Constructive models of quantum systems

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Quantum description of reversible evolution

Classical reversible evolution of X is a sequence of bijections $g_t \in G \leq \mathrm{Sym}(X)$ of X

Any set X can be "quantized" by assigning numbers

from a number system $\mathcal{F}$ to elements $x \in X$

$\implies$ X forms basis of module $M = \mathcal{F}^X$

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usually $\mathcal{F} = \mathbb{C}$; constructive version $\mathcal{F} = \mathbb{Q}_n$ — n-th cyclotomic field;

important primal case $\mathcal{F} = \mathbb{N}$ — semiring of naturals

classical to quantum:

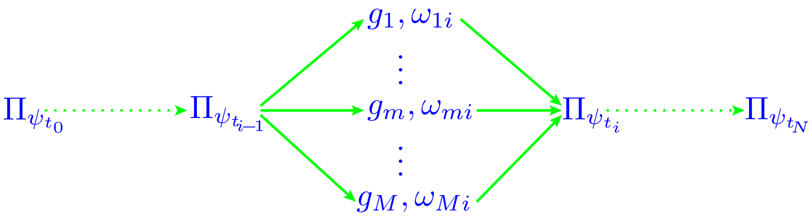
- states: set $X \longrightarrow$ Hilbert space $\mathcal{H}_X = \mathbb{C}^X$
- evolution: transformations $g_t \in G \longrightarrow$ unitary operators $U_t \in \mathrm{U}_{\mathcal{H}_X} \leq \mathrm{Aut}(\mathcal{H}_X)$

Remark: unitary evolution is not enough for physics

- single reversible evolution = symmetry transformation = change of coordinates is physically trivial:
  invariant relations among observables do not change in time
  nontriviality arises from
  - collections of evolutions — nontrivial holonomies in gauge theories
  - irreversible processes — measurements in quantum mechanics

Schematic model: quantum trajectory is a sequence of unitary evolutions interspersed with observations

- Times of observations $t_0, \ldots, t_N$
- Group $G = \{g_1, \ldots, g_M\}$ with representation U in space $\mathcal{H} \ni \psi_{t_i}$
- ω_{mi} is weight of g_m providing parallel transport $\mathsf{U}(g_m)\,\psi_{t_{i-1}} \equiv \varphi_{mi}$
- Projectors $\Pi_{\psi_{t_i}} = |\psi_{t_i}\rangle\langle\psi_{t_i}|$



Selection of Most Probable Trajectories and Principle of Least Action

- Probability to pass $\Pi_{\psi_{t_i}}$: $\mathbf{P}_{\psi_{t_{i-1}}\to\psi_{t_i}} = \sum_{m=1}^{M} \omega_{mi} \langle\varphi_{mi}| \Pi_{\psi_{t_i}} |\varphi_{mi}\rangle$
- Probability of trajectory: $\mathbf{P}_{\psi_{t_0}\to\cdots\to\psi_{t_N}} = \prod_{i=1}^{N} \mathbf{P}_{\psi_{t_{i-1}}\to\psi_{t_i}}$
- "Local" entropy: $\mathbf{S}_{\psi_{t_{i-1}}\to\psi_{t_i}} = \log \mathbf{P}_{\psi_{t_{i-1}}\to\psi_{t_i}}$
- Entropy of trajectory: $\mathbf{S}_{\psi_{t_0}\to\cdots\to\psi_{t_N}} = \sum_{i=1}^{N} \mathbf{S}_{\psi_{t_{i-1}}\to\psi_{t_i}}$

Continuum limit $N \to \infty, \quad t_i - t_{i-1} \to 0$

- $\psi_{t_i} \longrightarrow \psi\left(u_1\left(t\right), \ldots, u_K\left(t\right)\right)$; $\quad u_k\left(t\right)$ are continuous functions
- $\mathbf{S}_{\psi_{t_{i-1}}\to\psi_{t_i}} \longrightarrow$ Lagrangian $\mathcal{L} = A + B_{kk'}\left(\frac{du_k}{dt} - a_k\right)\left(\frac{du_{k'}}{dt} - a_{k'}\right)$
  negative definite quadratic form $B_{kk'}, a_k, A$ depend on $u_1, \ldots, u_K$
- $\mathbf{S}_{\psi_{t_0}\to\cdots\to\psi_{t_N}} \longrightarrow$ action $\mathcal{S} = \int \mathcal{L} dt$

Example: extracting Lagrangian from combinatorics I

$$P_{k_1,k_2,t} = \frac{t!}{k_1!k_2!}\alpha_1^{k_1}\alpha_2^{k_2} \quad — \quad \begin{cases}(1+1)\text{D random walk}\\ k_1+k_2=t,\ \alpha_1+\alpha_2=1\end{cases}$$

$$\Big\downarrow \quad \begin{aligned} x &:= k_1 - k_2 \\ v &:= \alpha_1 - \alpha_2 \quad \text{— “drift velocity”} \quad -1 \le v \le 1 \end{aligned}$$

$$P(x,t) = \frac{t!}{\left(\frac{t+x}{2}\right)!\left(\frac{t-x}{2}\right)!}\left(\frac{1+v}{2}\right)^{\frac{t+x}{2}}\left(\frac{1-v}{2}\right)^{\frac{t-x}{2}}$$

- fundamental (“Planck”) time $[0, 1, \ldots, T]$
- microscopic time (“observation times”) $[\tau_0 = 0, \ldots, \tau_{i-1}, \tau_i, \ldots, \tau_n = T]$
- observed values $[X_0, \ldots, X_{i-1}, X_i, \ldots, X_n]$

$\Delta\tau_i = \tau_i - \tau_{i-1}, \qquad 1 \ll \Delta\tau_i \ll T$

$\Delta X_i = X_i - X_{i-1}, \qquad v_i$ — drift velosity in $[\tau_{i-1}, \tau_i]$

Example: extracting Lagrangian from combinatorics II

$$\mathbf{P}_{X_{i-1}\to X_i} = \frac{\Delta\tau_i!}{\left(\frac{\Delta\tau_i+\Delta X_i}{2}\right)!\left(\frac{\Delta\tau_i-\Delta X_i}{2}\right)!}\left(\frac{1+v_i}{2}\right)^{\frac{\Delta\tau_i+\Delta X_i}{2}}\left(\frac{1-v_i}{2}\right)^{\frac{\Delta\tau_i-\Delta X_i}{2}}$$

$$\mathbf{S}_{X_{i-1}\to X_i} = \ln \mathbf{P}_{X_{i-1}\to X_i}$$

1. Stirling approximation: $\ln n! \approx n\ln n - n$
2. 2nd order expansion at stationary point $\Delta X_i^* = v_i\Delta\tau_i$
3. continuum approximation $X_i \to x(t)\,,\; v_i \to v(t)$

$$\Delta X_i \approx \dot{x}(t)\,\Delta\tau_i$$

$$\mathbf{S}_{X_{i-1}\to X_i} \approx -\frac{1}{2}\left(\frac{\dot{x}(t)-v}{\sqrt{1-v^2}}\right)^2\Delta\tau_i \Longrightarrow \text{Lagrangian } \mathcal{L} = \left(\frac{\dot{x}(t)-v}{\sqrt{1-v^2}}\right)^2$$

Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = 0 \Longrightarrow \ddot{x}\left(1-v^2\right) + 2\dot{x}v\frac{\partial v}{\partial t} - \left(1+v^2\right)\frac{\partial v}{\partial t} = 0$$

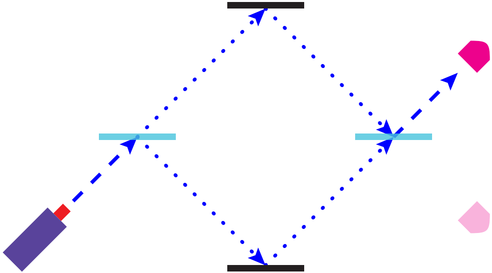
Mach–Zehnder interferometer

Beam-splitter S: $\begin{array}{lcl} |\nearrow\rangle & \to & \frac{1}{\sqrt{2}}\left(|\nearrow\rangle + \mathrm{i}\,|\searrow\rangle\right) \\ |\searrow\rangle & \to & \frac{1}{\sqrt{2}}\left(|\searrow\rangle + \mathrm{i}\,|\nearrow\rangle\right) \end{array}$ $S = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 1 \end{bmatrix}$

Mirror M: $\begin{array}{lcl} |\nearrow\rangle & \to & \mathrm{i}\,|\searrow\rangle \\ |\searrow\rangle & \to & \mathrm{i}\,|\nearrow\rangle \end{array}$ $M = \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}$ $M = S^2$

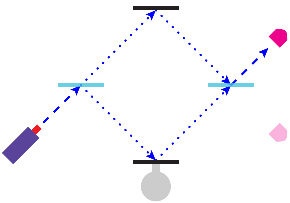
S generates group $\mathbb{Z}_8$

Scheme



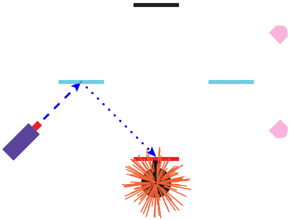
implements evolution $SMS\,|\nearrow\rangle = S^4\,|\nearrow\rangle = -\,|\nearrow\rangle$

Elitzur–Vaidman interaction-free measurements. Penrose bomb tester



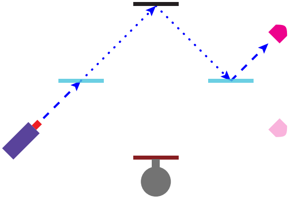
$$|\nearrow\rangle \xrightarrow{SMS} -|\nearrow\rangle \quad \mathbf{P} = 1$$

testing dud bomb



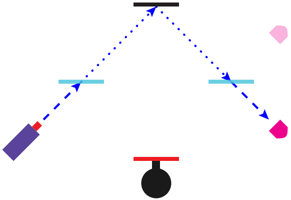
$$|\nearrow\rangle \xrightarrow{\Pi_{\searrow} S} \frac{i}{\sqrt{2}}|\searrow\rangle \quad \mathbf{P} = \frac{1}{2}$$

good bomb went off



$$|\nearrow\rangle \xrightarrow{\Pi_{\nearrow} S M \Pi_{\nearrow} S} -\frac{1}{2}|\nearrow\rangle \quad \mathbf{P} = \frac{1}{4}$$

bomb remains untested



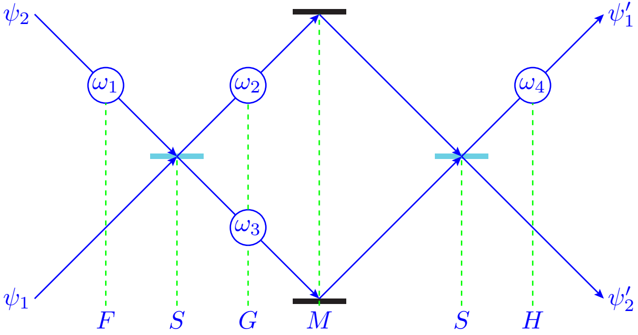
$$|\nearrow\rangle \xrightarrow{\Pi_{\searrow} S M \Pi_{\nearrow} S} \frac{i}{2}|\searrow\rangle \quad \mathbf{P} = \frac{1}{4}$$

bomb is good and intact

Mach–Zehnder interferometer implements any one-qubit gate I

$\dim \mathrm{U}(2) = 4 \Longrightarrow$ need to add 4 phase shifters $\omega_1, \omega_2, \omega_3, \omega_4$ to implement arbitrary unitary 2×2 matrix U

one of 16 possibilities:



$$U_{\mathrm{MZI}} = HSMGSF$$

$$F = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\omega_1} \end{bmatrix} \quad S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \quad G = \begin{bmatrix} e^{i\omega_2} & 0 \\ 0 & e^{i\omega_3} \end{bmatrix} \quad M = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \quad H = \begin{bmatrix} e^{i\omega_4} & 0 \\ 0 & 1 \end{bmatrix}$$

Mach–Zehnder interferometer implements any one-qubit gate II

$$U_{\mathrm{MZI}} = \begin{bmatrix} -\dfrac{e^{i(\omega_2+\omega_4)}+e^{i(\omega_3+\omega_4)}}{2} & -i\,\dfrac{e^{i(\omega_1+\omega_2+\omega_4)}-e^{i(\omega_1+\omega_3+\omega_4)}}{2} \\ i\,\dfrac{e^{i\omega_2}-e^{i\omega_3}}{2} & -\dfrac{e^{i(\omega_1+\omega_2)}+e^{i(\omega_1+\omega_3)}}{2} \end{bmatrix}$$

$$= e^{i\varphi}\begin{bmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} e^{i\delta} & 0 \\ 0 & e^{-i\delta} \end{bmatrix}$$

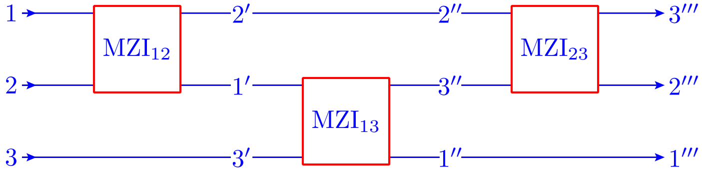
standard factorization of generic unitary matrix

$$\varphi = \frac{\omega_1+\omega_2+\omega_3+\omega_4}{2}, \quad \psi = \frac{\pi+\omega_4}{2}, \quad \theta = \frac{\omega_2-\omega_3}{2}, \quad \delta = \frac{\pi-\omega_1}{2}$$

MZI implementation of arbitrary matrix $U \in \mathrm{U}(n)$

$$U = \prod_{1\le i<j\le n} \mathrm{I}_{\{1,\ldots,\hat{i},\ldots,\hat{j},\ldots,n\}} \oplus U_{\mathrm{MZI}_{ij}}$$

- sequence of $\dfrac{n(n-1)}{2}$ Mach–Zehnder interferometers corresponding to two-dimensional subspaces of $\mathcal{H}_n$



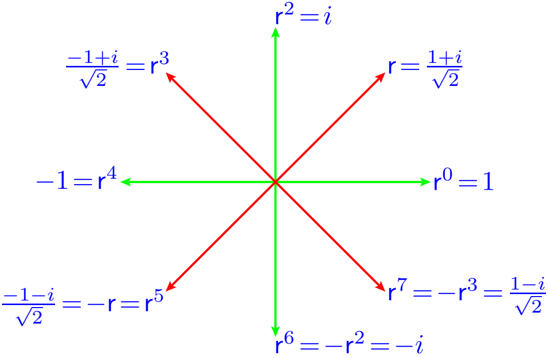
- $\dim \mathrm{U}(n) = n^2 \Longrightarrow \left[\begin{array}{l}\text{excess in number of parameters}\\ 4\dfrac{n(n-1)}{2} - n^2 = n^2 - 2n\end{array}\right.$

  a more economical scheme in:

  M. Reck, A. Zeilinger, H. J. Bernstein, P. Bertani "Experimental Realization of Any Discrete Unitary Operator" *Phys. Rev. Lett.* **73** (1994) 58

Constructive view on balanced Mach–Zehnder interferometer I

- Mirror is square of beam-splitter: $M = S^2$
  $\Longrightarrow$ any sequence of MZI's can be described by degrees of S
- S generates cyclic group $\mathbb{Z}_8$
  Cyclotomic polynomial $\Phi_8(\mathrm{r}) = 1 + \mathrm{r}^4$
  $\longrightarrow$ primitive and nonprimitive roots of unity



Constructive view on balanced Mach–Zehnder interferometer II

- embedding into permutations
  - smallest degree of faithful action $= 8$
  - generator $g = (1, 2, 3, 4, 5, 6, 7, 8) \longleftrightarrow S$
  - representation in 8D module of natural vectors $\mathbb{N}^8$
    $N = (n_1, n_2, n_3, n_4, n_5, n_6, n_7, n_8)^T \in \mathbb{N}^8$
  - permutation matrix

$$P(g) = \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

Constructive view on balanced Mach–Zehnder interferometer III

- $S(g) = T^{-1}P(g)\,T$ is similar matrix that contains splitter

$$S(g) = \begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathrm{r}^2 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -\mathrm{r}^2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \mathrm{r}^3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -\mathrm{r} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\mathrm{r}-\mathrm{r}^3}{2} & \frac{\mathrm{r}+\mathrm{r}^3}{2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\mathrm{r}+\mathrm{r}^3}{2} & \frac{\mathrm{r}-\mathrm{r}^3}{2} \end{bmatrix}$$

- quantum amplitude as projection of N into “splitter” subspace

$$|\psi\rangle = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -\mathrm{r}^3\,(n_1 + n_3 - n_5 - n_7) + \left(1 - \mathrm{r}^2\right)(n_2 - n_6) \\ \mathrm{r}\,(n_1 - n_3 - n_5 + n_7) + \left(1 + \mathrm{r}^2\right)(-n_4 + n_8) \end{bmatrix}$$

Quantum trajectories for Mach–Zehnder model

In coordinates on Bloch sphere

$$|\psi_{t_i}\rangle = |\alpha_i, \beta_i\rangle = \cos\left(\frac{\alpha_i}{2}\right)|\nearrow\rangle + e^{i\beta_i}\sin\left(\frac{\alpha_i}{2}\right)|\searrow\rangle$$

Assuming $\omega_{mi} = \omega_{m'i}$ for $g_m, g_{m'} \in \mathbb{Z}_8$

$$\mathbf{P}_{\psi_{t_{i-1}}\to\psi_{t_i}} \propto 1 + \sin\alpha_{i-1}\cos\beta_{i-1}\sin\alpha_i\cos\beta_i$$
$$|\alpha_i, \beta_i\rangle \in \mathrm{Orbit}\left(\mathbb{Z}_8, |\alpha_{i-1}, \beta_{i-1}\rangle\right)$$

Search for most likely trajectories is purely combinatorial problem

Continuum limit makes sense for groups whose orbits are ("empirically") dense on the Bloch sphere

Obrigado pela sua atenção