TWO-LOOP Renormalization in the Making

Giampiero PASSARINO

Torino

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The QED case

To understand renormalization at the two-loop level we consider first the case of pure QED where we have

$$\Pi_{QED}(s,m) = \frac{e^2}{16\,\pi^2}\,\Pi^{(1)}(s,m) + \frac{e^4}{256\,\pi^4}\,\Pi^{(2)}(s,m), \tag{1}$$

where $p^2 = -s$ and where we have indicated a dependence of the result on the (bare) electron mass. Suppose that we compute the two-loop contribution (3 diagrams) in the limit $m = 0$. The result is

$$\Pi^{(2)}(s,0) = -\frac{4}{\epsilon} + \mathcal{O}\,(1)\,, \tag{2}$$

where $n = 4 - \epsilon$. This is a well-known result which shows the cancellation of the double ultraviolet pole as well as of any non-local residue. The latter is related to the fact that the four one-loop diagrams with one-loop counterterms cancel due to a Ward identity. Let us repeat the calculation with a non-zero electron mass;

after *scalarization* of the result we consider the ultraviolet divergent parts of the various diagrams. Collecting all the terms we obtain

$$\Pi^{(2)}(s,m) = -\frac{1}{\epsilon}\left[4\left(1+24\,\frac{m^2}{s}\right)+192\,\frac{m^4}{s^2}\,\frac{1}{\beta(m)}\,\ln\frac{\beta(m)+1}{\beta(m)-1}\right]+\mathcal{O}(1)\,. \tag{3}$$

Note that the *m* dependent part is not only finite but also zero in the limit $s \to 0$; indeed, in the limit $s \to 0$ and with $\mu^2 = m^2/s - i\,\delta$ we have

$$\beta = 2\,i\,\mu - \frac{i}{2\,\mu} + \mathcal{O}\left(\mu^{-2}\right), \qquad \frac{1}{\beta}\,\ln\frac{\beta+1}{\beta-1} = -\frac{1}{2\,\mu^2}, \tag{4}$$

so that

$$\Pi^{(2)}(0,m) = -\frac{4}{\epsilon} + \Pi^{(2)}_{\mathrm{fin}}(0,m). \tag{5}$$

Eq.(5) is the main ingredient to build our renormalization equation and contains only bare parameters, in the true spririt of the fitting equations that express a measurable input, α in this case, as a function of bare parameters, e and m in this case, and of ultraviolet singularites.
To make a prediction, the running of α in this case, is a different issue: the scattering of two charged particles is proportional to

$$\frac{e^2}{1 - f(s)} = e^2 \left[1 + f(s) + f^2(s) + \cdots \right],$$

$$f(s) = \frac{e^2}{16\,\pi^2}\,\Pi^{(1)}(s) + \frac{e^4}{(16\,\pi^2)^2}\,\Pi^{(2)}(s) + \mathcal{O}\left(e^6\right). \qquad (6)$$

Renormalization

Renormalization amounts to substituting

$$e^2 = 4\,\pi\,\alpha - \alpha^2\,\Pi^{(1)}(0) + \frac{\alpha^3}{4\,\pi}\left\{\left[\Pi^{(1)}(0)\right]^2 - \Pi^{(2)}(0)\right\} + \mathcal{O}\left(\alpha^4\right), \qquad (7)$$

with the following result

$$\frac{e^2}{1 - f(s)} = 4\,\pi\,\alpha\Bigl\{1 + \frac{\alpha}{4\,\pi}\,\Pi^{(1)}_R(s) + \left(\frac{\alpha}{4\,\pi}\right)^2\,\Bigl[\Pi^{(1)}_R(s)\,\Pi^{(1)}_R(s) + \Pi^{(2)}_R(s) + \mathcal{O}\left(\alpha^3\right)\Bigr\},$$

$$\Pi^{(n)}_R(s) = \Pi^{(n)}(s) - \Pi^{(n)}(0). \qquad (8)$$

If our result has to be ultraviolet finite then the poles in $\Pi^{(n)}(s)$ should not depend on the scale s. This is obviously true for the one-loop result but what is the origin of the scale-dependent extra term in Eq.(3)? One should take into account that

$$\Pi^{(1)}(s,m) = -\frac{8}{3}\frac{1}{\epsilon} + \frac{4}{3}\left[\ln\frac{m^2}{M^2} + \left(1+2\,\frac{m^2}{s}\,\beta(m)\,\ln\frac{\beta(m)+1}{\beta(m)-1}\right] - \frac{20}{9} + \frac{4}{3}\Delta_{UV} - \frac{16}{3}\frac{m^2}{s}, \tag{9}$$

and that m is the bare electron mass. To proceed step-by-step we introduce a renormalized electron mass which is given by

$$m = m_R\left[1 + \frac{e^2}{16\,\pi^2}\left(-\frac{6}{\epsilon} + \text{finite part}\right)\right]. \tag{10}$$

If we write $m^2 = m_R^2\,(1+\delta)$ then

$$\beta(m) = \beta(m_R) - 2\,\frac{m_R^2}{\beta(m_R)\,s}\,\delta + \mathcal{O}\left(\delta^2\right),$$
$$\ln\frac{\beta(m)+1}{\beta(m)-1} = \ln\frac{\beta(m_R)+1}{\beta(m_R)-1} - \frac{\delta}{\beta(m_R)} + \mathcal{O}\left(\delta^2\right). \tag{11}$$

Inserting this expansion into our results we obtain

$$\begin{aligned}\Pi_{QED}(s,m_R) = \frac{e^2}{\pi^2}\Big[&-\frac{1}{6\,\epsilon} + \frac{1}{12}\,\ln\frac{m_R^2}{M^2} \\ &+ \frac{1}{3}\left(\frac{1}{4} - \frac{1}{2}\,\frac{m_R^2}{s} - 2\,\frac{m_R^4}{s^2}\right)\,\frac{1}{\beta(m_R)}\,\ln\frac{\beta(m_R)+1}{\beta(m_R)-1} - \\ &- \frac{5}{36} + \frac{1}{12}\,\Delta_{UV} - \frac{1}{3}\,\frac{m_R^2}{s}\Big] \\ &+ \frac{e^4}{\pi^4}\Big[-\frac{1}{64\,\epsilon} + \frac{1}{256}\,\Pi^{(2)}_{\rm fin}(s,m_R)\Big],\end{aligned} \tag{12}$$

showing cancellation of the ultraviolet poles in $\Pi_R^{(n)}(s, m_R)$ with $n = 1, 2$. Of course Eq.(10) is not yet a true renormalization equation since the latter should contain the physical electron mass m_e and not the intermediate parameter m_R but the relation between the two is ultraviolet finite. All of this is telling us that a renormalization equation has the structure

$$p_{\rm phys} = f\left(\frac{1}{\epsilon}\,,\,p_{\rm bare}\right)\,, \tag{13}$$

where the residue of the ultraviolet poles must be local. A prediction,

$$O\left(\frac{1}{\epsilon}\,,\,p_{\rm bare}\right) \equiv O(p_{\rm phys}), \tag{14}$$

gives a finite quantity that can be computed in terms of some input parameter set.

The SM case

In the full standard model the one-loop result is

$$\Pi^{(1)} = \Pi^{(1)}_{\text{bos}} + \sum_l \Pi^{(1)}_l + \Pi^{(1)}_{tb} + \Pi^{(1)}_{udcs}. \tag{15}$$

We introduce

$$x_w = \frac{M_w^2}{s}, \quad x_l = \frac{m_l^2}{M_w^2}, \quad \text{etc},$$

$$\Delta_{UV} = \gamma + \ln \pi + \ln \frac{M_w^2}{\mu^2}, \qquad L_\beta(x) = \ln \frac{\beta(x) + 1}{\beta(x) - 1}, \tag{16}$$

In the limit $s \to 0$ we have

$$\Pi^{(1)}_{\mathrm{bos}}(0) = -3\left(-\frac{2}{\epsilon} + \Delta_{UV}\right),$$
$$\Pi^{(1)}_{l}(0) = \frac{4}{3}\left(-\frac{2}{\epsilon} + \Delta_{UV}\right) + \frac{4}{9} + \frac{4}{3}\ln x_l,$$
$$\Pi^{(1)}_{tb}(0) = \frac{20}{9}\left(-\frac{2}{\epsilon} + \Delta_{UV}\right) + \frac{20}{27} + \frac{16}{9}\ln x_t + \frac{4}{9}\ln x_b. \qquad (17)$$

First we consider fermion mass renormalization, obtaining

$$m_f^2 = m_{f\,R}^2\left(1 + 2\,\frac{g^2}{16\,\pi^2}\,\frac{\delta Z_m^f}{\epsilon}\right), \qquad (18)$$

with renormalization constants given by

fermion mass renormalization

lepton

$$\delta Z_m^l = -\frac{3}{2}\,\frac{1}{c^4}\,x_H^{-1} - 3\,\frac{1}{c^2} + 3 + \frac{3}{4}\,x_L + 2\,\frac{x_L^2}{x_H} + 6\,\frac{x_B^2}{x_H} + 6\,\frac{x_T^2}{x_H} - \frac{3}{4}\,x_H - 3\,x_H^{-1}, \tag{19}$$

b quark

$$\delta Z^{b}_{m} = -\frac{3}{2}\,\frac{1}{c^4}\,x_H^{-1} + \frac{1}{3}\,\frac{1}{c^2} - \frac{1}{3} + \frac{3}{4}\,x_B - \frac{3}{4}\,x_T + 2\,\frac{x_L^2}{x_H} + 6\,\frac{x_B^2}{x_H} + 6\,\frac{x_T^2}{x_H} - \frac{3}{4}\,x_H - 3\,x_H^{-1}, \tag{20}$$

t quark

$$\delta Z^t_m = -\frac{3}{2}\frac{1}{c^4}x_H^{-1} - \frac{2}{3}\frac{1}{c^2} + \frac{2}{3} - \frac{3}{4}x_B + \frac{3}{4}x_T + 2\frac{x_L^2}{x_H} + 6\frac{x_B^2}{x_H} + 6\frac{x_T^2}{x_H} - \frac{3}{4}x_H - 3x_H^{-1}. \qquad (21)$$

Consider the fermionic part of $\Pi^{(1)}$ relative to one fermion generation (ν_l, l, t and b) and perform fermion mass renormalization; we obtain

$$\Pi^{(1)}_{\rm fer} \to \Pi^{(1)}_{\rm ferm} + \frac{g^2}{\pi^2\,\epsilon}\,\Delta\Pi^{(1)}_{\rm ferm}, \tag{22}$$

where

$$\begin{aligned}\Pi^{(1)}_{\rm fer} &= \frac{32}{9}\left(-\frac{2}{\epsilon} + \Delta_{UV}\right) + \frac{4}{3}\left(\ln x_L + \frac{1}{3}\ln x_B + \frac{4}{3}\ln x_T\right)\\ &- \frac{160}{27} - \frac{16}{3}\,x_W\,(x_L + \frac{1}{3}x_B + \frac{4}{3}x_T) + \frac{4}{3}(1 - 2\,x_W x_L - 8\,x_W^2 x_L^2)\\ &+ \frac{4}{3}\,\beta^{-1}(x_W x_L)\,L_\beta(x_W x_L) + \frac{4}{9}\,\beta^{-1}(x_W x_B)\,L_\beta(x_W x_B)\\ &+ \frac{16}{9}\,\beta^{-1}(x_W x_T)\,L_\beta(x_W x_T),\end{aligned} \tag{23}$$

$$
\begin{aligned}
\Delta\Pi^{(1)}_{\mathrm{ferm}} = {} & \frac{3}{2} c^{-4} x_W x_L x_H^{-1} + \frac{1}{2} c^{-4} x_W x_B x_H^{-1} + 2 c^{-4} x_W x_T x_H^{-1} + 3 c^{-2} x_W x_L \\
& - \frac{1}{9} c^{-2} x_W x_B + \frac{8}{9} c^{-2} x_W x_T - 6 x_W x_L x_B^2 x_H^{-1} - 6 x_W x_L x_T^2 x_H^{-1} + \cdots \\
& + 2 x_W^2 x_T^2 x_H - \frac{16}{9} x_W^2 x_T^2 - 2 x_W^2 x_T^3 - 16 x_W^2 x_T^4 x_H^{-1} \Big). \qquad (24)
\end{aligned}
$$

When we add the two-loop result we obtain

$$\frac{g^2}{16\,\pi^2}\,\Pi^{(1)}_{\rm fer} + \frac{g^4}{(16\,\pi^2)^2}\,\Pi^{(2)} = \text{one loop} + \frac{g^4}{\pi^4}\,\Big[R^{(2)}\,\epsilon^{-2} + R^{(1)}\,\epsilon^{-1} + \Pi_{\rm fin}\Big]. \tag{25}$$

The two residues are given by

$$\begin{aligned} R^{(2)} &= -\,\frac{11}{256}, \\ R^{(1)} &= \frac{11}{256}\,\Delta_{UV} + \frac{407}{27648} + \frac{9}{64} c^{-4} x_W x_H^{-1} - \frac{9}{128} c^{-2} x_W - \frac{131}{6912} c^{-2} \\ &+ \frac{3}{64} x_W x_L - \frac{3}{16} x_W x_L^2 x_H^{-1} + \frac{9}{64} x_W x_B - \frac{9}{16} x_W x_B^2 x_H^{-1} \end{aligned} \tag{26}$$

$$
\begin{aligned}
&+ \frac{9}{64}x_W x_T - \frac{9}{16}x_W x_T^2 x_H^{-1} + \frac{9}{32}x_W x_H^{-1} + \frac{9}{128}x_W x_H \\
&+ \frac{1}{32}x_{\boldsymbol{w}} + \frac{3}{512}x_L + \frac{7}{1536}x_B + \frac{13}{1536}x_T \\
&+ \beta^{-1}(x_W)L_\beta(x_W)\Big(-\frac{11}{768} + \frac{3}{64}c^{-4}x_W x_H^{-1} + \frac{9}{32}c^{-4}x_W^2 x_H^{-1} \\
&- \frac{1}{32}c^{-2}x_W - \frac{9}{64}c^{-2}x_W^2 + \frac{3}{128}x_W x_L - \frac{1}{16}x_W x_L^2 x_H^{-1} + \frac{9}{128}x_W x_B \\
&- \frac{3}{16}x_W x_B^2 x_H^{-1} + \frac{9}{128}x_W x_T - \frac{3}{16}x_W x_T^2 x_H^{-1} + \frac{3}{32}x_W x_H^{-1} \\
&+ \frac{3}{128}x_W x_H - \frac{13}{384}x_W + \frac{3}{32}x_W^2 x_L - \frac{3}{8}x_W^2 x_L^2 x_H^{-1} \\
&+ \frac{9}{32}x_W^2 x_B - \frac{9}{8}x_W^2 x_B^2 x_H^{-1} + \frac{9}{32}x_W^2 x_T - \frac{9}{8}x_W^2 x_T^2 x_H^{-1} \\
&+ \frac{9}{16}x_W^2 x_H^{-1} + \frac{9}{64}x_W^2 x_H + \frac{1}{16}x_W^2\Big). \qquad (27)
\end{aligned}
$$

Theorem

Therefore mass renormalization has removed

> *all logarithms in the residue of the simple ultraviolet pole for the fermionic part*
>
> *while a non-local residue remains in the the bosonic part.*

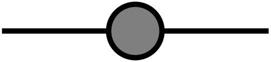
Unfortunately a simple procedure of W mass renormalization is not enough to get rid of logarithmic residues in the bosonic component and the reason is that in a bosonic loop we may have three different fields, the W, the ϕ and the charged ghosts
and only one mass is available.

Example

The situation is illustrated in Fig. 1 where the cross denotes insertion of a counterterm δZ_M; the latter is fixed to remove the ultraviolet pole in the W self-energy and one easily verifies that the total in the second and third line of Fig. 1 (ϕ and X self-energies, respectively) is not ultraviolet finite.

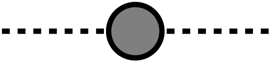

δZ_M

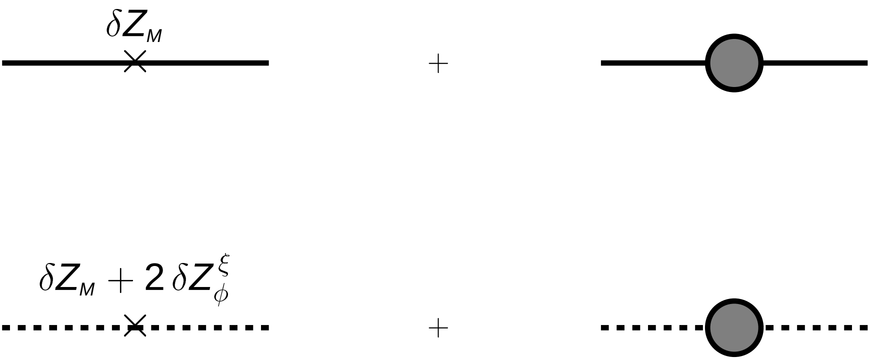
$+$




δZ_M

$+$




δZ_M
$+$
$\delta Z_M + 2\,\delta Z_{\phi}^{\xi}$
$+$

The procedure has to be changed if we want to make the result in the bosonic sector as similar as possible to the one in the fermionic sector. With this goal in mind we introduce the following counterterms

$$W_\mu = Z_W^{1/2}\, W_\mu^R, \qquad \phi = Z_\phi^{1/2}\, \phi^R, \qquad M_W = Z_M^{1/2}\, M_W^R. \tag{28}$$

Our solution is to work in a $R_{\xi\xi}$-gauge where the gauge-fixing term (limited to the charged sector) is

$$\mathcal{C} = -\frac{1}{\xi_W}\, \partial_\mu W_\mu + \xi_\phi\, M_W\, \phi. \tag{29}$$

We also introduce additional counter-terms for the gauge parameters,

$$\xi_W = Z_W^{\xi}\, \xi_W^R, \qquad \xi_\phi = Z_\phi^{\xi}\, \xi_\phi^R. \tag{30}$$

Our scheme is further specified by imposing the condition

$$\xi_W^R = \xi_\phi^R = 1. \tag{31}$$

Dropping from now on the index R for renormalized fields and parameters we define the counter-Lagrangian to be

$$\mathcal{L}_{\rm ct} = \frac{g^2}{16\,\pi^2}\left[\mathcal{L}^{WW}_{\rm ct} + \mathcal{L}^{\phi W}_{\rm ct} + \mathcal{L}^{\phi\phi}_{\rm ct}\right], \qquad \mathcal{L}^{ij}_{\rm ct} = \Phi^R_i\,\mathcal{O}_{ij}\,\Phi^R_i, \tag{32}$$

Φ_i being a vector or scalar field. We define δZ factors in the MS-scheme as

$$Z = 1 + \frac{g^2}{16\,\pi^2}\,\delta Z\,\frac{1}{\epsilon}, \tag{33}$$

and obtain

$$\epsilon\,\mathcal{O}^{WW}_{\mu\nu} = -\Big[\delta Z_W\,(p^2 + M^2_W) + \delta Z_M\,M^2_W\Big]\,\delta_{\mu\nu} + 2\,\delta Z^{\xi}_W\,p_\mu\,p_\nu,$$
$$\epsilon\,\mathcal{O}^{\phi\phi} = -\Big[\delta Z_\phi\,(p^2 + M^2_W) + M^2_W\,(\delta Z_M + 2\,\delta Z^{\xi}_\phi)\Big],$$
$$\epsilon\,\mathcal{O}^{W\phi}_{\mu} = (\delta Z^{\xi}_W - \delta Z^{\xi}_\phi)\,i\,M_W\,p_\mu. \tag{34}$$

These counter-terms are used to remove all poles from the transitions in the charged sector. After including the tadpole contribution and using Eq.(31) we find

$$
\begin{aligned}
\delta Z^{\xi}_{W} &= \frac{11}{6}, \\
\delta Z^{\xi}_{\phi} &= -\frac{2}{3} + \frac{3}{2}c^{-4}x_H^{-1} - \frac{5}{4}c^{-2} + x_L - 2x_L^2 x_H^{-1} \\
&\quad + 3x_B - 6x_B^2 x_H^{-1} + 3x_T - 6x_T^2 x_H^{-1} + 3/4x_H + 3x_H^{-1}, \\
\delta Z_W &= \frac{11}{3} \\
\delta Z_{\phi} &= 2 + c^{-2} - x_L - 3x_B - 3x_T, \\
\delta Z_M &= -\frac{2}{3} - 3c^{-4}x_H^{-1} + \frac{3}{2}c^{-2} - x_L + 4x_L^2 x_H^{-1} - 3x_B \\
&\quad + 12x_B^2 x_H^{-1} - 3x_T + 12x_T^2 x_H^{-1} - \frac{3}{2}x_H - 6x_H^{-1}. \qquad (35)
\end{aligned}
$$

Theorem

An important result follows, namely both

$$-Z_W^{1/2}\,(\xi_W Z_W^{\xi})^{-1}, \qquad +Z_M^{1/2}\,Z_\phi^{\xi}\,Z_\phi^{1/2}\,M\,\xi_\phi, \tag{36}$$

are ultraviolet finite so that the gauge-fixing term remains unrenormalized.

To continue our derivation we consider the ghost Lagrangian and the associated counter-terms,

$$\mathcal{L}_{\mathrm{g}} = Z_X\, \bar{X}^{\pm} \left[\frac{1}{Z^{\xi}_{W}\, \xi_W}\, \partial^2 - Z^{\xi}_{\phi}\, Z_M\, \xi_{\phi}\, M^2_W \right] X^{\pm}. \tag{37}$$

To this Lagrangian corresponds an operator

$$\epsilon\, \mathcal{O}^{gg} = - \Big[(\delta Z_X - \delta Z^{\xi}_{W})\, (p^2 + M^2_W) + (\delta Z_M + \delta Z^{\xi}_{W} + \delta Z^{\xi}_{\phi})\, M^2_W \Big]. \tag{38}$$

A simple calculation shows that, with the choice

$$\delta Z_x = \frac{23}{6}, \tag{39}$$

also the ghost Lagrangian is ultraviolet finite. The correct combination of mass counterterms is illustrated in Fig. 2. Note that in the $\overline{MS}$ scheme we define

$$Z = 1 + \frac{g^2}{16\,\pi^2}\,\delta Z\left[-\frac{2}{\epsilon} + \Delta_{UV}\right], \qquad \delta Z_{\overline{MS}} = -\frac{1}{2}\,\delta Z_{MS}. \tag{40}$$

Note that the two-loop part of Π remains unchanged since modifications are of $\mathcal{O}\left(g^6\right)$ while for $\Pi^{(1)}_{\mathrm{bos}}$ we have to repeat the calculation, working in the new gauge.

The bare propagators for charged fields in the $R_{\xi\xi}$ gauge are

$$
\begin{aligned}
\bar{\Delta}^{ww}_{\mu\nu} &= \frac{1}{p^2+M^2}\left[\delta_{\mu\alpha} + \frac{\xi_w^2-1}{p^2+\xi_w^2 M^2}\, p_\mu\, p_\alpha\right] \\
&\quad \times \left[\delta_{\alpha\nu} + (1-\frac{\xi_\phi}{\xi_w})^2\, \frac{\xi_w^2 M^2}{(p^2+\xi_w\xi_\phi M^2)^2}\, p_\alpha\, p_\nu\right], \\
\bar{\Delta}^{w\phi}_{\mu} &= i\, M\, p_\mu\, \frac{\xi_w\,(\xi_\phi-\xi_w)}{(p^2+\xi_w\xi_\phi M^2)^2}, \qquad \bar{\Delta}^{\phi\phi} = \frac{p^2+\xi_w^2 M^2}{(p^2+\xi_w\xi_\phi M^2)^2}, \\
\bar{\Delta}^{gg} &= \frac{\xi_w}{p^2+\xi_w\xi_\phi M^2},
\end{aligned}
\tag{41}
$$

where the last propagator refers to the ghost - ghost transition.

One example will be enough to describe the procedure. Consider the following integral, corresponding to a ϕ loop in the AA self-energy:

$$I_{\mu\nu} = \int d^n q \, \frac{(q^2 + \xi_W^2 M_W^2)((q+p)^2 + \xi_W^2 M_W^2)}{(q^2 + \xi_W \xi_\phi M_W^2)^2((q+p)^2 + \xi_W \xi_\phi M_W^2)^2} \times (2\, q_\mu + p_\mu)(2\, q_\nu + p_\nu). \tag{42}$$

We expand the propagators,

$$\begin{aligned}
(q^2 + \xi_W^2 M_W^2)^{-k} &= (q^2 + M_W^2)^{-k} \\
&\quad - 2\, k \, \frac{g^2}{16\, \pi^2 \epsilon} \, dZ_W^\xi \, M_W^2 \, (q^2 + M_W^2)^{-k-1} + \cdots, \\
(q^2 + \xi_W \xi_\phi M_W^2)^{-k} &= (q^2 + M_W^2)^{-k} \\
&\quad - k \, \frac{g^2}{16\, \pi^2 \epsilon} (dZ_W^\xi + dZ_\phi^\xi) \, M_W^2 \, (q^2 + M_W^2)^{-k-1} + \cdot
\end{aligned} \tag{43}$$

and obtain

$$I_{\mu\nu} = I_0\, \delta_{\mu\nu} + I_1\, p_\mu p_\nu, \tag{44}$$

with form factors

$$\begin{aligned} I_0 &= I_0(\xi = 1) + i\, \pi^2\, g^2\, \Delta I_0\, dZ_\phi^\xi, \\ \Delta I_0 &= \frac{1}{8}\, \frac{n-2}{n-1}\, A_0(1, M_w^2) - \frac{n-1}{2}\, M_w^2\, B_0(1, 1, p^2, M_w, M_w) \\ &+ \frac{1}{4}\, \frac{1}{n-1}\, M_w^2\, (p^2 + M_w^2)\, B_0(1, 2, p^2, M_w, M_w), \end{aligned} \tag{45}$$

where M_w is the bare W mass. Collecting all diagrams, renormalizing the W mass and inserting the solution for the renormalization constants we find the expression for the bosonic, one-loop, AA self-energy:

$$\Pi^{(1)}_{\rm bos} \rightarrow \frac{6}{\epsilon} + 6 - 3\Delta_{UV} + 8x_W + \cdots \tag{46}$$

Including both components and taking into account the additional contribution arising from renormalization we finally get residues for the ultraviolet poles which show the expected properties:

$$R^{(2)} = -\frac{55}{768},$$
$$R^{(1)} = \frac{11}{192}\,\Delta_{UV} + \frac{1199}{27648} - \frac{131}{6912}c^{-2} + \frac{3}{512}x_L + \frac{13}{1536}x_T + \frac{7}{1536}x_B. \tag{47}$$

Eq.(47) shows complete cancellation of poles with a logarithmic residue; furthermore the two residues in Eq.(47) are scale independent and cancel in the difference $\Pi(p^2) - \Pi(0)$.

Transitions

A final comment concerns the Z-photon transition which is not zero, at $p^2 = 0$, in any gauge where $\xi \neq 1$ even after the Γ_1 re-diagonalization procedure.

However, in our case, the non-zero result shows up only due to a different renormalization of the two bare gauge parameters and it is, therefore, of $\mathcal{O}\left(g^4\right)$; it can be absorbed into Γ_2 which does not modify our result for Π since there are no Γ_2-dependent terms in the AA transition (only Γ_1^2 appears).

renormalization procedure

One should observe that our procedure is completely equivalent to consider one-loop diagrams with the insertion of one-loop counterterms and one may wonder why
we have not included $\delta Z_W, \delta Z_\phi, \delta Z_X$ and also a δZ_e,
arising from charge renormalization and a δZ_A from the renormalization of the photon field.

about counterterms

The argument goes as follows: first we consider the relevant vertices with counterterms:

$$
\begin{aligned}
AWW &= Z_{W}\, Z_{A}^{1/2}\, Z_e \;\otimes\; \text{Born},\\
A\phi\phi &= Z_{\phi}\, Z_{A}^{1/2}\, Z_e \;\otimes\; \text{Born},\\
AW\phi &= (Z_{W}\, Z_{\phi}\, Z_{A}\, Z_{M})^{1/2}\, Z_e \;\otimes\; \text{Born},\\
A\overline{X}^{\pm}X^{\pm} &= Z_{X}\, Z_{A}^{1/2}\, Z_e \;\otimes\; \text{Born}.
\end{aligned}
\tag{48}
$$

Next, we consider the ultraviolet divergent part of the corresponding one-loop diagrams and obtain:

$$V_{UV} = \frac{g^2}{16\,\pi^2}\,\frac{\delta V}{\epsilon}, \tag{49}$$

where

$$\begin{aligned}
\delta V^{AWW}_{\alpha\beta\gamma} &= -\frac{11}{3}\,\delta_{\alpha\beta}\,(p_2 + 2\,p_1)_\gamma + \frac{11}{3}\,\delta_{\alpha\gamma}\,(p_1 + 2\,p_2)_\beta \\
&+ \frac{11}{3}\,\delta_{\beta\gamma}\,(p_1 - p_2)_\alpha \\
\delta V^{A\phi\phi}_{\alpha} &= \Big(2 + c^{-2} - x_{L} - 3\,x_{T} - 3\,x_{B}\Big)\,(p_1 - p_2)_\alpha, \\
\delta V^{AXX}_{\alpha} &= 2\,p_{1\alpha}, \\
\delta V^{AW\phi}_{\alpha\gamma} &= i\,\delta_{\alpha\gamma}\,M_{W}\,\Big(\frac{3}{2}c^{-4}\frac{1}{x_H} - \frac{5}{4}c^{-2} - 2\frac{x_L^2}{x_H} - 6\frac{x_T^2}{x_H} - 6\frac{x_B^2}{x_H} + \frac{3}{x_H} + \frac{3}{4}x_H \\
&+ x_L + 3x_T + 3x_B - \frac{5}{2}\Big).
\end{aligned} \tag{50}$$

With these results we can prove that

$$\delta Z_e + \frac{1}{2}\,\delta Z_A = 0, \tag{51}$$

i.e. that, like in QED, charge renormalization is only due to vacuum polarization. Note that the Γ_1 prescription is crucial for proving the Ward identity of Eq.(51). Consider now the one-loop photon self-energy in our gauge; for instance, the diagrams with a ghost loop have vertices proportional to Z_x (thanks to Eq.(51)) and ghost propagators given by

$$\Delta^{gg} = \frac{1}{Z_x}\,\frac{\xi_W}{p^2 + \xi_W\,\xi_\phi\,mw^2}. \tag{52}$$

Clearly, δZ_x gives no contribution. The same holds for all other diagrams and for the remaining counterterms, δZ_ϕ and δZ_W. In conclusion, in computing Π we can forget about one-loop diagrams with field and charge counterterms and only worry about mass renormalization which we do, in some unconventional way, by expanding the explicit expression for $\Pi^{(1)}(s)$.

Inclusion of Δ_{UV}

In the previous section we have performed renormalization in the *MS* scheme and here we proceed by extending the same procedure to the $\overline{MS}$ scheme. The counterterms in the two schemes are connected by the simple relation $\delta Z_{\overline{MS}} = -\frac{1}{2}\,\delta Z_{MS}$ and what we may show that not only the double and single ultraviolet poles of $\Pi(s)$ have scale independent, local, residues but also the terms proportional to powers of Δ_{UV} have the same property.

Fermion mass fitting equations

For the complete answer we need fitting equations that relate the bare masses to the physical ones since the renormalized mass is only an intermediate parameter which is bound to disappear in the expresion for any physical observable. For a generic $u-d$ doublet we obtain

$$
\begin{aligned}
m_f &= m_f^{\mathrm{phys}} + \frac{g^2}{16\,\pi^2}\,\Sigma_f\Big|_{m=m^{\mathrm{phys}}},\\
m^2_{f\,\mathrm{ren}} &= m^2_{f\,\mathrm{phys}}\left\{1+\frac{g^2}{8\,\pi^2}\left[\frac{\Sigma_f}{m_f^2}\Big|_{m=m^{\mathrm{phys}}} - \delta Z_m^f\right]\right\}
\end{aligned}
\tag{53}
$$

W mass fitting equations

The relation between renormalized and physical W mass is

$$M^2_{W\,\mathrm{ren}} = M^2_{W\,\mathrm{phys}} \left\{ 1 + \frac{g^2}{16\,\pi^2} \left[\frac{\mathrm{Re}\,\Sigma_{WW}(-M^2_{W\,\mathrm{phys}})}{M^2_{W\,\mathrm{phys}}} - \delta Z_M \right] \right\}, \tag{54}$$

where the quantity within square brackets is ultraviolet finite by construction and where

$$\Sigma_{WW} = \sum_{\mathrm{gen}} \Sigma^f_{WW} + \Sigma^b_{WW} - 2\,(\beta_{t1} + \Gamma_1). \tag{55}$$

Part II

Introduction to the Fermi Coupling Constant

Definitions

Writing a renormalization equation that involves G_F should not be confused with making a prediction with the muon life-time.

In the following section we present few examples that are relevant in evaluating Δg (see Eq.(58)) up to two-loops and therefore in contructing one of our renormalization equations.

– The Lagrangian of the Fermi theory which is relevant for our pourposes can be written as:

$$\mathcal{L}_F = \mathcal{L}_{QED} + \frac{G_F}{\sqrt{2}}\, \overline{\psi}_{\nu_m u}\, \gamma^\mu\, \gamma_+\, \psi_\mu\, \overline{\psi}_e\, \gamma^\mu\, \gamma_+\, \psi_{\nu_e}, \tag{56}$$

where $\gamma_+ = 1 + \gamma_5$.

To leading order in G_F and to all orders in α the muon lifetime takes the form

$$\frac{1}{\tau_\mu} = \Gamma_0 \, (1 + \Delta q), \qquad \Gamma_0 = \frac{G_F^2 \, m_\mu^5}{192 \, \pi^3}. \tag{57}$$

The standard model weak corrections to τ_μ are conventionally parametrized by the relation

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 \, M^2} \, (1 + \Delta g). \tag{58}$$

Our goal will be to derive an explicit expression for Δg so that one can use Eq.(58) as a relation where on the left hand side there is a quantity whose value is obtained by experiment and where on the right hand side we have bare quantities.

Ther quantity Δg may be written as the sum of various contributions, which are

$$\Delta g = \Delta g^{WF} + \Delta g^{V} + \Delta g^{B} + \Delta g^{S}. \tag{59}$$

The various terms arise from wave-function renormalization factors, weak vertices, boxes and the W self-energy. Self-energy corrections always play a special role and will be dicussed separately, although they are crucial in establishing gauge parameter independence.

Strategy of the calculation

In the standard model and in the $\xi = 1$ gauge the lowest order amplitude is

$$
\begin{aligned}
\mathcal{M}_{SM\,;\,0} &= (2\pi)^4\, i\, \frac{g^2}{8}\, \frac{1}{Q^2 + M^2}\, \overline{u}(p_{\nu_\mu})\, \gamma^\alpha\, \gamma_+\, u(p_\mu)\, \overline{u}(p_e)\, \gamma^\alpha\, \gamma_+\, v(p_{\nu_e}) \\
&\approx \frac{G_F}{\sqrt{2}}\, \overline{u}(p_{\nu_\mu})\, \gamma^\alpha\, \gamma_+\, u(p_\mu)\, \overline{u}(p_e)\, \gamma^\alpha\, \gamma_+\, v(p_{\nu_e}) \equiv \mathcal{M}_F\,, \qquad (60)
\end{aligned}
$$

where we have introduced $Q = p_\mu - p_e$.

Note that at one loop we have

$$\frac{1}{\tau_\mu} = \frac{m_\mu^5}{192\,\pi^3}\,\frac{g^4}{32\,M^2}\,(1 + 2\,\Delta g^{(1)} + \Delta q^{(1)}), \tag{61}$$

and we have to separate the pure e.m. corrections evaluated in the Fermi theory to obtain $\Delta g^{(1)}$. To obtain the amplitude which generates the one-loop weak correction we consider first

$$\mathcal{M}_{W\,;\,1} = \mathcal{M}_{SM\,;\,1} - \mathcal{M}_{\mathrm{sub}\,;\,1}, \tag{62}$$

where $\mathcal{M}_{\mathrm{sub}\,;\,1}$ is obtained by
grouping the one-loop SM corrections with one photon line connected to external fermions and one W line,
by shrinking the W line to a point and by replacing the corresponding W propagator with $1/M^2$.

At the one-loop level and after the substitution $g^2/(8\,M^2) \to G_F/\sqrt{2}$ we obtain

$$\mathcal{M}_{\mathrm{sub}\,;\,1} \equiv \mathcal{M}_{F\,;\,1}, \tag{63}$$

where the latter generates $\Gamma_0\, \Delta q^{(1)}$. In the subtracted amplitude the soft terms have disappeared and we generate $\Delta g^{(1)}$ with the help of

$$\mathcal{M}^{\mathrm{leading}}_{W\,;\,1} = \lim_{p_i, m_i \to 0} \mathcal{M}_{\mathrm{sub}\,;\,1}, \tag{64}$$

i.e. we only retain the lading part, with vanishing lepton masses and external momenta, which amounts to neglect corrections of $\mathcal{O}\left(\alpha\, m^2/M^2\right)$. One-loop diagrams with no photons only have an hard component and do not need a subtraction.

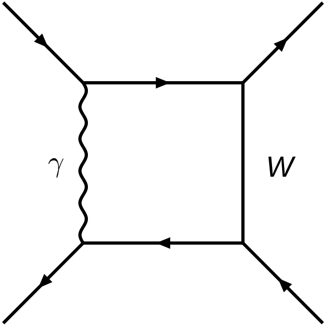


Figure: Infrared divergent one-loop box.

This amplitude contains two structures,

$$M_0 = \overline{u}\,\gamma^\alpha\,\gamma_+\,u\,\overline{u}\,\gamma^\alpha\,\gamma_+\,v, \qquad M_1 = \overline{u}\,\gamma^\alpha\,\gamma^\mu\,\gamma^\beta\gamma_+\,u\,\overline{u}\,\gamma^\beta\,\gamma^\mu\,\gamma^\alpha\,\gamma_+\,v. \tag{65}$$

However, M_1 is simply related to the current $\otimes$ current structure as it will be illustrated by considering the case of the one-loop box with W, γ exchange. We neglect for the moment all coupling constants and write

$$\begin{aligned}\mathcal{M}^{\rm sub}_{{\rm box}_{\gamma W}} &= -\int d^nq\,\frac{q_\lambda\,q_\sigma}{(q^2+M^2)\,(q^2)^2}\,J^{\alpha\lambda\beta}\,J^{\beta\sigma\alpha},\\ J^{\alpha\lambda\beta} &= \overline{u}(p_{\nu_\mu})\,\gamma^\alpha\,\gamma_+\,\gamma^\lambda\,\gamma^\beta\,u(p_\mu), \quad J^{\beta\sigma\alpha} = \overline{u}(p_e)\,\gamma^\beta\,\gamma^\sigma\,\gamma^\alpha\,\gamma_+\,v(p_{\nu_e}).\end{aligned} \tag{66}$$

After integration we obtain

$$\mathcal{M}^{\mathrm{sub}}_{\mathrm{box}_{\gamma W}} = -\,i\,\pi^2\,B_0(2,1\,;\,0,0,M)\,J^{\alpha\lambda\beta}\,J^{\beta\lambda\alpha}. \tag{67}$$

It can be shown that

$$J^{\alpha\lambda\beta}\,J^{\beta\lambda\alpha} = B^{(1)}\,M_0, \tag{68}$$

where $B^{(1)}$ is obtained with the help of a projection operator,

$$\sum_{\text{spin}} \mathcal{P}\,\left(J^{\alpha\lambda\beta}\,J^{\beta\lambda\alpha} - B^{(1)}\,M_0\right) = 0,$$
$$\mathcal{P} = \overline{v}(p_{\nu_e})\,\gamma^\rho\,\gamma_+\,u(p_{\nu_\mu})\,\overline{u}(p_\mu)\,\gamma^\rho\,\gamma_+\,u(p_e). \tag{69}$$

After a straightforward algebraic manipulation one obtains (in the limit $Q^2 \to 0$)

$$B^{(1)} = (n-2)^2, \tag{70}$$

which, after multiplication by $B_0(2,1\,;\,0,0,M)$ and in the limit $n \to 4$ reproduces the correct result, proportional to $B_0(2,1\,;\,0,0,M) - 1/2$.

Alternatively we start from the expression for the γ, W box without nullifying the soft scales,

$$\mathcal{M}_{\mathrm{box}_{\gamma W}} = \int d^q \frac{1}{d_0 d_1 d_2 d_3} \overline{u}(p_{\nu_\mu})\, \gamma^\alpha \gamma_+ \left[-i\, (\not{q} + \not{p}_\mu) + m_\mu \right] \gamma^\beta u(p_\mu) \times \overline{u}(p_e)\, \gamma^\beta \left[-i\, (\not{q} + \not{p}_e) + m_e \right] \gamma^\alpha \gamma_+ v(p_{\nu_e}), \tag{71}$$

where we introduce

$$d_0 = q^2, \quad d_1 = (q+p_\mu)^2+m_\mu^2, \quad d_2 = (q+P)^2+M^2, \quad d_3 = (q+p_e)^2+m_e^2, \tag{72}$$

$$\left(p_\mu - p_{\nu_\mu}\right)^2 = P^2, \qquad (p_\mu - p_e)^2 = Q^2. \tag{73}$$

A standard decomposition gives

$$\frac{1}{d_0 d_1 d_2 d_3} = \frac{1}{P^2+M^2}\left[\frac{1}{d_0 d_1 d_3} - \frac{1}{d_1 d_2 d_3} - 2\,\frac{q\cdot P}{d_0 d_1 d_2 d_3}\right]. \tag{74}$$

- The first term in the decomposition (in the limit $|P^2| \ll M^2$) is the QED vertex in the local Fermi theory that can be computed with standard techniques;
- The last two terms inside the square bracket of Eq.(74) are finite in the soft limit so that the extra contribution from the infrared SM box can be evaluated for $m_\mu, m_e = 0$ and $Q^2, P^2 = 0$.

In this limit only the term with three propagators survives and gives the well-known result.

With this technique (extracting instead of subtracting) we circumvent the puzzling procedure of Eq.(64) where the subtracted term is zero in dimensional regularization. However, the two procedures are totally equivalent.

If we neglect, for the moment, issues related to gauge parameter independence it is convenient to define a G constant that is totally process independent,

$$\Delta g = \delta_G + \Delta g^S, \qquad G = G_F \left(1 - \frac{g^2}{8\,M^2}\,\delta_G\right), \quad \delta_G = \sum_{n=1} \left(\frac{g^2}{16\,\pi^2}\right)^n \delta_G^{(n)}. \tag{75}$$

Alternatively, but always neglecting issues related to gauge parameter independence, we could resum δ_G by defyning $G_R = G_F/(1+\delta_G)$.

In one case we obtain

$$G = \frac{g^2}{8\,M^2}\left[1 - \frac{g^2}{16\,\pi^2\,M^2}\,\Sigma_{WW}(0)\right]^{-1},$$

$$\Sigma_{WW}(0) = \Sigma^{(1)}_{WW}(0) + \frac{g^2}{16\,\pi^2}\,\Sigma^{(2)}_{WW}(0), \tag{76}$$

where Σ_{WW} is the W self-energy, whereas with resummation we get

$$
G_R = \frac{g^2}{8\,M^2}\left[1 - \frac{g^2}{16\,\pi^2\,M^2}\,\overline{\Sigma}_{WW}(0)\right]^{-1},
$$

$$
\overline{\Sigma}_{WW}(0) = \Sigma^{(1)}_{WW}(0) + \frac{g^2}{16\,\pi^2}\left[\Sigma^{(2)}_{WW}(0) - \Sigma^{(2)}_{WW}(0)\,\delta^{(1)}_{G}\right]. \tag{77}
$$