Application of the Lambert-W Function in pQCD

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1. Introduction

QCD running coupling at the 2-loop order in MS-like (massless) renormalization schemes can be solved explicitly as a function of the scale in terms of the Lambert W function.

[1] Magradze B 1998 Proc. of the 10th Int. Seminar "QUARKS-98" (Suzdal) vol 1

[2] Gardi E, Grunberg G and Karliner M 1998 JHEP07(1998)007

[3] Appelquist T, Ratnaweera A, Terning J and Wijewardhana L C R 1998 Phys. Rev. D58 105017

The Lambert W function is the multivalued solution of

$$W_k(z)\,\mathrm{exp}\{W_k(z)\} = z, \tag{1}$$

the branches of W are denoted $W_k(z)$, $k = 0, \pm 1, \ldots$ For a real positive Q^2 ($Q^2 = -q^2 = -(q^0)^2 + \vec{q}^2$, $Q^2 > 0$ and $Q^2 > Q_L^2$. the 2-loop coupling takes the form

$$\alpha_{\mathsf{s}}^{(2)}(Q^2, n_{\mathsf{f}}) = \begin{cases} -(\beta_0/\beta_1)(1 + W_{-1}(z_Q))^{-1}, & \text{if } 0 \leq n_{\mathsf{f}} \leq 8 \\ -(\beta_0/\beta_1)(1 + W_0(z_Q))^{-1}, & \text{if } 9 \leq n_{\mathsf{f}} \leq 16 \end{cases} \quad (2)$$

where

$$z_Q = -(eb_1)^{-1}(Q^2/\Lambda^2)^{-1/b_1},$$

β_0 and β_1 are the first two β-function coefficients, $b_1 = \beta_1/\beta_0^2$ and $\Lambda \equiv \Lambda_{\overline{\mathsf{MS}}}$ is the conventional $\overline{\mathsf{MS}}$ scheme QCD parameter.

The solution to RGE, at the three-loop order, with Padé transformed beta-function

$$\beta_{Padé} = -\beta_0 \alpha_s^2 \left(1 + \frac{\beta_1 \alpha_s}{\beta_0 - \frac{\beta_0 \beta_2}{\beta_1} \alpha_s} \right),$$

has the form

$$\alpha^{(3)}_{Padé}(Q^2, f) = -\frac{\beta_0}{\beta_1}\frac{1}{1 - \beta_0\beta_2/{\beta_1}^2 + W_{-1}(\xi)} :$$

$$\xi = -\frac{1}{eb_1}\exp\left(\frac{\beta_0\beta_2}{{\beta_1}^2}\right)\left(\frac{Q^2}{\Lambda^2}\right)^{-\frac{1}{b_1}} .$$

The running coupling in higher order ($k \geq 3$) in arbitrary MS-like renormalization scheme was expanded as

$$\alpha^{(k)}_s(Q^2) = \sum_{n=1}^{\infty} c^{(k)}_n \alpha^{(2)n}_s(Q^2). \qquad (3)$$

[3] D.S. Kourashev, hep-ph/9912410

[4] D.S. Kourashev and B.A. Magradze, Theor. Math. Phys. **135**, 531-540 (2003)

[5] Howe D M and Maxwell C J 2002

■ Dispersive (Renormalization group invariant analytic) approach in QFT: P. Redmond, Phys.Rev. (1958)

N.N. Bogoliubov, A.A. Logunov and D.V. Shirkov JETP,(1959)

■ Dispersive approach in QCD:

D.V. Shirkov and I.L. Solovtsov, Phys. Rev. Lett. **79**, 1209 (1997)

Yu. Dokshitzer, G. Marchesini and B.R. Webber, Nucl. Phys. B**469**, 93 (1996);

G. Grunber, JHEP 9811, 006 (1998); JHEP 9903, 024 (1999);

■ The Analytic Perturbation Theory (APT) of Shirkov and Solovtsov:

[8] D.V. Shirkov and I.L. Solovtsov, Phys. Rev. Lett. **79**, 1209 (1997)

[9]K.A. Milton, I.L. Solovtsov and O.P Solovtsova, Phys. Lett. B**415**, 104 (1997)

in the time-like region APT is equivalent to the “contour improved” perturbation theory:

[10] A.A. Pivovarov, Sov. J. Nucl. Phys. 54 (1991) 676; Z. Phys. C53 (1992) 461.

■ More sophisticated non-perturbative modifications of the (minimal) analytic QCD model:

A.I. Alekseev and B. A. Arbuzov, Mod. Phys. Lett. A13, (1998)

Cvetič G, Valenzuela C and Schmidt I 2005 *Preprint* hep-ph/0508101 B.A. Magradze, Int. J. of Mod. Phys. A**15**,(2000);

■ Applications to the 3-point functions:

Bakulev, A. P., Mikhailov, S. V., Stefanis, N. G.: Phys. Rev. **D72**, 074014 (2005); Bakulev, A. P., Karanikas, A. I., Stefanis, N. G.: Phys. Rev. **D72** 074015 (2005); Bakulev, A. P., Stefanis, N. G.: Nucl. Phys. **B721** 50 (2005);

■ Main ingradients of the APT of Shirkov and Solovtsov:

Adler D-function related to some timelike process in PT:

$$D_{pt}(Q^2) = D_0(1 + \sum_{n=1}^{\infty} d_n \alpha_s^n(Q^2)),$$

the corresponding APT image is given by non-power expansion:

$$D_{an}(Q^2) = D_0(1 + \sum_{n=1}^{\infty} d_n \mathcal{A}_n(Q^2))$$

The physical quantity $R(s)$ determined through $D(Q^2)$ in the timelike domain in APT has the representation

$$R(s) = R_0(1 + r(s)) \quad r(s) = \sum_{n=1}^{\infty} d_n \mathfrak{A}_n(s, f).$$

The functions $\mathfrak{A}_n$ are defined through the transformation

$$\mathfrak{A}_n(s) = -\frac{1}{2\pi\imath} \int_{s-\imath\epsilon}^{s+\imath\epsilon} \frac{dz}{z} \mathcal{A}_n(-z),$$

$$A_n(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{\rho_n(\sigma, f)}{\sigma + Q^2}, \qquad \mathfrak{A}_n(s) = \frac{1}{\pi}\int_s^\infty \frac{d\sigma}{\sigma}\rho_n(\sigma)d\sigma$$

where the spectral function:

$$\rho_n(\sigma) = \Im\{\alpha_s(-\sigma - \imath 0)\}^n$$

■ The spectral functions are to be calculated in perturbation theory. The analytical structure of the coupling must be determined in the complex momentum plane.

The talk is organized as follows:

1. I shall discuss the singularity structure of the $\overline{MS}$ coupling in higher orders in the complex Q^2 plane,
2. I shall give the proof of the convergence of the series
$$\alpha_s^{(k)}(Q^2) = \sum_{n=1}^{\infty} c_n^{(k)} \alpha_s^{(2)n}(Q^2).$$
in the $\overline{MS}$ scheme to all orders in perturbation theory.
3. The radii of convergence of series at 3- and 4-loops as a function of n_f.
4. Applications of the series in the Shirkov-Solovtsov Analytic Perturbation Theory approach.

2. The Analytic Structure of the Coupling to Higher Orders

The RG equation, to the kth order:

$$\frac{\mathrm{d}\alpha_{\mathrm{s}}(Q^2)}{\mathrm{d}\ln Q^2} = \beta^{(k)}(\alpha_{\mathrm{s}}(Q^2)) = -\sum_{n=0}^{k-1} \beta_n \{\alpha_{\mathrm{s}}(Q^2)\}^{n+2}. \tag{4}$$

$$\alpha_s(\mu^2) = g^2/(4\pi),$$

The results for the coefficients β_k in the $\overline{MS}$ scheme are known up to 4-loops. With a new variable:

$$u = Q^2/\Lambda^2$$

and a modified running coupling,

$$a_{\mathrm{s}}(u) = \beta_0 \alpha_{\mathrm{s}}(Q^2)$$

the RGE equation:

$$u\frac{da_s(u)}{du} = \bar{\beta}^{(k)}(a_s(u)) = -\sum_{n=0}^{k-1} b_n a_{\mathsf{s}}^{n+2}(u) \tag{5}$$

where $\bar{\beta}^{(k)}(a_{\mathsf{s}}) = \beta_0 \beta^{(k)}(a_{\mathsf{s}}/\beta_0)$ and $b_n = \beta_n/\beta_0^{n+1}$. at the singularity

$$a_{\mathsf{s}}^{(k)}(u) \approx (u_L^{(k)}/(u - u_L^{(k)}))^{1/k} \quad \text{as} \quad u \to u_L^{(k)}.$$

Let us integrate the RGE

$$t = T^{(k)}(a) \tag{6}$$

where $t = \ln(u)$.

$$\begin{aligned} t &= 1/a - b_1 \ln(b_1 + 1/a) + \int_0^a g^{(k)}(a')\mathrm{d}a' : \qquad (7) \\ g^{(k)}(a) &= 1/\bar{\beta}^{(k)}(a) - 1/\bar{\beta}^{(2)}(a), \end{aligned}$$

where $\bar{\beta}^{(2)}(a)$ is the 2-loop $\bar{\beta}$-function.

The function $T^{(k)}(a)$ can be expressed in terms of the elementary functions. In the 3-loop case, we find

$$T^{(3)}(a) = 1/a + b_1 \ln(a) - 0.5 b_1 \ln(b_2 a^2 + b_1 a + 1) + \tilde{T}_1^{(3)}(a) - \tilde{T}_1^{(3)}(0), \tag{8}$$

where

$$\tilde{T}_1^{(3)}(a) = \begin{cases} \dfrac{2b_2 - b_1^2}{\sqrt{\Delta^{(3)}}} \arctan\left(\dfrac{b_1 + 2b_2 a}{\sqrt{\Delta^{(3)}}}\right) & \text{if } 0 \le n_f \le 5 \\ \dfrac{2b_2 - b_1^2}{2\sqrt{-\Delta^{(3)}}} \ln\left(\dfrac{a - a_1}{a_2 - a}\right) & \text{if } 6 \le n_f \le 16, \end{cases} \tag{9}$$

here $\Delta^{(3)} = 4b_2 - b_1^2$, and $a_{1,2} = (-b_1 \pm \sqrt{-\Delta^{(3)}})/(2b_2)$. In the $\overline{MS}$ scheme $\Delta^{(3)} > 0$ (< 0) if $0 \le n_f \le 5$ ($6 \le n_f \le 16$) (see tables 1 and 2).

In the 4-loop case, we find

$$T^{(4)}(a) = 1/a + b_1 \ln(a) - b_3^{-1} \sum_{i=1}^{3} E_i \ln(a - a_i) + \tilde{t}_0 \tag{10}$$

where a_i (i=1..3) denote roots of the algebraic equation, for $k = 4$,

$$\bar{\beta}^{(k)}(a)/a^2 = -\sum_{n=0}^{k-1} b_n a^n = 0, \tag{11}$$

and

$$E_i = \{a_i^2 (a_i - a_j)(a_i - a_k)\}^{-1}, \quad i \neq j \neq k$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. we may rewrite equation (10), for $0 \leq n_f \leq 7$, as

$$t = 1/a + b_1 \ln(a) + T_1^{(4)}(a) - T_1^{(4)}(0), \tag{12}$$

where $T_1^{(4)}(a)$ is a regular at zero function

$$T_1^{(4)}(a) = b_3^{-1}\left\{E_1 \ln(a-a_1) - \mathrm{Re}(E_2)\ln[(a-a_2)(a-a_3)] \right.$$
$$\left. -2\mathrm{Im}(E_2)\arctan\left[(a-\mathrm{Re}a_2)|\mathrm{Im}a_2|^{-1}\right]\right\}. \quad (13)$$

In the 4-loop case for $8 \le n_f \le 16$, may now be rewritten, for $8 \le n_f \le 16$,

$$t = 1/a + b_1 \ln(a) - b_3^{-1}\sum_{i=1}^{3} E_i \ln(a_i^{-1}(a_i - a)), \quad (14)$$

We have first to discuss the analytical properties of the inverse function $t = T^{(k)}(a)$ in the complex coupling plane.

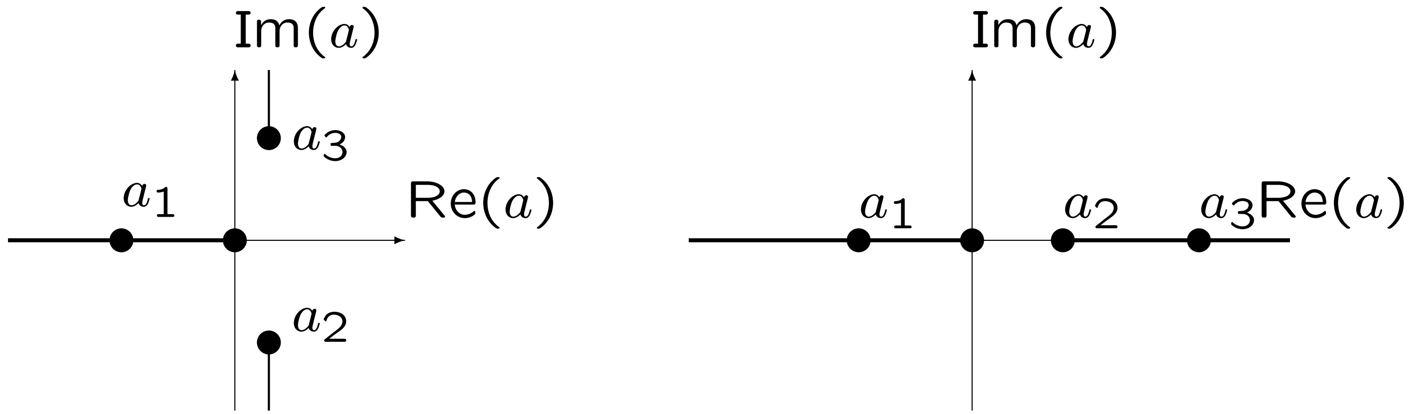
For $0 < n_f \le 5$ at 3-loops and $0 < n_f \le 7$ at 4-loops the cuts are chosen as

$$\{a : \mathrm{Im}(a_3) < \mathrm{Im}(a) < \infty, \mathrm{Re}(a) = \mathrm{Re}(a_3)\}$$

$$\{a : -\infty < \mathrm{Im}(a) < \mathrm{Im}(a_2), \mathrm{Re}(a) = \mathrm{Re}(a_2)\}$$

Consider now the cases with real roots ($6 < n_f \leq 16$ at 3-loops $7 < n_f \leq 16$ at 4-loops). Let a_1 be the negative root, and a_2 be the positive one (in the 4-loop case a_2 is the smallest positive root). The branch cuts then can be chosen along the real intervals

$$\{a : -\infty < a < 0\} \quad and \quad \{a : a_2 < a < \infty\}$$



(a) $0 < n_f \leq 7$ (b) $8 \leq n_f \leq 16$

Evidently, the singular points are determined by the limiting values of the function $T^{(k)}(a)$ as a tends to infinity. The 3-loops case for $6 \leq n_f \leq 16$

$$t_{\pm}^{(3)}(n_f) \ = -0.5 b_1 \ln|b_2| + \frac{b_2 - 0.5 b_1^2}{\sqrt{-\Delta^{(3)}}} \ln|a_2/a_1| \pm \imath \left(0.5 b_1 + \frac{b_2 - 0.5 b_1^2}{\sqrt{-\Delta^{(3)}}}\right) \pi$$

$6 \leq n_f \leq 16$

In the 4-loop case for $8 \leq n_f \leq 16$:

$$t_{\pm}^{(4)}(n_f) = b_3^{-1} \sum_{k=1}^{3} E_k \ln|a_k| \pm \imath (E_2 + E_3)\pi. \qquad (15)$$

It is important to determine whether or not the singular points $t_{\pm}^{(k)}(n_f)$ are located inside the strip $-\pi < \mathrm{Im(t)} \leq \pi$.

1) If the points lie inside the strip, then the unphysical Landau singularities appear in the first sheet. Then perturbation theory is incomplete: This case corresponds to real-world QCD, where $n_f \leq 6$.

2) In the second case, the singular points may arise beyond the strip. So that there are not real or complex singularities on the first sheet of the momentum squared variable, and thus perturbation theory is consistent with causality.

The value of n_f above which the causal analytical structure of the coupling is restored can be found from the equation

$$\mathrm{Im}\{\mathsf{t}^{(\mathsf{k})}(\mathsf{n}_\mathsf{f}^\star)\} = \pm\pi. \tag{16}$$

We find

$$n_f^{*(2)} \approx 9.68, \quad n_f^{*(3)} \approx 8.460 \quad n_f^{*(4)} \approx 8.455$$

Note that for $n_f > n_f^*$ the β-function has a positive infrared stable fixed point.

[8] T. Banks and A. Zaks, Nucl. Phys.B196, 189-204.

■ Conformal window in QCD:

[11] Miransky V A 1999 Phys. Rev.59 105003

[12] Appelquist T, Ratnaweera A, Terning J and Wijewardhana L C R 1998 Phys. Rev. D58 105017

Non-perturbative studies show that there is a phase transition in QCD with respect to n_f inside the range $0 \leq n_f \leq 16$:

For small values of n_f below the critical point ($n_f < N_f^{\text{cr}} < 16$) the theory is defined via the confining phase. Above this point, there is a conformal window $N_f^{\text{cr}} < n_f \leq 16$, where the theory is defined via the non-Abelian Coulomb phase with no color confinement and dynamical chiral symmetry breaking.

■ Oehme-Zimmermann criterion for the gluon confinement, the superconvergence rule for the transverse gluon propagator, determines N_f^{cr}

[11] R. Oehme and W. Zimmerman, Phys. Rev. D21 (1980) 471;

$$N_f^{\text{cr}} = 13N_{\text{c}}/4(= 9.75 \quad N_{\text{c}} = 3 \quad \text{colours})$$

Other possibility is to apply arguments of dynamical chiral symmetry breaking

[12] Miransky V A 1999 Phys. Rev.59 105003

[13] Appelquist T, Ratnaweera A, Terning J and Wijewardhana L C R 1998 Phys. Rev. D58 105017

This gives slightly higher value

$$N_f^{\text{cr}} \approx 4N_{\text{c}}.$$

It has been confirmed that the perturbative running coupling beyond the 1-loop approximation always is causal inside the conformal window, i.e.

$$n_f^* < N_f^{\mathrm{cr}}.$$

This condition is scheme independent.

[14] E. Gardi and M. Karliner, Nucl. Phys.B529 (1,2) (1998) 383-423.

Consider now the cases where the β-function has complex roots. In the $\overline{MS}$ scheme this takes place at 3-loops if

$$0 \leq n_f \leq 5 \quad \text{at} \quad 3-\text{loops}$$

$$0 \leq n_f \leq 7 \quad \text{at} \quad 4-\text{loops}.$$

Using the methods of complex analysis we find real singularities

$$t^{(3)}_{\text{rhp.}}(n_f) = -0.5 b_1 \ln(b_2) + \frac{2b_2 - b_1^2}{\sqrt{\Delta^{(3)}}} \left(\frac{\pi}{2} - \arctan\left(\frac{b_1}{\sqrt{\Delta^{(3)}}} \right) \right), \quad (17)$$

$$t^{(4)}_{\text{rhp.}}(n_f) = b_3^{-1} \left(-2\text{Im}(E_2) \left(0.5\pi + \arctan\left(\frac{\text{Re}a_2}{|\text{Im}a_2|} \right) \right) + E_1 \ln|a_1| + 2\text{Re}(E_2) \ln|a_2| \right), \quad (18)$$

as well as complex singularities

$$t^{(3)}_{\text{lhp.}\pm}(n_f) = t^{(3)}_{\text{rhp.}}(n_f) - \pi(2b_2 - b_1^2)/\sqrt{\Delta^{(3)}} \pm \imath\pi b_1, \quad (19)$$

$$t^{(4)}_{\text{lhp.}\pm}(n_f) = t^{(4)}_{\text{rhp.}}(n_f) + 2\pi\text{Im}(E_2)/b_3 \pm \imath\pi(b_1 - E_1/b_3). \quad (20)$$

singularities always present in the first sheet of the Q^2-plane.

n_f	$a_1 = \bar{a}_2$	$t^{(3)}_{rhp.}$	$t^{(3)}_{lhp.\pm}$
0	$-0.39 + 0.88\imath$	0.84	$-1.54 \pm 2.65\imath$
1	$-0.40 + 0.89\imath$	0.84	$-1.51 \pm 2.63\imath$
2	$-0.42 + 0.92\imath$	0.83	$-1.43 \pm 2.58\imath$
3	$-0.45 + 0.97\imath$	0.81	$-1.29 \pm 2.48\imath$
4	$-0.53 + 1.07\imath$	0.77	$-1.03 \pm 2.32\imath$
5	$-0.82 + 1.35\imath$	0.65	$-0.42 \pm 2.07\imath$

n_f	a_1	a_2	$t^{(3)}_{\pm}$
6	-1.49	7.09	$0.17 \mp 0.08\imath$
7	-0.88	1.24	$-0.16 \mp 1.05\imath$
8	-0.65	0.66	$-0.02 \mp 2.36\imath$
9	-0.51	0.41	$0.62 \mp 4.26\imath$
10	-0.40	0.26	$2.17 \mp 7.21\imath$
12	-0.25	0.1	$13.6 \mp 21.3\imath$
14	-0.12	0.03	$98 \mp 90\imath$
16	-0.02	0.001	$6216 \mp 2852\imath$

n_f	a_1	$a_2 = \bar{a}_3$	$t^{(4)}_{rhp.}$	$t^{(4)}_{lhp.\pm}$
0	-0.80	$0.13 - 0.78\imath$	1.16	$-1.29 \pm 0.96\imath$
1	-0.79	$0.13 - 0.78\imath$	1.16	$-1.28 \pm 0.93\imath$
2	-0.79	$0.14 - 0.79\imath$	1.16	$-1.26 \pm 0.87\imath$
3	-0.80	$0.16 - 0.81\imath$	1.15	$-1.20 \pm 0.77\imath$
4	-0.80	$0.19 - 0.84\imath$	1.11	$-1.10 \pm 0.59\imath$
5	-0.81	$0.26 - 0.91\imath$	1.04	$-0.94 \pm 0.29\imath$
6	-0.79	$0.44 - 1.01\imath$	0.82	$-0.71 \mp 0.22\imath$
7	-0.74	$1.12 - 0.97\imath$	-0.05	$-0.42. \mp 1.07\imath$

n_f	a_1	a_2	a_3	$t^{(4)}_{\pm}$
8	-0.62	0.70	6.48	$-0.09 \mp 2.37\imath$
9	-0.48	0.43	4.81	$0.52 \mp 4.29\imath$
10	-0.36	0.28	1.94	$1.95 \mp 7.32\imath$
12	-0.19	0.11	0.55	$13 \mp 22\imath$
14	-0.09	0.03	0.20	$98 \mp 92\imath$
16	-0.01	0.001	0.03	$6219 \mp 2858\imath$

3. The Proof of the Convergence of the series

Inserting the series

$$\alpha_s^{(k)}(Q^2) = \sum_{n=1}^{\infty} c_n^{(k)} \alpha_s^{(2)n}(Q^2). \tag{21}$$

into the RG equation (4), we recursively determine the coefficients $\{c_n^{(k)}\}_{n=3}^{\infty}$ in terms of c_2 ($c_1 = 1$) and the β-function coefficients. We have that: $c_2^{(k)} = 0$. Let us change the variable according to

$$Q^2 \to \theta = \beta_0 \alpha_s^{(2)}(Q^2)$$

and consider the new function

$$w(\theta) \equiv w^{(k)}(\theta) = \alpha_s^{(k)}(Q^2)/\alpha_s^{(2)}(Q^2) - 1.$$

The RG equation (4) may be rewritten as

$$\theta \frac{\mathrm{d}w}{\mathrm{d}\theta} = f^{(k)}(\theta, w), \tag{22}$$

where

$$f^{(k)}(\theta, w) = \frac{(w+1)^2}{1+b_1\theta} \sum_{n=0}^{k-1} b_n \theta^n (1+w)^n - (w+1), \tag{23}$$

and $b_n = \beta_n/\beta_0^{n+1}$. The function of two variables $f^{(k)}(w,\theta)$ has the Taylor expansion

$$f^{(k)}(w,\theta) = \sum_{m,n=0}^{\infty} \eta_{m,n}^{(k)} w^m \theta^n, \tag{24}$$

with $\eta_{0,0}^{(k)} = 0$, $\eta_{1,0}^{(k)} = 1$ and $\eta_{0,1}^{(k)} = 0$. In the 4-loop case, the expansion is

$$f^{(4)}(w,\theta) = w + b_1\theta w + b_2\theta^2 + w^2 + (b_3 - b_1 b_2)\theta^3 + \ldots \tag{25}$$

With the initial condition

$$w(0) = 0,$$

this equation has a singularity: for $\theta = 0$ and $w = 0$ the ratio

$f^{(k)}(\theta, w)/\theta$ is undefined. Nevertheless, in the special case where

$$\eta_{0,0}^{(k)} = 0, \quad \eta_{1,0}^{(k)} = 1 \quad \eta_{0,1}^{(k)} = 0$$

an analytic solution satisfying the initial condition $w(0) = 0$ may be found. The expansion (24) converges in the domain $D = \{0 < |w| < r_1, 0 < |\theta| < r_2\}$, where r_1 and r_2 are some positive numbers chosen in the range $\{r_1, r_2 : r_1 < \infty, \quad r_2 < 1/|b_1|\}$. It follows then from the classical theory that there exists a positive number $M^{(k)}$ such that $|f^{(k)}(w,\theta)| \leq M^{(k)}$ for $(w,\theta) \subset D$, and the coefficients $\eta_{m,n}^{(k)}$ satisfy the inequalities

$$|\eta_{m,n}^{(k)}| \leq \xi_{m,n}^{(k)}, \quad \text{where} \quad \xi_{m,n}^{(k)} = M^{(k)} r_1^{-m} r_2^{-n}. \tag{26}$$

Under these conditions, we may show that there exists a regular solution to equation (22)

$$w(\theta) \equiv w^{(k)}(\theta) = \sum_{n=2}^{\infty} \bar{c}_n^{(k)} \theta^n, \tag{27}$$

where $\bar{c}_n^{(k)} = \beta_0^{-n} c_{n+1}^{(k)}$, with $c_n^{(k)}$ being the coefficients in the original series (21).

Consider now the auxilary function $\tilde{w} = \tilde{w}(\theta)$ satisfying the equation

$$\tilde{w} = f_1^{(k)}(\tilde{w}, \theta) \equiv \frac{M^{(k)}}{(1 - \tilde{w}/r_1)(1 - \theta/r_2)} - M^{(k)} \left(1 + \frac{\tilde{w}}{r_1} + \frac{\theta}{r_2}\right), \tag{28}$$

it has the Taylor expansion

$$f_1^{(k)}(\tilde{w}, \theta) = \sum_{m=0,n=0} \xi_{m,n}^{(k)} \tilde{w}^m \theta^n, \tag{29}$$

with the coefficients $\xi_{m,n}^{(k)}$ defined in (26). Equation (28) has a series solution

$$\tilde{w}(\theta) \equiv \tilde{w}^{(k)}(\theta) = \sum_{n=2}^{\infty} \gamma_n^{(k)} \theta^n. \tag{30}$$

Considering the recurrence formulas for the coefficients for $\bar{c}_n^{(k)}$ and $\gamma_n^{(k)}$ one may derive

$$|\bar{c}_n^{(k)}| < \gamma_n^{(k)} \quad \mathrm{f}or \quad n = 2, 3 \ldots$$

So that the original series converges.

4 Determination of the Radius of Convergence of the Series

By a change of variable $Q^2 \to \theta = a_{\mathsf{s}}^{(2)}(u)$ $(u = (Q/\Lambda)^2)$ equation (5) can be rewritten

$$\frac{\mathrm{d}a}{\mathrm{d}\theta} = \frac{\sum_{n=0}^{k-1} b_n a^{n+2}}{\theta^2 + b_1\theta^3}, \tag{31}$$

where a$=A^{(k)}(\theta) = a_{\mathsf{s}}^{(k)}(u) = \beta_0 \alpha_{\mathsf{s}}^{(k)}(Q^2)$. In the preceding section, we have shown that the series (27) or equivalently the series

$$a = A(\theta) = \sum_{n=1}^{\infty} \tilde{c}_n \theta^n, \qquad (\tilde{c}_n = \beta_0^{-n+1} c_n) \tag{32}$$

has a positive convergence radius. It is possible then to define the inverse function $\theta = \Theta(a)$, which can be expanded in powers of a

$$\Theta(a) = \sum_{n=1}^{\infty} d_n a^n. \tag{33}$$

The series (33) is also convergent. Under this condition, we may apply the classical method for estimating the convergence radius of series

There are two possible cases to be considered. First, suppose that θ_0 be a finite singularity of $A(\theta)$,

$$\theta_0 = \Theta(a_0), \qquad \left.\frac{\mathrm{d}\Theta(a)}{\mathrm{d}a}\right|_{a=a_0} = 0. \tag{34}$$

Using (31) at $\theta = \theta_0$, we may rewrite (34) in the form

$$\left.\frac{\mathrm{d}\theta}{\mathrm{d}a}\right|_{a=a_0} = \frac{\theta_0^2(1 + b_1\theta_0)}{\sum_0^{k-1} b_n a_0^{n+2}} = 0, \tag{35}$$

for a finite a_0 (which is not a root of $\sum_0^{k-1} b_n a_0^n = 0$) this equation has only two solutions $\theta_0 = 0$, which should be rejected, and

$$\theta_0 = -1/b_1 \qquad (= -81/64 \quad \text{for} \qquad \mathrm{n_f} = 3). \tag{36}$$

Secondly, suppose that there exists a curve C going to infinity in the domain of analyticity of $\Theta(a)$ along which

$$\Theta(a) \to \theta_0 < \infty \quad \text{as} \quad a \to \infty \quad \text{while} \quad a \in C, \tag{37}$$

then θ_0 is a singular point.

Consider the 3- and 4-loop cases with

$$0 \leq n_f \leq 5 \qquad \text{and} \quad 0 \leq \mathrm{n_f} \leq 7$$

Let us integrate equation (31) in the real range

$$1/\theta - b_1 \ln(b_1 + 1/\theta) = 1/a - b_1 \ln(b_1 + 1/a) + \int_0^a g^{(k)}(a')\mathrm{d}a', \tag{38}$$

where

$$g^{(k)}(a) = (\bar{\beta}^{(k)}(a))^{-1} - (\bar{\beta}^{(2)}(a))^{-1} \tag{39}$$

Equation (38) may be continued for complex values of a and θ. Fortunately, we may solve the transcendental equation (38) for θ explicitly as a function of a in terms of the Lambert-W function

$$\theta = \Theta(a) = -b_1^{-1}(1 + W_n(z))^{-1}, \tag{40}$$

where $z = \zeta(a)$ with $\zeta(a) = -(eb_1)^{-1} \exp(-T(a)/b_1)$

$$T(a) = 1/a - b_1 \ln(b_1 + 1/a) + \int_0^a g(a')\, \mathrm{d}a'. \tag{41}$$

In the region $a > 0$ inside the convergence disc of the series (33)

$$\theta = \Theta^{(k)}(a) = -b_1^{-1}(1 + W_{-1}(\zeta^{(k)}(a)))^{-1}. \tag{42}$$

Formula (42) can be continued beyond the convergence circle on the positive a-axis. On using (42), we calculate the limit of $\Theta(a)$ as $a \to \infty$ along positive a-axis, determining thereby the singularity of the function $a = A(\theta)$

$$\theta_{\mathrm{s.1}}^{(k)} = -b_1^{-1}(1 + W_{-1}(\zeta^{(k)}(\infty))^{-1} = a_{\mathrm{s}}^{(2)}(u_{\mathrm{rhp.}}^{(k)}), \tag{43}$$

since

$$\zeta^{(k)}(\infty) = \lim_{a \to +\infty} \zeta^{(k)}(a) = -(b_1 e)^{-1}(u_{\mathrm{rhp.}}^{(k)})^{-1/b_1}, \tag{44}$$

and $u_{\mathrm{rhp.}}^{(k)} = \exp(t_{\mathrm{rhp.}}^{(k)})$ being the Landau singularity located on the positive u-axis (see the 3-and 4-loop formulas (17) and (18)).

We find exactly one real negative solution for θ inside the negative interval $a \in (\tilde{a}, 0)$, where $\tilde{a} \to -\infty$ in the 3-loop order and it is the finite negative root of (11) in the 4-loop order. This solution is determined in terms of the branch $W_0(z)$

$$\Theta^{(k)}(a) = -b_1^{-1}(1 + W_0(\tilde{\zeta}^{(k)}(a)))^{-1}, \tag{45}$$

where $\tilde{\zeta}^{(k)}(a) = (eb_1)^{-1} \exp(-\tilde{T}^{(k)}(a)/b_1)$ and

$$\tilde{T}^{(k)}(a) = 1/a - b_1 \ln(-1/a - b_1) + \int_0^a g(a')\, da'. \tag{46}$$

In general, the singularities of $T(a)$ are, at the same time, singularities of $\Theta(a)$. Nevertheless, $\Theta(a)$ is regular at $a = 0$, where $T(a)$ is singular. On the other side, $\Theta(a)$ may have additional singularities $a_{b\pm}$ arising due to the common branch point of $W_0(z)$ and $W_{\pm 1}(z)$ at $z = -1/e$. To determine locations of these singularities we numerically solve the equation

$$z = \zeta^{(k)}(a) = -1/e \tag{47}$$

at the 3- and 4 loop orders.

To define the analytical continuation we demand that the function $\theta = \Theta(a) \equiv \tilde{\Theta}(\zeta(a))$ will be continuous as a function of the phase of a. This will be achieved if we use the rules of counter-clockwise continuity to select the branches of W when the curve crosses the branch cut.

$$W_{-1}(x+\imath 0) = W_1(x-\imath 0) \quad \text{if} \quad -1/e < x < 0 \tag{48}$$

$$W_1(x-\imath 0) = W_0(x+\imath 0) \quad \text{if} \quad -\infty < x < -1/e$$

$$W_n(x+\imath 0) = W_{n+1}(x-\imath 0) \quad \text{if} \quad -\infty < x < 0 \quad \text{and} \quad n \geq 1.$$

Having the analytical structure of $\Theta(a)$ established, we can construct explicit expressions for $\Theta(a)$ in the entire cut complex a-plane.

We may now define the analytical continuation along negative a-axis determining the relevant branch on the negative a-axis. Making

$a \rightarrow -\infty$ in (45), we determine the singular point in the 3-loops

$$\Theta^{(3)}_{\mathrm{s.2}} = \lim_{a \rightarrow -\infty} \Theta^{(3)}(a) = -(b_1(1 + W_0(\tilde{\zeta}^{(3)}(-\infty)))^{-1} \quad , \tag{49}$$

$$\tilde{\zeta}^{(3)}(-\infty) = (\mathrm{e}b_1)^{-1} \exp(-\tilde{T}^{(3)}(-\infty)/b_1),$$

with

$$\tilde{T}^{(3)}(-\infty) = \lim_{a \rightarrow -\infty} \tilde{T}^{(3)}(a) = -b_1 \ln b_1 - p.v. \int_{-\infty}^{0} g^{(3)}(a) \mathrm{d}a = \mathrm{Re}(t^{(3)}_{\mathrm{lhp.}\pm}), \tag{50}$$

In the 4-loop case, the limiting values of this analytic function from above and below the left-hand cut, i.e. the limits of $\tilde{T}^{(4)}(a)$ as $\mathrm{Im(a)} \rightarrow 0^{\pm}$ for $\mathrm{Re(a)} < \mathrm{a}_1$, may be determined as

$$\tilde{T}^{(4)}_{\pm}(a) = 1/a - b_1 \ln(|b_1 - 1/|a||) + p.v. \int_0^a g^{(4)}(s) \mathrm{d}s \pm \mathrm{i}\kappa\pi, \tag{51}$$

where κ stands for the residue

$$\kappa = \lim_{a\to a_1}(a-a_1)g^{(4)}(a) = (b_2+b_3a_1)\{b_3(1+b_1a_1)(a_1-a_2)(a_1-a_3)\}^{-1},$$

and a_i, i=1..3, denote the roots of (11).

Then we find

$$\theta^{(4)}_{\mathrm{s.2}\pm} = -b_1^{-1}(1+W_0(\tilde{\zeta}^{(4)}_{\pm}(-\infty)))^{-1} \quad for \quad 0 \leq n_f \leq 5 \tag{52}$$

$$\theta^{(4)}_{\mathrm{s.2}\pm} = -b_1^{-1}(1+W_{\pm 1}(\tilde{\zeta}^{(4)}_{\pm}(-\infty)))^{-1} \quad \text{for} \quad 6 \leq n_f \leq 7$$

where $\tilde{\zeta}^{(4)}_{\pm}(-\infty) = (\mathrm{e}b_1)^{-1}\exp(-\tilde{T}^{(4)}_{\pm}(-\infty)/b_1)$, and

$$\tilde{T}^{(4)}_{\pm}(-\infty) = \lim_{a\to -\infty}\tilde{T}^{(4)}(a \pm \imath 0) = \mathrm{Re}(t_{\mathrm{lhp}\pm}) \pm \imath\pi\kappa, \tag{53}$$

here the subscript "$\pm$" shows that the limits were evaluated keeping the upper (lower) side of the cut. Evidently, $\theta^{(4)}_{\mathrm{s.2}-} = \overline{\theta}^{(4)}_{\mathrm{s.2}+}$.

n_f	0	1	2	3	4	5
$\theta_{s.1}^{(3)}$	0.627	0.635	0.653	0.691	0.776	1.029
$\theta_{s.2}^{(3)}$	-0.594	-0.601	-0.618	-0.653	-0.731	-0.956

n_f	$\theta_{s.1}$	n_1		$\theta_{s.2\pm}$	$\|\theta_{s.2\pm}\|$	n_2	$\tilde{\rho}$
0	0.485	-1		$-0.545 \mp 0.334\imath$	0.639	0	0.485
1	0.488	-1		$-0.544 \mp 0.341\imath$	0.642	0	0.488
2	0.497	-1		$-0.546 \mp 0.354\imath$	0.650	0	0.497
3	0.516	-1		$-0.550 \mp 0.380\imath$	0.668	0	0.516
4	0.554	-1		$-0.552 \mp 0.429\imath$	0.699	0	0.554
5	0.641	-1		$-0.533 \mp 0.526\imath$	0.748	0	0.641
6	0.934	-1		$-0.394 \pm 0.672\imath$	0.779	± 1	0.779
7	$-0.887 \mp 1.531\imath$	∓ 1		$-0.105 \pm 0.614\imath$	0.623	± 1	0.623

In the 3-loops:

$$\tilde{\rho}^{(3)} = |\theta_{\mathrm{s.2}}^{(3)}(n_f)| \quad \text{for} \quad 0 \le n_f \le 5.$$

In the 4-loops:

$$\tilde{\rho}^{(4)} = |\theta^{(4)}_{\mathrm{s.1}}(n_f)| \quad if \quad 0 \le n_f \le 5,$$

$$\tilde{\rho}^{(4)} = |\theta^{(4)}_{\mathrm{s.2}}(n_f)| \quad if \quad 6 \le n_f \le 7.$$

The radius of convergence of the original series (21) is

$$\rho^{(k)} = \tilde{\rho}^{(k)}/\beta_0$$

$$(\rho^{(3)} = 0.965 \quad and \quad \rho^{(4)} = 0.720 \quad for \quad n_f = 3).$$

Next consider the cases with large n_{f} values where the β-function has non-trivial real zeros. This takes place in the 3-loop case for $n_{\mathrm{f}} = \{6 - 16\}$. From now on we shall confine ourselves to the 3-loop case. Then we determine the singular points on the θ-plane:

n_f	$W_{\pm n}$	$\theta_0 = -1/b_1$	$\lvert\theta_{s.1\pm}\rvert$	$\tilde{\rho}$
6	$W_{\pm 1}$	-1.885	2.114	1.885
7	$W_{\pm 1}$	3.008	0.664	0.664
8	$W_{\pm 19}$	48.17	0.417	0.417
9	$W_{\mp 1}$	2.08	0.291	0.291
10	$W_{\mp 1}$	0.761	0.208	0.208
11	$W_{\mp 1}$	0.360	0.148	0.148
12	$W_{\mp 1}$	0.180	0.102	0.102
13	$W_{\mp 1}$	0.087	0.067	0.067
14	$W_{\mp 1}$	0.037	0.039	0.037
15	$W_{\mp 1}$	0.011	0.018	0.011
16	$W_{\mp 1}$	0.001	0.003	0.001

6 The Momentum Scale Associated With the Convergence radius of the Series

The convergence region of the series (32) in the momentum squared space may be easily determined, since the function $\theta = a^{(2)}(Q^2)$ for real positive $Q^2 > Q_L^2 \geq 0$ is monotonic (Q_L^2 being the real Landau singularity of the 2-loop coupling which appear if $0 \leq n_{\mathsf{f}} \leq 8.05$). First, we consider the series for large n_{f} values. Note that the quantity $\theta_0 = -b_1^{-1}$ in the Banks-Zaks domain ($n_{\mathsf{f}} > 8.05$) is the infrared fixed point of the 2-loop coupling $\theta = a^{(2)}(u)$. So that the restriction, $0 < \theta < |b_1|^{-1}$, holds for all $Q^2 \in (0, \infty)$. From the Table 10, we see that $\tilde{\rho} = \theta_0$ for $n_{\mathsf{f}} = \{14 - 16\}$. This means that the series (32) at 3-loops for $n_{\mathsf{f}} = \{14 - 16\}$ converges in the whole interval $Q^2 \in (0, \infty)$. Let n_{f}^{**} be the lowest value for which this condition holds ($n_{\mathsf{f}}^{**} = 14$ in the $\overline{\text{MS}}$ scheme). For $n_{\mathsf{f}} < n_{\mathsf{f}}^{**}$, the series (32) converges in the more restricted domain $Q^2_{\rm min} < Q^2 < \infty$ ($Q^2_{\rm min} > 0$). The value of $Q^2_{\rm min}$ may be determined from the equation

$$\theta = a^{(2)}(u) = -b_1^{-1}(1 + W_n(z_Q))^{-1} = \tilde{\rho} \tag{54}$$

where $z_Q = -(eb_1)^{-1}u^{-1/b_1}$ and $u = Q^2/\Lambda^2$ (see Eq. (2)). Solving (54), we obtain

$$u_{\rm min} = Q^2_{\rm min}/\Lambda^2 = (b_1 + \tilde{\rho}^{-1})^{-b_1} \exp(\tilde{\rho}^{-1}).$$

The results for the dimensionless quantity $\sqrt{u_{\rm min}} = Q_{\rm min}/\Lambda$ ($Q_{\rm min} = \sqrt{Q^2_{\rm min}}$) to the 3- and 4-loop orders for $n_{\rm f} = \{0 - 6\}$ are tabulated in Table 11. We compare $\sqrt{u_{\rm min}}$ with the ratio $\sqrt{u_{\rm rhp}} = Q_{\rm rhp}/\Lambda$ (the value $Q_{\rm rhp} = \sqrt{Q^2_{\rm rhp}}$ corresponds to the real space-like Landau singularity of the coupling).

The ratios $\sqrt{u^{(k)}_{\rm min}} = Q^{(k)}_{\rm min}/\Lambda$ and $\sqrt{u^{(k)}_{\rm rhp}} = Q^{(k)}_{\rm rhp}/\Lambda$ in the $\overline{\rm MS}$ scheme to the 3-and 4-loop orders for $n_{\rm f} = \{0 - 6\}$.

n_f	$\sqrt{u_{min}^{(3)}}$	$\sqrt{u_{rhp}^{(3)}}$	$\sqrt{u_{min}^{(4)}}$	$\sqrt{u_{rhp}^{(4)}}$
0	1.571	1.525	1.790	1.790
1	1.566	1.521	1.788	1.788
2	1.558	1.514	1.784	1.784
3	1.541	1.500	1.773	1.773
4	1.505	1.467	1.745	1.745
5	1.416	1.384	1.678	1.678
6	1.283	–	1.623	1.507

It is seen from the Table that in general the quantity Q^2_{min} can not be identified with the real Landau singularity Q^2_{rhp}. The equality $Q^2_{\text{min}} = Q^2_{\text{rhp}}$ holds only in the cases where the convergence radius $\tilde{\rho}$ is determined via the real (space-like) Landau singularity. This happens, for example, in the $\overline{\text{MS}}$ scheme in the 4-loop case for $n_f = \{0-5\}$. However, in the cases where $\tilde{\rho}$ is determined via the complex Landau singularities, $Q^2_{\text{lhp}\pm}$, the relation between Q^2_{min} and $Q^2_{\text{lhp}\pm}$ is not so simple, and then the inequality holds $Q^2_{\text{min}} > Q^2_{\text{rhp}}$. Such a situation occurs, for instance, in the $\overline{\mathrm{MS}}$ scheme in the 3-loop case for $n_\mathrm{f} = \{0 - 5\}$.

It is reasonable to compare Q_min with the infrared boundary of QCD, the momentum scale μ_C that separates the perturbative and non-perturbative regimes of the theory in the confining phase. Several estimates for this quantity was suggested using different nonperturbative methods. In recent work

Alekseev, A. I., Arbuzov, B. A.: Mod. Phys. Lett. **A20**, 103 (2005)

an useful nonperturbative approximation for the QCD β-function was constructed. This model, with the perturbative $\overline{\mathrm{MS}}$ scheme component of the β-function to the 3- and 4-loop orders at $n_\mathrm{f} = 3$, predicts that

$$(\mu_{\rm c}/\Lambda_{\rm QCD})_{3-loop} \approx 3.204,$$

$$(\mu_{\rm c}/\Lambda_{\rm QCD})_{4-loop} \approx 3.526.$$

Another way to estimate the infrared boundary is to use arguments based on dynamical chiral symmetry breaking in QCD. There are the results obtained within the nonperturbative framework of Schwinger-Dyson equations

Fomin, P. I., et al.: Riv. Nuovo Cimento **6**, 1 (1983)

According to this the critical value of the coupling needed to generate the chiral condensate is $\alpha_c = \pi/4$ (for $N_{\rm c} = 3$ QCD). It is reasonable to identify the corresponding scale with the infrared boundary.

One way to obtain approximations to μ_c is to use the perturbative expressions for the coupling in the $\overline{\mathrm{MS}}$ scheme. Then the equation $\alpha_s^{(k)}(\mu_c^2) = \pi/4$ to the 3- and 4-loop orders yields the estimates, at $n_{\mathrm{f}} = 3$,

$$(\mu_{\mathrm{c}}/\Lambda_{\overline{\mathrm{MS}}})_{3-\mathrm{loop}} = 1.972 \quad \text{and} \quad (\mu_{\mathrm{c}}/\Lambda_{\overline{\mathrm{MS}}})_{4-\mathrm{loop}} = 2.115.$$

At least the above considered estimates are consistent with the inequality $Q^2_{\mathrm{min}} < \mu_{\mathrm{c}}^2$.

This enable us to suppose that the series expansion (32) in the $\overline{\mathrm{MS}}$ scheme may be safely used in the whole perturbative region $\mu_{\mathrm{c}}^2 < Q^2 < \infty$.

6 Application to Analytic Perturbation Theory

In the Analytic Perturbation Theory (APT) approach of Shirkov and Solovtsov, Euclidean and Minkowskian QCD observables (which depend on the single scale) are represented by asymptotic expansions over non-power sets of specific functions $\{\mathcal{A}_n^{(k)}(u)\}_{n=1}^{\infty}$ and $\{\mathfrak{A}_n^{(k)}(\bar{s})\}_{n=1}^{\infty}$ respectively, here $u = Q^2/\Lambda^2$ and $\bar{s} = s/\Lambda^2$. These sets are constructed via the integral representations in the following way

$$\mathcal{A}_n^{(k)}(u) = \frac{1}{\pi}\int_0^{\infty} \frac{\varrho_n^{(k)}(\varsigma)d\varsigma}{\varsigma + u}, \quad \mathfrak{A}_n^{(k)}(\bar{s}) = \frac{1}{\pi}\int_{\bar{s}}^{\infty} \frac{\varrho_n^{(k)}(\varsigma)}{\varsigma}d\varsigma, \tag{55}$$

where the spectral densities to the kth order are determined from powers of the running coupling: $\varrho_n^{(k)}(\varsigma) = -\Im(a^{(k)n}(-\varsigma + \imath 0))$. In APT the power series (32) give rise to the following series of functions

$$\mathcal{A}_m^{(k)}(u) = \sum_{n=m}^{\infty} \mathcal{C}_{m,n}^{(k)} \mathcal{A}_n^{(2)}(u) \quad m = 1, 2 \ldots \tag{56}$$

$$\mathfrak{A}_m^{(k)}(\bar{s}) = \sum_{n=m}^{\infty} \mathcal{C}_{m,n}^{(k)} \mathfrak{A}_n^{(2)}(\bar{s}) \quad m = 1, 2 \ldots \tag{57}$$

$$\varrho_m^{(k)}(\varsigma) = \sum_{n=m}^{\infty} \mathcal{C}_{m,n}^{(k)} \varrho_n^{(2)}(\varsigma) \quad m = 1, 2 \ldots, \tag{58}$$

where $\mathcal{C}_{m,m}^{(k)} = 1$. The sets of coefficients $\{\mathcal{C}_{m,n}^{(k)}\}_{n=m}^{\infty}$, $m = 1, 2 \ldots$, are constructed from the set of coefficients of the original series, $\{\tilde{c}_n^{(k)}\}_{n=1}^{\infty}$, according to the rules for products of power series: $\mathcal{C}_{1,n}^{(k)} = \tilde{c}_n^{(k)}$, $\mathcal{C}_{2,n}^{(k)} = \sum_{j=1}^{n-1} \tilde{c}_{n-j}^{(k)} \tilde{c}_j^{(k)}$ etc. The spectral densities at the 2-loop order can be expressed analytically in closed form

$$\varrho_n^{(2)}(\varsigma) = b_1^{-n} \Im (1 + W_{-1}(z_\varsigma))^{-n} \quad \text{with}$$

$$z_\varsigma = (e b_1)^{-1} \varsigma^{-1/b_1} \exp\left(-\imath\pi(1/b_1 - 1)\right).$$

Now we are going to prove that the series of functions (56), (57) and (58) are uniformly convergent over whole ranges of the corresponding variables: $0 < u < \infty$, $0 < \bar{s} < \infty$ and $0 < \varsigma < \infty$. Evidently, it is sufficient to prove that the series (58) is uniformly convergent. Let us now write $W_{-1}(z_\varsigma) = \mathcal{W} = \mathcal{X} + \imath\mathcal{Y}$, $(1+\mathcal{W})^{-1} = \mathcal{R}\exp(\imath\Psi)$, where $\mathcal{R} = ((\mathcal{X}+1)^2 + \mathcal{Y}^2)^{-1/2}$ and $\Psi = \arcsin(-\mathcal{Y}\mathcal{R})$ (for the branch W_{-1}, we have $-3\pi < \mathcal{Y} < 0$). According to this, we may rewrite the 2-loop spectral densities (59) as

$$\varrho_n^{(2)}(\varsigma) = (\mathcal{R}/b_1)^n \sin(n\Psi), \quad n = 1, 2 \ldots . \tag{59}$$

It is seen from Eq. (59) that the modulus of the spectral densities are bounded above

$$|\varrho_n^{(2)}(\varsigma)| < (\theta_{\text{max}})^n, \tag{60}$$

where $\theta_{\text{max}} = \mathcal{R}_{\text{max}}/b_1$ and $\mathcal{R}_{\text{max}}$ is the maximal value of $\mathcal{R}$ in the range $0 < \varsigma < \infty$. We find useful to use the "Maple 7" for determining $\mathcal{R}_{\mathsf{max}}$ numerically. In Table 12, we listed numerical values of θ_{max} in the phenomenologically interesting cases $n_f = \{0 - 6\}$.

n_{f}	0	1	2	3	4	5	6
θ_{max}	0.237	0.237	0.238	0.240	0.243	0.249	0.259
$\tilde{\rho}^{(3)}$	0.594	0.601	0.618	0.653	0.731	0.956	1.885
$\tilde{\rho}^{(4)}$	0.485	0.488	0.497	0.516	0.554	0.641	0.779

Note that all the power series $\sum_{n=m}^{\infty} \mathcal{C}_{m,n}^{(k)} \theta^n$, $m = 1, 2 \ldots$, have the same radius of convergence, $\tilde{\rho}^{(k)}$, as the original series (32). This follows from the definition

$$\sum_{n=m}^{\infty} \mathcal{C}_{m,n}^{(k)} \theta^n = \left(\sum_{l=1}^{\infty} \tilde{c}_l^{(k)} \theta^l \right)^m . \tag{61}$$

Consider now the set of numerical series of positive terms

$$\sum_{n=m}^{\infty} |\mathcal{C}_{m,n}^{(k)}| \theta_{max}^n \quad m = 1, 2 \ldots , \tag{62}$$

looking at the numbers in Table 12, we see that θ_{max} is inside the convergence disk of the series (61): $0 < \theta_{\text{max}} < \tilde{\rho}^{(k)}$, both in the 3- and 4-loop cases. Hence all the numerical series (62) are convergent. Combining this fact with the bounding conditions (60), we find that the series of functions $\sum_{n=m}^{\infty} |\mathcal{C}_{m,n}^{(k)} \varrho_n^{(2)}(\varsigma)|$, $m = 1, 2 \ldots$, are uniformly convergent by the comparison test due to Weierstrass. Then all the series (58) are uniformly convergent. Hence by the arguments given above, the series of functions (56) and (57) are also uniformly convergent.

The expansions (56) and (57) enable us to calculate the infrared limits of the APT expansion functions. Thus we may reproduce remarkable results of Shirkov and Solovtsov in a mathematically rigorous way.

Since

$$\lim_{u \to 0^+} \mathcal{A}_n^{(2)}(u) = \lim_{\bar{s} \to 0^+} \mathfrak{A}_n^{(2)}(\bar{s}) = \delta_{n,1}.$$

These relations may be extended to higher orders by means of the expansions (56) and (57). Thus we can write

$$\lim_{u \to 0^+} \mathcal{A}_m^{(k)}(u) = \sum_{n=m}^{\infty} \mathcal{C}_{m,n}^{(k)} \lim_{u \to 0^+} \mathcal{A}_n^{(2)}(u) = \mathcal{C}_{m,1}^{(k)} \equiv \delta_{m,1}, \tag{63}$$

The universality of $\mathcal{A}_1^{(k)}(0)$ and $\mathfrak{A}_1^{(k)}(0)$ (the scheme independence and invariance with respect to higher-loop corrections) is evident.

Conclusion

- We systematically investigate the analyticity structure of the modified coupling $a_s(Q^2/\Lambda^2)$ at 3- and 4-loops in the complex $u = Q^2/\Lambda^2$ plane for all n_f values in the range of AF $0 \leq n_f \leq 16$. In the confining phase of the theory, for relatively small n_f values, we have found a pair of complex conjugate singularities (besides of the real Landau singularity) in the first Riemann sheet. Just these complex singularities determine the radius of convergence of the series solution for most values of n_f.

- we have proved that in the $\overline{MS}$-like schemes the power series solution has a finite radius of convergence to all orders in perturbation theory for $n_f = 1 - 16$.

- We have determined the analytical structure of the higher order coupling in the complex plane of the 2-loop running coupling θ ($\theta = a_s^{(2)}(Q^2/\Lambda^2)$). We have considered the 3- and 4-loop cases for $0 \leq n_f \leq 16$ and $0 \leq n_f \leq 7$ respectively. The correspondence between the singularities of the coupling in the Q^2 and θ planes have been established. Comparing the singularities of the coupling in the θ-plane, we have determined the radii of convergence of the series solution. The radii have been found to be sufficiently large from practical point of view,

- We have studied the convergence properties of the non-power series constructed from the series (32) according to the rules of the QCD Analytic Perturbation Theory of Shirkov and Solovstov in both the space- and time-like regions. We have shown that the Euclidean and Minkowskian variants of these non-power series are uniformly convergent over whole domains of the corresponding momentum squared variables. A mathematically rigorous proof of the finiteness and universality of the analytic coupling at zero momentum has also been presented.