Fermionic NNLO Corrections to $b \to s\,\gamma$

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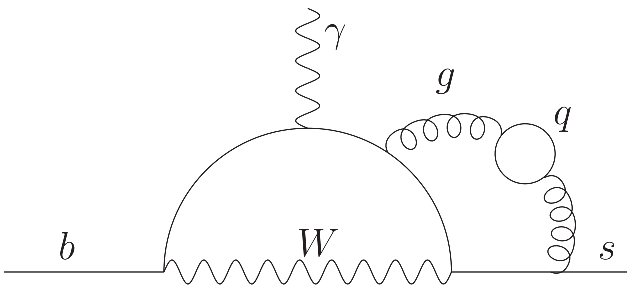
Introduction

Inclusive decay ($B \to X_s\,\gamma$):

- good agreement of experimental data with SM predictions
- precision tests of SM
- Sensitivity to effects of “new physics”
- nonperturbative effects are small (5%)

- well approximated by partonic decay $b \to s\gamma$

Aim: Fermionic NNLO corrections to $b \to s\gamma$



Effective Theory

$$H_{eff} = -\frac{4G_F}{\sqrt{2}}\lambda_t \sum_{i=1}^{8} C_i(\mu)\, O_i(\mu)$$

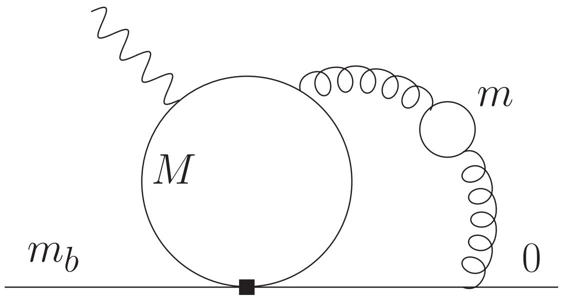
- Resummation of large logarithms
- Heavy particles (W,t) are integrated out.
- Matching with full theory at $\mu = m_W$

1. Computation of the matching coefficients $C_i(\mu)$ to $\mathcal{O}(\alpha_s^2)$
2. Evolution of the $C_i(\mu)$ from $\mu = m_W$ to $\mu = m_b$
3. $\mathcal{O}(\alpha_s^2)$ calculation of $< s\gamma|O_i(\mu)|b >$

- Numerically important: $O_2 = (\bar{s}_L \gamma_\mu c_L)(\bar{c}_L \gamma^\mu b_L)$

Known Results

- Matching coefficients at NNLO [Misiak et al. 2004] [Bobeth et al. 2000]
- Mixing at NNLO [Gorbahn et al. 2005]
- Matrix elements:



- 2 loops, $M = m_c$ [Greub et al. 1996]
- 3 loops, $M = m_c$, $m = 0$ [Bieri et al. 2003]
- new: 3 loops, $M = 0$, $m = m_b$

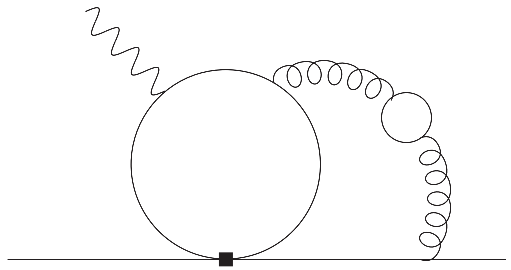
Outline of the Calculation

- Generation of the relevant 3 loop diagrams
- Reduction to master integrals
  - ▷ Laporta algorithm
- Calculation of the master integrals
  - ▷ Dimensional regularization
  - ▷ Mellin-Barnes method
- Renormalization

Dimensional Regularization:

- integrals in $D = 4 - 2\varepsilon$ dimensions.
- Expansion in ε
- UV divergences $\rightarrow$ poles in ε

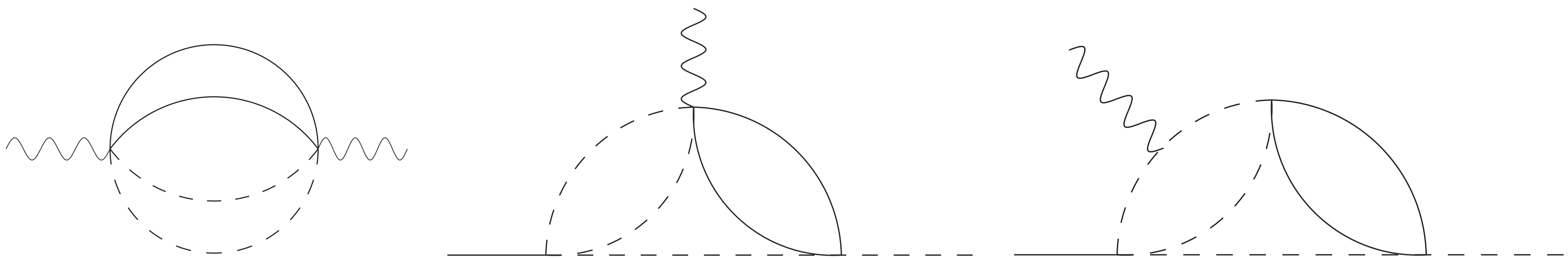
Reduction to Master integrals



Map complicated integrals to a set of simpler ones (master integrals) using *integration by parts* relations

$\rightarrow$ *Laporta algorithm* [Laporta 2000]

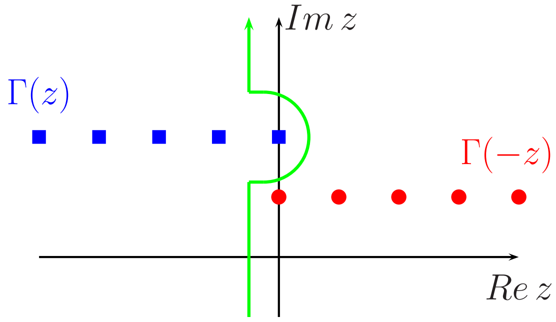
We use a new program by P. Marquard and D. Seidel.



Mellin-Barnes-Method

Mellin-Barnes integral: [Smirnov 2004]

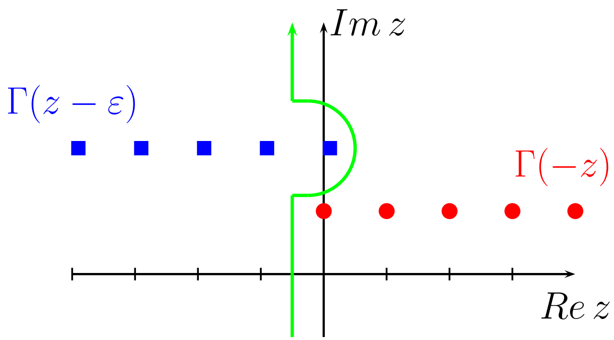
$$\frac{1}{(X+Y)^\lambda} = \frac{1}{\Gamma(\lambda)}\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\,\Gamma(\lambda+z)\,\Gamma(-z)\,\frac{Y^z}{X^{\lambda+z}}$$



- transforms sums to products
- e.g. massive propagator to massless propagator
- introduces complex integrations

Analytical Continuation of MB integrals

MB integrals have singularities for $\varepsilon \to 0$ ($\leftrightarrow$ UV poles)
$\to$ analytical continuation in ε necessary.
E.g. by shifting contours:



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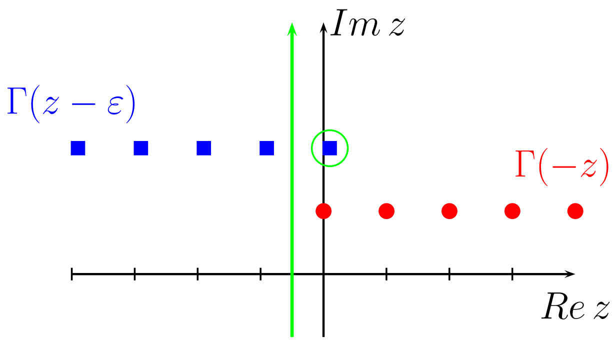
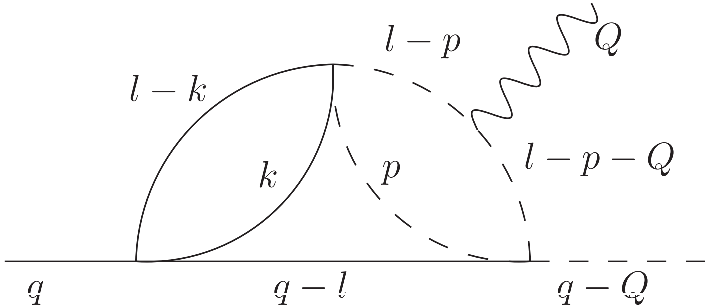


+ Residue($z = \varepsilon$)

$$\int_{C_1}\int_{C_2} f(z_1, z_2) = \int_{C_1}\int_{C_2^*} f(z_1, z_2) + \int_{C_1} \tilde{f}(z_1)$$

Automatized using *Tausk's* prescription [Tausk 1999]
[Czakon 2005], [Anastasiou 2005]

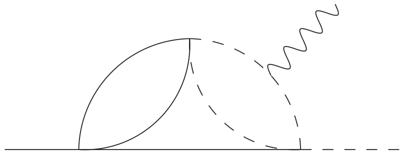
Calculation of Master Integrals



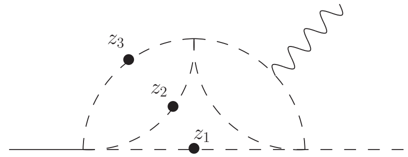
$$q^2 = m^2, \quad Q^2 = 0, \quad (q-Q)^2 = 0$$

$$= \int \frac{1}{(k^2+m^2)\,[(l-k)^2+m^2]\,(l-p)^2\,p^2\,[(q-l)^2+m^2]\,(l-p-Q)^2}$$

- massive propagators $\rightarrow$ massless, 3 MB-integrations



$$= \int dz_1 \int dz_2 \int dz_3\, (m^2)^{z_1+z_2+z_3}$$



- Feynman parameterization for “massless” diagram

$$\frac{1}{A^\alpha\, B^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} \int_0^1 dx_1 \int_0^1 dx_2\, x_1^{\alpha-1}\, x_2^{\beta-1}\, \frac{\delta(x_1+x_2-1)}{(x_1 A + x_2 B)^{\alpha+\beta}}$$

Master Integrals (2)

- Evaluate momentum integrals

$$\int \frac{d^d l}{i\,\pi^{d/2}} \frac{1}{(l^2 - \Delta + i0)^n} = e^{-ni\pi}\,(\Delta - i0)^{\frac{d}{2}-n}\,\frac{\Gamma(n-\frac{d}{2})}{\Gamma(n)}$$

- MB representation of sums in denominator
- Evaluate Feynman parameter integrals

$$\int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

$$\int_{-i\infty}^{+i\infty} dz_j\,(-1)^{-2(3\varepsilon+z_2+z_3+z_4)}\Gamma(-\varepsilon)\Gamma(-\varepsilon-z_1)\Gamma(-z_1)\Gamma(z_1+1)\times$$
$$\Gamma(\varepsilon+z_1+1)\Gamma(-2\varepsilon-z_1-z_2)\Gamma(-z_2)\Gamma(-\varepsilon-z_3+1)\Gamma(-z_3)\Gamma(-z_4)\ldots$$

Master Integrals (3)

- ▷ Analytical continuation and expansion in ε (MB)
  Numerical evaluation of MB-integrals
- ▷ Evaluation by Barnes' Lemma

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} dz\Gamma(a+z)\,\Gamma(b+z)\,\Gamma(c-z)\,\Gamma(d-z) = \frac{\Gamma(a+c)\,\Gamma(a+d)\,\Gamma(b+c)\,\Gamma(b+d)}{\Gamma(a+b+c+d)}$$

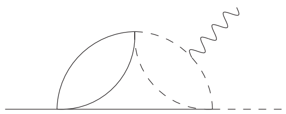
- ▷ Evaluation via residue theorem (infinite sum over residues)
- ▷ Last step: 1-dimensional MB-integral

$$\int_{-i\infty}^{+i\infty} dz(-1)^{-2z}2^{-8\varepsilon-2z+1}\Gamma\left(\tfrac{1}{2}-\varepsilon\right)\Gamma(1-\varepsilon)^2\Gamma(-\varepsilon)\Gamma(\varepsilon)\Gamma(-3\varepsilon-z+1)^2\times$$

$$\frac{\Gamma\left(-3\varepsilon-z+\frac{3}{2}\right)\Gamma(-\varepsilon-z+1)^2\Gamma(-z)\Gamma(\varepsilon+z)\Gamma(3\varepsilon+z)}{\pi\Gamma(1-2\varepsilon)^2\Gamma(-2\varepsilon-2z+2)\Gamma(-4\varepsilon-z+2)\Gamma(-3\varepsilon-z+2)}$$

Master Integrals (4)

- Analytical Continuation
- Residues $\rightarrow$ Sums of n^k, $\Gamma(n)$, $\psi^m(n)$
- Symbolic Summation
  - ▷ `SUMMER, XSUMMER, nestedsums, HypExp,...`
- Numerical Summation
  - ▷ improvement by convergence acceleration
  - ▷ PSLQ
- Numerical result for the example:



$$= 0.33333\,\varepsilon^{-3} + 1.35506\,\varepsilon^{-2} + 2.65791\,\varepsilon^{-1} + 13.9472 + 3.97114\,\varepsilon$$

Renormalization

Counterterm contributions:

1. Renormalization of α_s
2. Mixing of O_2 and O_4 at order α_s
3. Mixing of O_2 and O_4 at order $\alpha_s^2 n_f$
4. Mixing of O_2 and O_7 at order $\alpha_s^2 n_f$

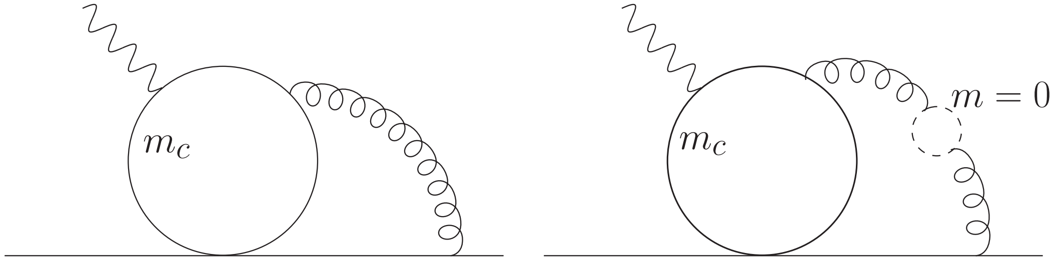
1, 3 and 4 could be taken from known results.

2 had to be newly calculated.

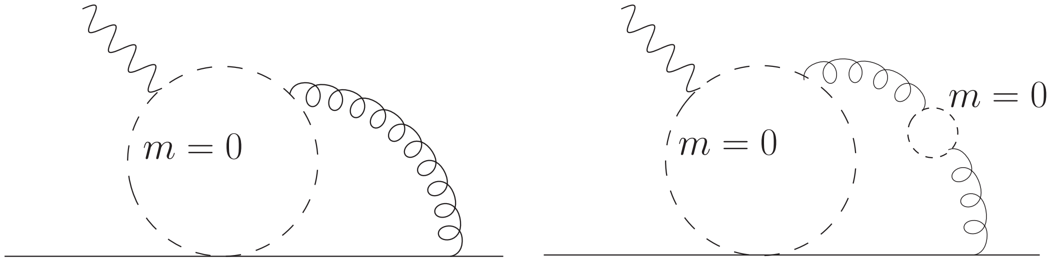
Check with known results

Recalculation of the two loop results [Greub et al. 1996] and the 3 loop $m = 0$ results [Bieri et al. 2003] for the case $M = 0$.

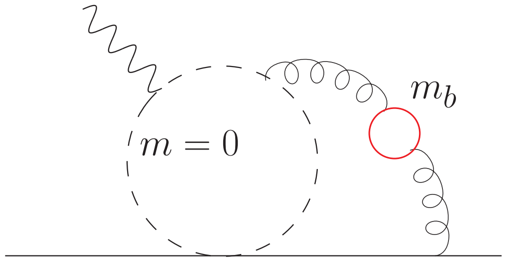
known:



checked:



new calculation:



Check (2)

$$< s\gamma|O_2|b > = < s\gamma|O_7|b > \times$$

$$\frac{\alpha_s}{4\pi}\left\{Q_u\left(\frac{100}{81}+\frac{16}{27}L+\frac{8i\pi}{27}\right)+Q_d\left(-\frac{29}{3}+8L-\frac{4i\pi}{3}\right)\right\}$$

$$+\left(\frac{\alpha_s}{4\pi}\right)^2 n_f\left\{Q_u\left(\frac{1021}{54}+\frac{\pi^2}{9}-20L+8L^2+\frac{47i\pi}{9}-\frac{8i\pi}{3}L\right)\right.$$

$$\left.+Q_d\left(-\frac{6160}{729}+\frac{80\pi^2}{81}+\frac{80}{27}L+\frac{32}{27}L^2-\frac{16i\pi}{9}+\frac{32i\pi}{27}L\right)\right\}$$

$$L = \ln\left(\frac{m_b}{\mu}\right)$$

Status

- Known results could be reproduced
- All Master integrals could be evaluated numerically
- Some integrals could be evaluated analytically
- satisfactory numerical precision for the matrix element
  - ▷ can be improved

- New results nearly completed

$< s\gamma|O_2|b >$ = very preliminary

Conclusion

- $b \to s\gamma$
- Mellin-Barnes Method
- New results are on the way

Convergence Acceleration

$$s = \sum_{i=1}^{\infty} a_i \qquad\qquad s \cong s_n = \sum_{i=1}^{n} a_i$$

$\hookrightarrow$ (nonlinear) *sequence transformations*

$$\{s_n\} \to \{s'_n\}, \qquad\qquad \lim_{n\to\infty} s'_n = \lim_{n\to\infty} s_n$$

$\longrightarrow$ iterative application to improve the accuracy

Example: *Wynn's Rho algorithm*

$$\rho_{-1}^{(n)} = 0 \qquad\qquad \rho_0^{(n)} = s_n$$

$$\rho_{k+1}^{(n)} = \rho_{k-1}^{(n+1)} + \frac{k+1}{\rho_k^{(n+1)} - \rho_k^{(n)}}, \qquad k, n \in \mathbb{N}_0,$$

Example: $\zeta(2)$

$$\zeta(2) = \sum_{i=1}^{\infty} \frac{1}{i^2}$$

Results for $\zeta(2)$ $(n = 2000)$			
method	value	error	digits
sum	1.644 434 191 827 393	$5 \cdot 10^{-4}$	4
Theta	1.644 934 066 848 056	$1.7 \cdot 10^{-13}$	13
Epsilon	1.644 929 401 946 069	$4.7 \cdot 10^{-6}$	5
Aitken	1.644 934 037 602 495	$2.9 \cdot 10^{-8}$	8
Rho	1.644 934 066 848 226	$3.5 \cdot 10^{-265}$	264
Levin v	1.644 934 066 804 612	$4.4 \cdot 10^{-11}$	11
exact	1.644 934 066 848 226		

PSLQ

$$N = 12.0806776188920853608008722501$$

Analytical result? $\rightarrow$ ansatz: linear combination of $\zeta(n)$

$$N = a_1\,\zeta_2 + a_2\,\zeta_3 \; + a_3\,\zeta_4 + a_4\,\zeta_5 + a_5\,\zeta_2\,\zeta_3 + \ldots, \qquad a_i \in \mathbb{Z}$$

Determination of a_i: *„Integer Relation Detection“*

$\hookrightarrow$ PSLQ algorithm [Ferguson et al. 1992]

$$N = 2\,\zeta_2 + 3\,\zeta_3 + 5\,\zeta_5$$