singularities. This amounts to subtracting a term

$$4\pi B_{\rm YFS}(s,m) = \left(4\ln\frac{m_0}{m}+1\right)\left(\ln\frac{s}{m^2}-1-i\pi\right)-\ln^2\left(\frac{s}{m^2}\right)-1+\frac{4\pi^2}{3}+i\pi\left(2\ln\frac{s}{m^2}-1\right)$$

from the form factor f_0. At NLL order, we would have

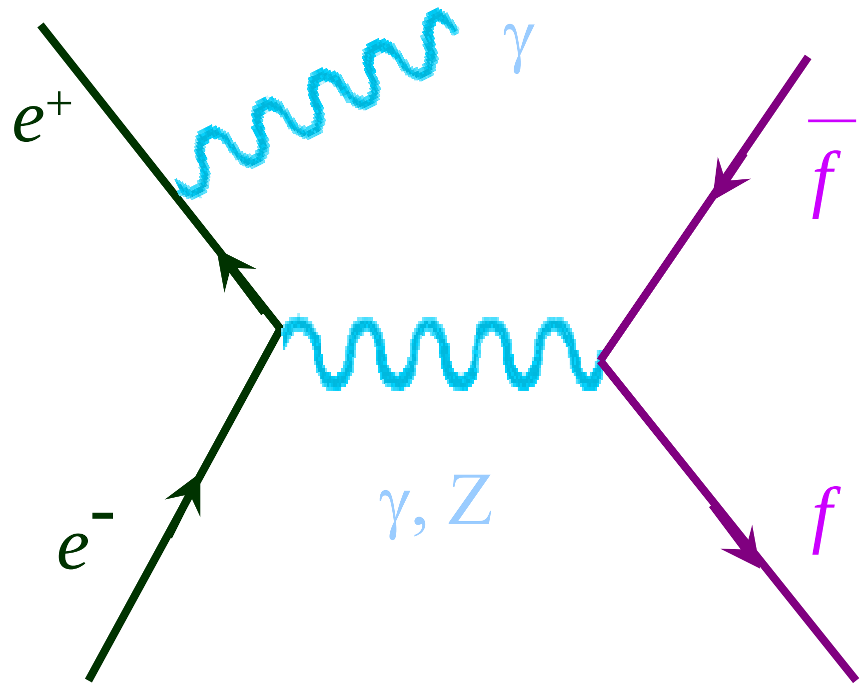
$$\overline{\beta}_1^{(2)} = \overline{\beta}_1^{(1)}\left(1+\frac{\alpha}{2\pi}\langle f_0\rangle^{\rm NLL}\right)$$

In our comparisons we will actually subtract this NLL term and look at the NNLL behavior of each expression.

Application: Radiative Return

Radiating a hard photon from the initial state can be a convenient way to explore a range of energies in a fixed-energy collider.

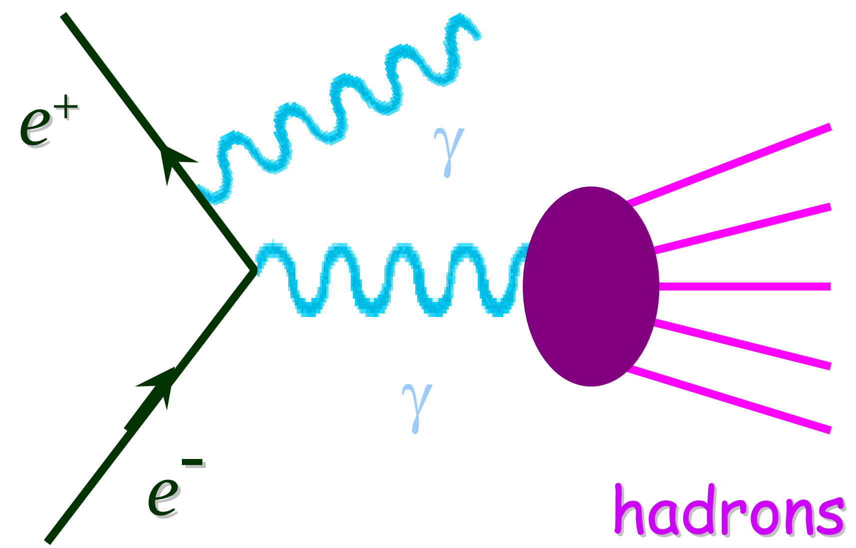
For example, the Z resonance can be probed in detail using a collider with $s > M_Z^2$.



Radiative Return

Hadronic final states can also be studied using the radiative return method.

J. Kuhn *et al* have developed a MC called PHOKHARA to calculate this process for hadronic and leptonic (*ie* muon pair) final states.



[Czyz, Grzelinska, Kuhn, Szopa, Eur. Phys. J. C33 (2004) 333 and refs.]

Comparison of Radiative Corrections

PHOKHARA incorporates initial state radiative corrections via a "Leptonic Tensor".

The work of Kuhn *et al* also an important cross-check on the radiative corrections calculated for the KK MC. [Rodrigo, Gehrmann-De Ridder, Guilesaume, Kuhn, Eur. Phys. J. C22, 81 (2001); Kuhn, Rodrigo, Eur. Phys. J. C25, 215 (2002)].

Since the initial state radiation is calculated exactly at order α^2 in both PHOKHARA and KK, and since both expressions are available analytically, it is useful to know how the expressions compare.

Form of the Comparison Result

The new comparison is to the leptonic tensor of Kuhn and Rodrigo, which was constructed for radiative return in hadron production, but can be adapted to fermion pairs by changing the final state tensor. The ISR result is expressed as a leptonic tensor

$$L_0^{\mu\nu} = \frac{e^6}{ss'^2}\{a_{00}s\eta^{\mu\nu} + a_{11}p_1^\mu p_1^\nu + a_{22}p_2^\mu p_2^\nu + a_{22}(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu) + i\pi a_{-1}(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)\}$$

which can be contracted with a final state tensor to get the squared matrix element:

$$H^{\mu\nu} = e^2(p_3^\mu p_4^\nu + p_3^\nu p_4^\mu - p_3 \cdot p_4 \eta^{\mu\nu})$$

$$|\mathcal{M}^{\rm ISR}|^2 = z L_0^{\mu\nu} H_{\mu\nu}.$$

Coefficient Functions in Comparison

The coefficients in the leptonic tensor can be separated into a tree-level term and a one-loop correction,

$$a_{ij} = a_{ij}^{(0)} + \frac{\alpha}{\pi} a_{ij}^{(1)}$$

An infrared term is subtracted to get a finite result: $c_{ij} = a_{ij}^{(1)} - a_{ij}^{\mathrm{IR}}$

with
$$a_{ij}^{\mathrm{IR}} = a_{ij}^{(0)} \left[2(L-1)\ln v_{\min} + \frac{1}{2}L - 1 + \frac{\pi^2}{3} \right]$$

cutoff on $v = 2E_\gamma/\sqrt{s}$

Calculating the coefficients requires considerable attention to numerical stability! We have re-expressed them in terms of Lf_n and Sp_n to prevent problems in the collinear limits.

Analytical Comparison

The expression for the leptonic tensor is very different from our earlier exact result, but it is possible to show that in the NLL limit, they agree analytically, in the following sense. First, consider the result *without* explicit mass terms.

The virtual correction to the YFS residual can be expressed in the NLL limit as

$$\overline{\beta}_1^{(2)} = \overline{\beta}_1^{(1)} \left(1 + \frac{\alpha}{2\pi} C_1\right) + C_2$$

with coefficients

$$C_1 = \frac{1}{2a_{00}^{(0)}} \left(\frac{c_{11}}{z} + z c_{22} - 2c_{12}\right),$$

$$C_2 = \frac{c_{11}}{4z} + \frac{z c_{22}}{4} - \frac{c_{12}}{2} - c_{00},$$

We find that

$$C_1 = \langle f_0 \rangle^{\mathrm{NLL}}$$

Analytic Comparison

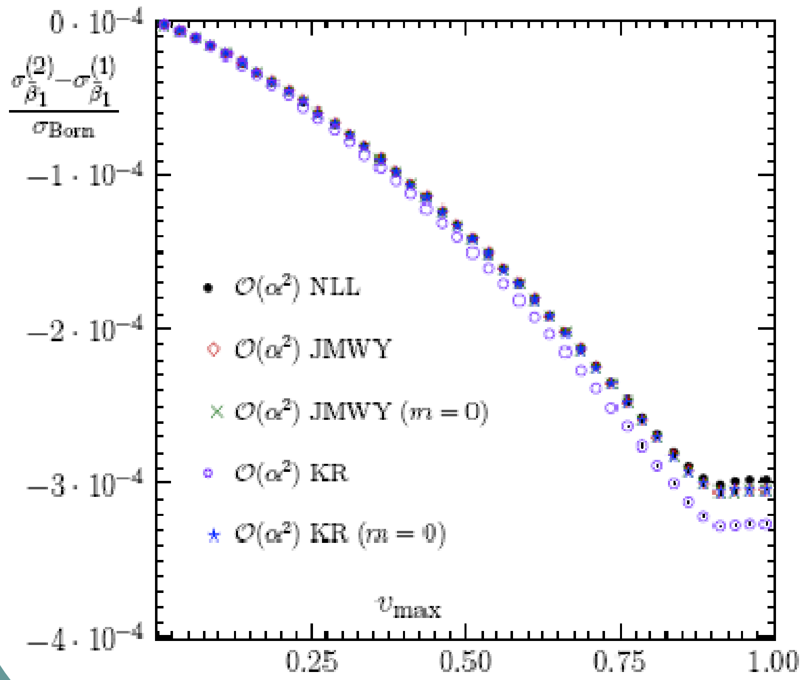
The analytic comparison of mass terms is more complicated. The result of Kuhn and Rodrigo contains terms of order $1/(p_i k)^3$ and $1/(p_i k)^4$, which actually cancel when carefully rewritten using the functions Lf_n and Sp_n, leaving the same $1/(p_i k)^2$ collinear behavior as in our expression.

A direct comparison shows that the results are identical to NLL order, and that the NNLL terms agree in the soft collinear limit (but *not* the general collinear limit).

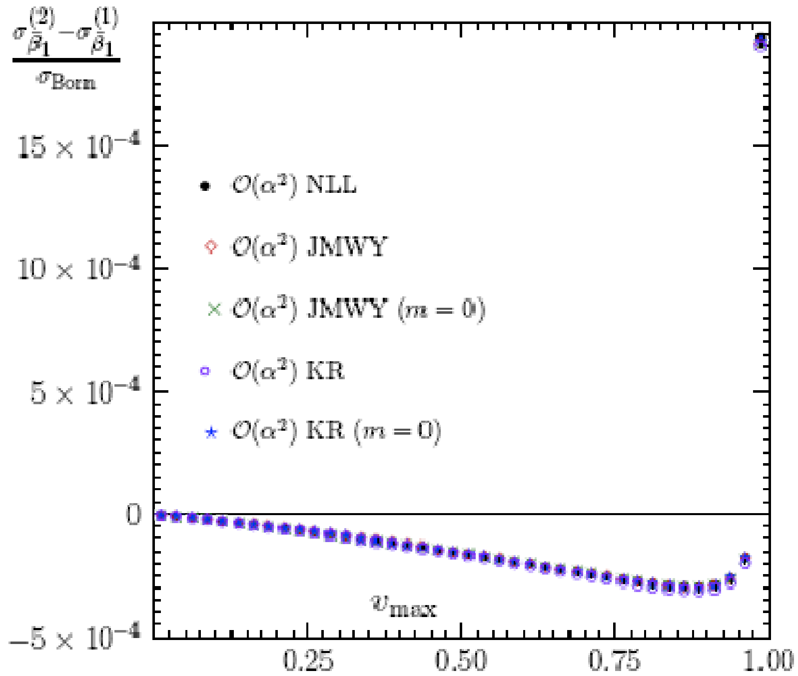
Monte Carlo Results

Results of KK Monte Carlo runs with 10^8 events at $E_{\mathrm{CMS}} = 1$ GeV and 500 GeV, showing only the virtual correction, with $4\pi B_{\mathrm{YFS}}$ subtracted.

1 GeV



500 GeV



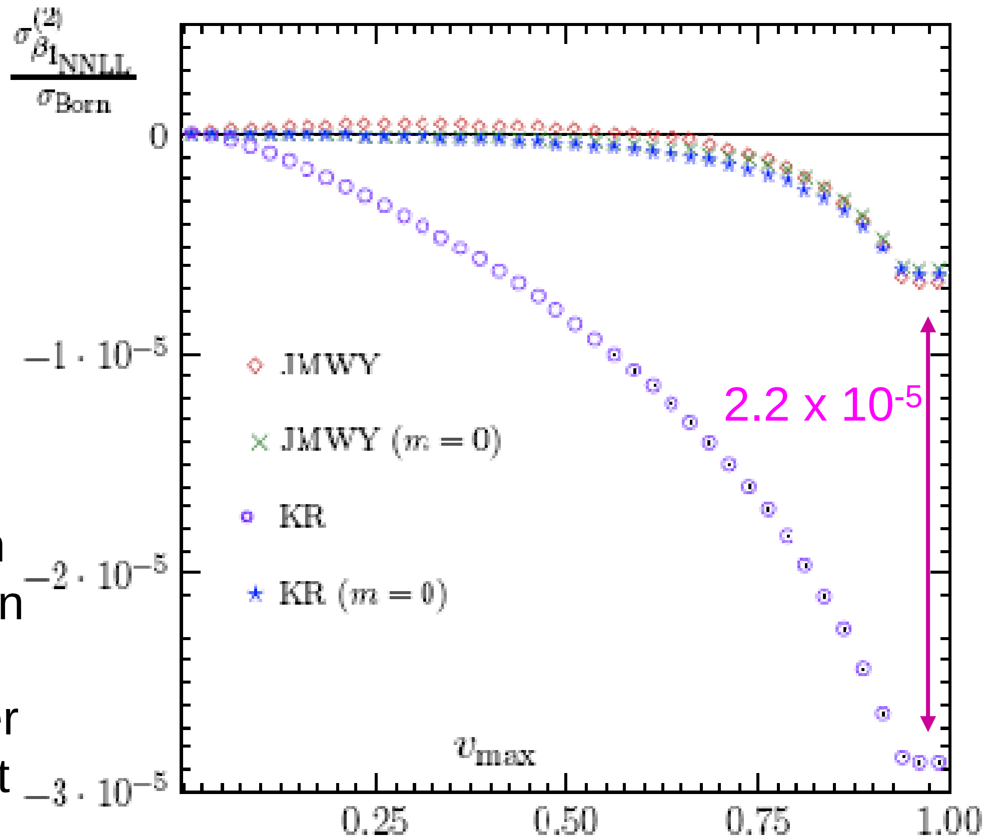
NNLL Comparison: 1 GeV

Results of a KK Monte Carlo run with 10^8 events at E_{CMS} = 1 GeV.

This figure shows the sub-NLL contribution to the real + virtual photon correction to muon pair production.

The "massless" NLL expression of JMWY has been subtracted in each case. Agreement of the massless results is on the order of 10^{-6}. The JMWY NNLL result is at most 7 x 10^{-6}.

$\sigma_1^{ISR} = 0.113\ \sigma_{Born}$



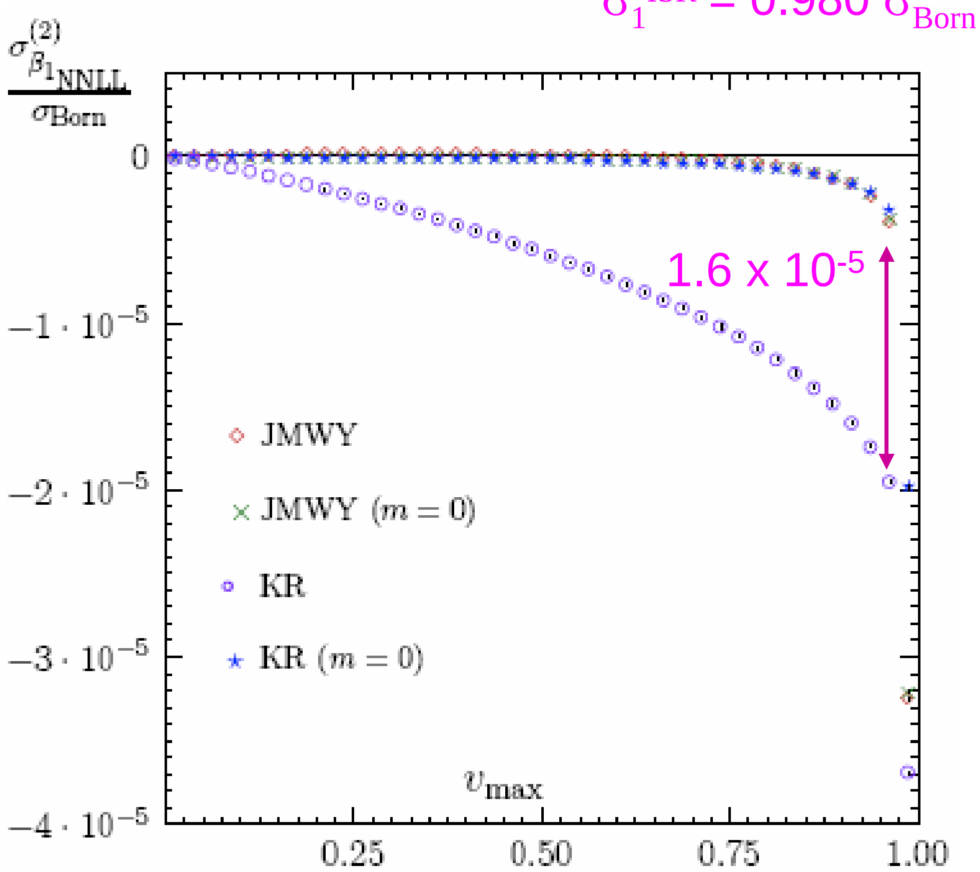
NNLL Comparison: 500 GeV

Results of a KK Monte Carlo run with 10^8 events at E_{CMS} = 500 GeV.

This figure shows the sub-NLL contribution to the real + virtual photon correction to muon pair production.

The "massless" NLL expression of JMWY has been subtracted in each case. Agreement of the massless results is 10^{-6} or better. The JMWY NNLL result is at most 3.2×10^{-5}.

$\sigma_1^{ISR} = 0.980\ \sigma_{Born}$



Summary

- YFS-exponentiation provides a very convenient framework for treating soft and collinear photon emission in a Monte Carlo framework.
- Helicity amplitude techniques provide a compact and efficient means for calculating the hard photon residuals to the desired order.
- Compact expressions for the most essential mass corrections can be added to the massless helicity amplitudes using techniques developed by Berends *et al* (the CALCUL collaboration).
- The YFS residuals can be evaluated for arbitrarily soft or collinear photons only if sufficient care is taken to express them in a numerically stable manner. Analytic cancellations should be used wherever possible.