QED with x-electric potential steps

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Contents

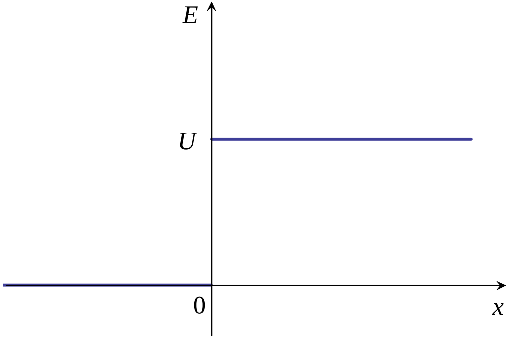


Figure: A potential step

t-electric potential steps

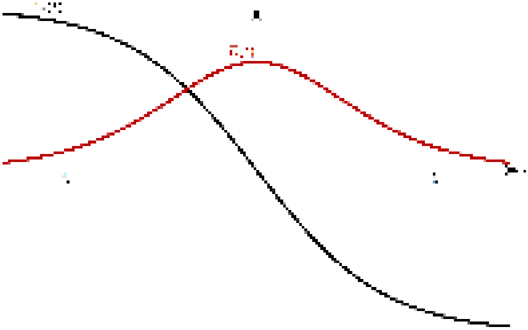


Figure:

In- and out- solutions

Two complete sets of solutions of Dirac equation:
in-set $\{ {}_\zeta\psi_n(X), \ \zeta = \pm\}$ classified as electrons $(+)$ and positrons $(-)$ at $t \in (-\infty, t_{\rm in})$,
out-set $\{ {}^\zeta\psi_n(X), \ \zeta = \pm\}$, classified as electrons $(+)$ and positrons $(-)$ at $t \in (-\infty, t_{\rm in})$.
Decomposing the Heisenberg Dirac field in these solutions, we introduce in- and out-operators;
in- and out-operators are related by a linear canonical transformation (Bogolubov transformation)
All the characteristics of quantum processes can be expressed via coefficients of these transformations.
Gitman, J. Phys. A 10 (1977)
Fradkin, Gitman, Shvartsman, **Quantum Electrodynamics with Unstable Vacuum** (Springer-Verlag, 1991)

x-electric potential steps

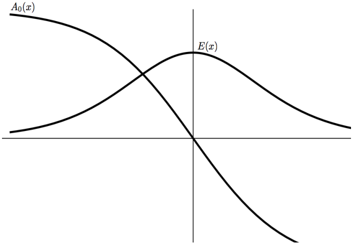


Figure:

$$A_0(x) = -aE \tanh(x/a), \quad a > 0, \text{ Sauter step } (\text{Z.Phys.}\mathbf{73}, 547, 1931),$$
$$E(x) = E\cosh^{-2}(x/a), \; U(x) = -eA_0(x) = eE\alpha \tanh(x/\alpha).$$

Gitman,Gavrilov, Quantization of charged fields in the presence of critical potential steps, http://arxiv.org/abs/1506.01156, [hep-th]

x-electric potential steps

In $3+1$ dim., space-time with coordinates $x^0 = t$, $\mathbf{r} = (x, y, z)$ potentials A^μ that correspond to an x-electric potential step are chosen to be $A^\mu = A^0(x)$, $\mathbf{A} = 0$, such that the magnetic field $\mathbf{B}$ be zero and

$$\mathbf{E} = (E_x(x), 0, 0), \quad E_x(x) = -A_0'(x) = E(x).$$

The electric field is inhomogeneous in the x-direction, and does not depend on time t ($\mathbf{E}$ is a constant field).

It is supposed that

$$A_0(x) \overset{x\to\pm\infty}{\longrightarrow} A_0(\pm\infty) = \mathrm{const}, \quad E(x) \overset{|x|\to\infty}{\longrightarrow} 0, \ \text{or}$$
$$A_0(x)|_{x\in S_\mathrm{L}} = A_0(-\infty), \quad E(x)|_{x\in S_\mathrm{L}} = 0, \quad S_\mathrm{L} = (-\infty, x_\mathrm{L}],$$
$$A_0(x)|_{x\in S_\mathrm{R}} = A_0(+\infty), \quad E(x)|_{x\in S_\mathrm{R}} = 0, \quad S_\mathrm{R} = [x_\mathrm{R}, \infty).$$

$\mathbb{U}$ is the magnitude of the electric step,

$$\mathbb{U} = U_{\mathrm{R}} - U_{\mathrm{L}}, \quad U_{\mathrm{L}} = U(-\infty), \quad U_{\mathrm{R}} = U(+\infty).$$

If $\mathbb{U} < \mathbb{U}_c = 2m$ we deal with **noncritical steps: the range** Ω_3 (Klein zone) does not exist.
If $\mathbb{U} > \mathbb{U}_c = 2m$ we deal with **critical steps: the range** Ω_3, there exists the Klein zone.
$\pi_0(\mathrm{L}) = p_0 - U_{\mathrm{L}}$ asymptotic kinetic energy in the region $S_{\mathrm{L}} = (-\infty, x_{\mathrm{L}}]$,
$\pi_0(\mathrm{R}) = p_0 - U_{\mathrm{R}}$ asymptotic kinetic energy in the region $S_{\mathrm{R}} = [x_{\mathrm{R}}, \infty)$,

Dirac equation with x-electric potential steps

$$i\partial_t\psi = \hat{H}\psi, \hat{H} = -i\boldsymbol{\alpha}\boldsymbol{\nabla} + \beta m + U(x), \quad U(x) = -eA_0(x),$$

$\hat{H}$ is the one-particle (time independent) Dirac Hamiltonian, $U(x)$ is the potential energy of an electron in a x-electric potential step.

Stationary solutions

$$\begin{aligned}
&\psi_n = \exp\left(-ip_0t + i\mathbf{p}_\perp\mathbf{r}_\perp\right)\tilde{\psi}_n(x), \ \mathbf{p}_\perp = (p_y, p_z), \\
&\tilde{\psi}_n(x) = \left\{\gamma^0\left[p_0 - U(x)\right] - \gamma^1\hat{p}_x - \boldsymbol{\gamma}_\perp\mathbf{p}_\perp + m\right\}\varphi_n(x)v_\sigma, \\
&\left\{\hat{p}_x^2 + iU'(x) - \left[p_0 - U(x)\right]^2 + \mathbf{p}_\perp^2 + m^2\right\}\varphi_n(x) = 0, \\
&\alpha^1 v_\sigma = v_\sigma, \ i\gamma^2\gamma^3 v_\sigma = \sigma v_\sigma, \quad \sigma = \pm 1, \\
&\boldsymbol{\gamma}_\perp = \left(\gamma^2, \gamma^3\right), \ \hat{p}_x = -i\partial_x, \quad n = (p_0, \mathbf{p}_\perp, \sigma).
\end{aligned}$$

In the asymptotic regions S_{L} and S_{R} solutions ψ_n are eigenfunctions of the kinetic energy operator $\hat{H}^{\mathrm{kin}} = \hat{H} - U(x)$,

$$\hat{H}^{\mathrm{kin}}\psi_n(X)\Big|_{x\to\pm\infty} = \pi_0(\mathrm{R/L}) \ \psi_n(X)\big|_{x\to\pm\infty}.$$

Solutions of Dirac equation with special left and right asymptotics

In such solutions functions $\varphi_n(x)$ are denoted as $_\zeta\varphi_n(x)$ or $^\zeta\varphi_n(x)$ respectively,

S_{L}:

$$_\zeta\varphi_n(x) = \varphi^{\mathrm{L}}_{n,\zeta}(x) \text{ as } x \in S_{\mathrm{L}} = (-\infty, x_{\mathrm{L}}] ,$$

$$\left\{\hat{p}_x^2 - [\pi_0(\mathrm{L})]^2 + \pi_\perp^2\right\} \varphi^{\mathrm{L}}_n(x) = 0, \quad \varphi^{\mathrm{L}}_{n,\zeta}(x) = {}_\zeta\mathcal{N} \exp\left(ip^{\mathrm{L}} x\right),$$

$$p^{\mathrm{L}} = \zeta\sqrt{[\pi_0(\mathrm{L})]^2 - \pi_\perp^2}, \quad \zeta = \mathrm{sgn}\left(p^{\mathrm{L}}\right) = \pm , \quad \pi_\perp = \sqrt{\mathbf{p}_\perp^2 + m^2}.$$

S_{R} :

$$^\zeta\varphi_n(x) = \varphi^{\mathrm{R}}_{n,\zeta}(x) \text{ as } x \in S_{\mathrm{R}} = [x_{\mathrm{R}}, \infty),$$

$$\left\{\hat{p}_x^2 - [\pi_0(\mathrm{R})]^2 + \pi_\perp^2\right\} \varphi^{\mathrm{R}}_n(x) = 0, \quad \varphi^{\mathrm{R}}_{n,\zeta}(x) = {}^\zeta\mathcal{N} \exp\left(ip^{\mathrm{R}} x\right),$$

$$p^{\mathrm{R}} = \zeta\sqrt{[\pi_0(\mathrm{R})]^2 - \pi_\perp^2}, \quad \zeta = \mathrm{sgn}\left(p^{\mathrm{R}}\right) = \pm .$$

Solutions of Dirac equation with special left and right asymptotics

The corresponding solutions of the Dirac equation, are denoted as ${}_\zeta\psi_n(X)$ and ${}^\zeta\psi_n(X)$. They are states with definite momenta p^{L} as $x \to -\infty$, or with p^{R} as $x \to +\infty$,

$$\begin{aligned} \hat{p}_x \, {}_\zeta\psi_n(X) &= p^{\mathrm{L}} \, {}_\zeta\psi_n(X), \quad x \to -\infty, \\ \hat{p}_x \, {}^\zeta\psi_n(X) &= p^{\mathrm{R}} \, {}^\zeta\psi_n(X), \quad x \to +\infty. \end{aligned}$$

Nontrivial solutions ${}_\zeta\psi_n(X)$ exist only for certain n,

$$[\pi_0(\mathrm{L})]^2 > \pi_\perp^2 \Longleftrightarrow \left\{ \begin{array}{l} \pi_0(\mathrm{L}) > \pi_\perp \\ \pi_0(\mathrm{L}) < -\pi_\perp \end{array} \right. .$$

Nontrivial solutions ${}^\zeta\psi_n(X)$ exist only for certain n,

$$[\pi_0(\mathrm{R})]^2 > \pi_\perp^2 \Longleftrightarrow \left\{ \begin{array}{l} \pi_0(\mathrm{R}) > \pi_\perp \\ \pi_0(\mathrm{R}) < -\pi_\perp \end{array} \right. .$$

Ranges of quantum numbers

There exist five ranges Ω_k, $k = 1, ..., 5$ of quantum numbers n where solutions $\varphi_n^{\mathrm{L/R}}(x)$ have similar properties and forms,

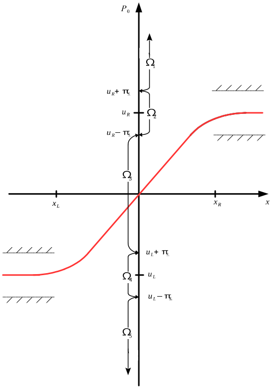


Figure: Potential energy $U(x)$ of an electron in an x-electric step and ranges of quantum numbers

The range Ω_1 includes quantum numbers n_1 that obey the inequality $p_0 \geq U_{\mathrm{R}} + \pi_\perp$. Two complete sets $\left\{ {}^{\zeta}\psi_{n_1} \right\}$ and $\left\{ {}_{\zeta}\psi_{n_1} \right\}$ can be interpreted as electron solutions.

The range Ω_5 includes quantum numbers n_5 that obey the inequality $p_0 \leq U_{\mathrm{L}} - \pi_\perp$. Two complete sets $\left\{ {}^{\zeta}\psi_{n_5} \right\}$ and $\left\{ {}_{\zeta}\psi_{n_5} \right\}$ can be interpreted as positron solutions.

The range $n_2 \in \Omega_2$ exists for any $\mathbb{U}$, quantum numbers n_2 obey the inequalities $U_{\mathrm{R}} - \pi_\perp < p_0 < U_{\mathrm{R}} + \pi_\perp$.

Any solution ψ_{n_2} has zero **right** asymptotics and zero Dirac current in x-direction. This fact imposes restrictions on solutions ψ_{n_2},

$$\psi_{n_2}(X) = {}_{+}\psi_{n_2}(X)\, c_{+} + {}_{-}\psi_{n_2}(X)\, c_{-} \,,$$
$$|c_{+}| = |c_{-}| \implies \psi_{n_2}(X) = {}_{+}\psi_{n_2}(X)\, e^{+i\theta_{n_2}} + {}_{-}\psi_{n_2}(X)\, e^{-i\theta_{n_2}} \,.$$

Complete set $\left\{ \psi_{n_2} \right\}$ represents solutions that **are sums of two electron waves travelling in opposite directions, with equal in magnitude currents, which means that we deal with a total reflection.** Similarly, we can treat the range Ω_4

Third range (Klein zone)

The Klein zone exists if $\mathbb{U} > 2m$. Here quantum numbers $\mathbf{p}_\perp$ are restricted by $2\pi_\perp \leq \mathbb{U}$,

$$U_{\mathrm{L}} + \pi_\perp \leq p_0 \leq U_{\mathrm{R}} - \pi_\perp \quad \Longrightarrow \begin{cases} \pi_0\left(\mathrm{L}\right) \geq \pi_\perp \\ \pi_0\left(\mathrm{R}\right) \leq -\pi_\perp \end{cases},$$

and there exist the following two complete sets of solutions

$$\left\{ {}_\zeta\psi_{n_3}(X) \right\}, \quad \left\{ {}^\zeta\psi_{n_3}(X) \right\}, \quad \zeta = \pm .$$

In contrast to the ranges Ω_1 and Ω_5, the naive one-particle interpretation of these solutions becomes erroneous. E.g. the following contradiction: **from the point of view of the left asymptotic area, only electron states are possible in the Klein zone, whereas from the point of view of the right asymptotic area, only positron states are possible in the Klein zone**. QED consideration shows that solutions ${}^\zeta\psi_{n_3}(X)$ describe electrons, whereas ${}_\zeta\psi_{n_3}(X)$ describe positrons.

Orthogonality and normalization

Solutions ${}_{\zeta}\psi_{n}(X)$ and ${}^{\zeta}\psi_{n}(X)$, $n \in \Omega_1 \cup \Omega_3 \cup \Omega_5$, can be subjected to the following orthonormality conditions

$$
\begin{aligned}
\left({}_{\zeta}\psi_{n}, {}_{\zeta'}\psi_{n'} \right)_x &= \zeta \eta_{\mathrm{L}} \delta_{\zeta,\zeta'} \delta_{n,n'}, \quad \eta_{\mathrm{L}} = \operatorname{sgn} \pi_0(\mathrm{L}), \\
\left({}^{\zeta}\psi_{n}, {}^{\zeta'}\psi_{n'} \right)_x &= \zeta \eta_{\mathrm{R}} \delta_{\zeta,\zeta'} \delta_{n,n'}, \quad \eta_{\mathrm{R}} = \operatorname{sgn} \pi_0(\mathrm{R}) \ , \\
(\psi, \psi')_x &= \int \psi^{\dagger}(X) \gamma^0 \gamma^1 \psi'(X) \, dt d\mathbf{r}_{\perp} \ .
\end{aligned}
$$

Coefficients g :

$$
g\left({}_{\zeta} \middle| {}^{\zeta'} \right) = \left({}_{\zeta}\psi_{n}, {}^{\zeta'}\psi_{n} \right)_x
$$

define mutual decompositions of these solutions

$$
\begin{aligned}
\eta_{\mathrm{L}} \, {}^{\zeta}\psi_{n}(X) &= {}_{+}\psi_{n}(X) g\left({}_{+} \middle| {}^{\zeta} \right) - {}_{-}\psi_{n}(X) g\left({}_{-} \middle| {}^{\zeta} \right), \\
\eta_{\mathrm{R}} \, {}_{\zeta}\psi_{n}(X) &= {}^{+}\psi_{n}(X) g\left({}^{+} \middle| {}_{\zeta} \right) - {}^{-}\psi_{n}(X) g\left({}^{-} \middle| {}_{\zeta} \right).
\end{aligned}
$$

Orthogonality and normalization

$$\left(\psi_n, \psi'_{n'}\right) = 0,\ \forall n \neq n';\quad \left(\psi, \psi'\right) = \int \psi^{\dagger}\psi' d\mathbf{r}.$$

$$\left({}_{\zeta}\psi_n,\ {}_{\zeta}\psi_{n'}\right) = \left({}^{\zeta}\psi_n,\ {}^{\zeta}\psi_{n'}\right) = \delta_{\sigma,\sigma'}\delta\left(p_0 - p_0'\right)\delta\left(\mathbf{p}_{\perp} - \mathbf{p}'_{\perp}\right)\mathcal{M}_n,$$

$$\mathcal{M}_n = \left|g\left({}_{+}\left|{}^{+}\right.\right)\right|^2,\quad n \in \Omega_1 \cup \Omega_5;\quad \mathcal{M}_3 = \left|g\left({}_{+}\left|{}^{-}\right.\right)\right|^2,\quad n \in \Omega_3;$$

$$\left({}_{\zeta}\psi_n,\ {}^{-\zeta}\psi_n\right) = 0,\quad n \in \Omega_1 \cup \Omega_5\ ,\quad {}_{\zeta}\psi_n \text{ and } {}^{-\zeta}\psi_n \text{ independent},$$

$$\left({}_{\zeta}\psi_n,\ {}^{\zeta}\psi_n\right) = 0,\quad n \in \Omega_3\ ,\ {}_{\zeta}\psi_n \text{ and } {}^{\zeta}\psi_n \text{ independent}.$$

Then we identify:

$$\left.\begin{array}{lll} {}_{+}\psi_n\left(X\right), & {}^{-}\psi_n\left(X\right) & \text{in} - \text{solutions}, \\ {}_{-}\psi_n\left(X\right), & {}^{+}\psi_n\left(X\right) & \text{out} - \text{solutions}, \end{array}\right\},\quad n \in \Omega_1 \cup \Omega_5\ ,$$

$$\left.\begin{array}{lll} {}_{-}\psi_{n_3}\left(X\right), & {}^{-}\psi_{n_3}\left(X\right) & \text{in} - \text{solutions} \\ {}_{+}\psi_{n_3}\left(X\right), & {}^{+}\psi_{n_3}\left(X\right) & \text{out} - \text{solutions} \end{array}\right\},\quad n \in \Omega_3\ .$$

Quantized Dirac field and in- and out-operators

$$\Psi(X) \Longrightarrow \hat{\Psi}(X), \quad \left[\hat{\Psi}(X), \hat{\Psi}(X')\right]_+\Big|_{t=t'} = 0,$$

$$\left[\hat{\Psi}(X), \hat{\Psi}(X')^\dagger\right]_+\Big|_{t=t'} = \delta(\mathbf{r}-\mathbf{r}'); \quad \hat{\Psi}(X) = \sum_{i=1}^{5} \hat{\Psi}_i(X),$$

$$\hat{\Psi}_1(X) = \sum_{n_1} \mathcal{M}_{n_1}^{-1/2}\left[\,{}_+a_{n_1}(\mathrm{in})\,{}_+\psi_{n_1}(X) + {}^-a_{n_1}(\mathrm{in})\,{}^-\psi_{n_1}(X)\right]$$

$$= \sum_{n_1} \mathcal{M}_{n_1}^{-1/2}\left[\,{}^+a_{n_1}(\mathrm{out})\,{}^+\psi_{n_1}(X) + {}_-a_{n_1}(\mathrm{out})\,{}_-\psi_{n_1}(X)\right],$$

$$\hat{\Psi}_2(X) = \sum_{n_2} \mathcal{M}_{n_2}^{-1/2} a_{n_2}\psi_{n_2}(X), \quad \hat{\Psi}_4(X) = \sum_{n_4} \mathcal{M}_{n_4}^{-1/2} b_{n_4}^\dagger \psi_{n_4}(X),$$

$$\hat{\Psi}_3(X) = \sum_{n_3} \mathcal{M}_{n_3}^{-1/2}\left[\,{}^-a_{n_3}(\mathrm{in})\,{}^-\psi_{n_3}(X) + {}_-b_{n_3}^\dagger(\mathrm{in})\,{}_-\psi_{n_3}(X)\right]$$

$$= \sum_{n_3} \mathcal{M}_{n_3}^{-1/2}\left[\,{}^+a_{n_3}(\mathrm{out})\,{}^+\psi_{n_3}(X) + {}_+b_{n_3}^\dagger(\mathrm{out})\,{}_+\psi_{n_3}(X)\right],$$

$$\hat{\Psi}_5(X) = \sum_{n_5} \mathcal{M}_{n_5}^{-1/2} \left[{}^{+}b^{\dagger}_{n_5}(\mathrm{in})\ {}^{+}\psi_{n_5}(X) + {}_{-}b^{\dagger}_{n_5}(\mathrm{in})\ {}_{-}\psi_{n_5}(X) \right]$$

$$= \sum_{n_5} \mathcal{M}_{n_5}^{-1/2} \left[{}_{+}b^{\dagger}_{n_5}(\mathrm{out})\ {}_{+}\psi_{n_5}(X) + {}^{-}b^{\dagger}_{n_5}(\mathrm{out})\ {}^{-}\psi_{n_5}(X) \right],$$

where all a and b are Fermi annihilation operators and all $a^{\dagger}$ and $b^{\dagger}$ are Fermi creation operators. Kinetic energy operator:

$$\widehat{\mathbb{H}}^{\mathrm{kin}} = \int \hat{\Psi}(X)^{\dagger}\, \hat{H}^{\mathrm{kin}}\, \hat{\Psi}(X)\, d\mathbf{r} - \mathbb{H}_0 = \sum_{i=1}^{5} \sum_{n_i} \widehat{\mathbb{H}}_{n_i},$$

$$\mathbb{H}_0 = \sum_{n_3} {}_{+}E_{n_3} + \sum_{n_4} \pi_0(\mathrm{R}) + \sum_{n_5} \left({}_{+}\mathcal{E}_{n_5} + {}^{-}\mathcal{E}_{n_5} \right),$$

$$\widehat{\mathbb{H}}_{n_1} = {}_{+}\mathcal{E}_{n_1}\ {}_{+}a^{\dagger}_{n_1}(\mathrm{in})\ {}_{+}a_{n_1}(\mathrm{in}) + {}^{-}\mathcal{E}_{n_1}\ {}^{-}a^{\dagger}_{n_1}(\mathrm{in})\ {}^{-}a_{n_1}(\mathrm{in})$$

$$= {}_{-}\mathcal{E}_{n_1}\ {}_{-}a^{\dagger}_{n_1}(\mathrm{out})\ {}_{-}a_{n_1}(\mathrm{out}) + {}^{+}\mathcal{E}_{n_1}\ {}^{+}a^{\dagger}_{n_1}(\mathrm{out})\ {}^{+}a_{n_1}(\mathrm{out}),$$

$$\widehat{\mathbb{H}}_{n_2} = \pi_0(\mathrm{L})\, a^{\dagger}_{n_2} a_{n_2}, \quad \widehat{\mathbb{H}}_{n_4} = -\pi_0(\mathrm{R})\, b^{\dagger}_{n_4} b_{n_4},$$

$$
\begin{aligned}
\widehat{\mathbb{H}}_{n_3} &= {}^{+}\mathcal{E}_{n_3}\ {}^{+}a^{\dagger}_{n_3}(\mathrm{out})\ {}^{+}a_{n_3}(\mathrm{out}) - {}_{+}\mathcal{E}_{n_3}\ {}_{+}b^{\dagger}_{n_3}(\mathrm{out})\ {}_{+}b_{n_3}(\mathrm{out}) \\
&= {}^{-}\mathcal{E}_{n_3}\ {}^{-}a^{\dagger}_{n_3}(\mathrm{in})\ {}^{-}a_{n_3}(\mathrm{in}) - {}_{-}\mathcal{E}_{n_3}\ {}_{-}b^{\dagger}_{n_3}(\mathrm{in})\ {}_{-}b_{n_3}(\mathrm{in})\ , \\
\widehat{\mathbb{H}}_{n_5} &= -\ {}_{+}\mathcal{E}_{n_5}\ {}_{+}b^{\dagger}_{n_5}(\mathrm{out})\ {}_{+}b_{n_5}(\mathrm{out}) - {}^{-}\mathcal{E}_{n_5}\ {}^{-}b^{\dagger}_{n_5}(\mathrm{out})\ {}^{-}b_{n_5}(\mathrm{out}) \\
&= -\ {}_{-}\mathcal{E}_{n_5}\ {}_{-}b^{\dagger}_{n_5}(\mathrm{in})\ {}_{-}b_{n_5}(\mathrm{in}) - {}^{+}\mathcal{E}_{n_5}\ {}^{+}b^{\dagger}_{n_5}(\mathrm{in})\ {}^{+}b_{n_5}(\mathrm{in}),
\end{aligned}
$$

Conditions of the Hamiltonian positivity

$$
\begin{gathered}
\mathcal{E}_n > 0, \quad \forall n \in \Omega_1 \cup \Omega_2\ ; \quad \mathcal{E}_n < 0, \quad \forall n \in \Omega_4 \cup \Omega_5\ , \\
{}^{\zeta}\mathcal{E}_{n_3} - {}_{\zeta}\mathcal{E}_{n_3} = \mathbb{U}\left(1 - \left|g\left({}_{+}\middle|{}^{-}\right)\right|^{-2}\right) \geqslant 0,\ n \in \Omega_3 .
\end{gathered}
$$

Relations between in- and out-operators

In the range Ω_1 :

$$
\begin{aligned}
{}^{+}a_n(\text{out}) &= \eta_{\mathrm{L}} g\left({}_{+}|{}^{+}\right)^{-1}\; {}_{+}a_n(\text{in}) + g\left({}^{-}|{}_{-}\right)^{-1} g\left({}^{+}|{}_{-}\right)\; {}^{-}a_{n_1}(\text{in}),\\
{}_{-}a_n(\text{out}) &= g\left({}_{+}|{}^{+}\right)^{-1} g\left({}_{-}|{}^{+}\right)\; {}_{+}a_n(\text{in}) - \eta_{\mathrm{R}} g\left({}^{-}|{}_{-}\right)^{-1}\; {}^{-}a_{n_1}(\text{in});\\
{}_{+}a_n(\text{in}) &= g\left({}_{-}|{}^{-}\right)^{-1} g\left({}_{+}|{}^{-}\right)\; {}_{-}a_n(\text{out}) + \eta_{\mathrm{R}} g\left({}^{+}|{}_{+}\right)^{-1}\; {}^{+}a_n(\text{out}),\\
{}^{-}a_{n_1}(\text{in}) &= -\eta_{\mathrm{L}} g\left({}_{-}|{}^{-}\right)^{-1}\; {}_{-}a_n(\text{out}) + g\left({}^{+}|{}_{+}\right)^{-1} g\left({}^{-}|{}_{+}\right)\; {}^{+}a_n(\text{out}).
\end{aligned}
$$

In the range Ω_5 similar relations can be obtained by the substitution

$$
\begin{aligned}
&{}_{+}a_{n_1}(\text{in}) \to {}_{+}b^{\dagger}_{n_5}(\text{out}), \quad {}^{-}a_{n_1}(\text{in}) \to {}^{-}b^{\dagger}_{n_5}(\text{out}),\\
&{}^{+}a_{n_1}(\text{out}) \to {}^{+}b^{\dagger}_{n_5}(\text{in}), \quad {}_{-}a_{n_1}(\text{out}) \to {}_{-}b^{\dagger}_{n_5}(\text{in}).
\end{aligned}
$$

Relations between in- and out-operators

In the range Ω_3

$$
\begin{aligned}
{}^{+}a_n(\mathrm{out}) &= -g\left({}_{-}|{}^{+}\right)^{-1}\;{}_{-}b_n^{\dagger}(\mathrm{in}) + g\left({}^{-}|{}_{+}\right)^{-1} g\left({}^{+}|{}_{+}\right)\;{}^{-}a_n(\mathrm{in}),\\
{}_{+}b_n^{\dagger}(\mathrm{out}) &= g\left({}_{-}|{}^{+}\right)^{-1} g\left({}_{+}|{}^{+}\right)\;{}_{-}b_n^{\dagger}(\mathrm{in}) + g\left({}^{-}|{}_{+}\right)^{-1}\;{}^{-}a_n(\mathrm{in}),\\
{}_{-}b_n^{\dagger}(\mathrm{in}) &= g\left({}_{+}|{}^{-}\right)^{-1} g\left({}_{-}|{}^{-}\right)\;{}_{+}b_n^{\dagger}(\mathrm{out}) - g\left({}^{+}|{}_{-}\right)^{-1}\;{}^{+}a_n(\mathrm{out}),\\
{}^{-}a_n(\mathrm{in}) &= g\left({}_{+}|{}^{-}\right)^{-1}\;{}_{+}b_n^{\dagger}(\mathrm{out}) + g\left({}^{+}|{}_{-}\right)^{-1} g\left({}^{-}|{}_{-}\right)\;{}^{+}a_n(\mathrm{out}),
\end{aligned}
$$

show us that vacuum vectors $|0,\mathrm{in}\rangle$ and $|0,\mathrm{out}\rangle$,

$$
a(\mathrm{in})\,|0,\mathrm{in}\rangle = b(\mathrm{in})\,|0,\mathrm{in}\rangle = 0, \quad a(\mathrm{out})\,|0,\mathrm{out}\rangle = b(\mathrm{out})\,|0,\mathrm{out}\rangle = 0,
$$

are different.

The vacua are not charged and has zero kinetic energy,

$$
\langle 0,\mathrm{in}|\widehat{\mathbb{H}}^{\mathrm{kin}}\,|0,\mathrm{in}\rangle = 0, \;\; \langle 0,\mathrm{in}|\,\hat{Q}\,|0,\mathrm{in}\rangle = \langle 0,\mathrm{out}|\,\hat{Q}\,|0,\mathrm{out}\rangle = 0.
$$

In and out particles

Using QFT operators,

$$\hat{F}(x) = \frac{1}{T}\int \hat{\Psi}(X)^{\dagger}\gamma^{0}\gamma^{1}\hat{H}^{\mathrm{kin}}\,\hat{\Psi}(X)\,dtd\mathbf{r}_{\perp},\ \text{energy flux},$$
$$\hat{Q} = \frac{q}{2}\int\left[\hat{\Psi}(X)^{\dagger},\hat{\Psi}(X)\right]_{-}d\mathbf{r},\ \text{charge operator},$$
$$\hat{J} = -\frac{e}{T}\int \hat{\Psi}^{\dagger}(X)\gamma^{0}\gamma^{1}\hat{\Psi}(X)\,dtd\mathbf{r}_{\perp},\ \text{electric current}.$$

we can calculate all the characteristics of one particle states and justify in- and out-interpretations.
E.g. **all a are electrons, whereas all b are positrons,**

$$\begin{aligned}\langle 0,\mathrm{in}|a(\mathrm{in})\hat{Q}a^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle &= \langle 0,\mathrm{out}|a(\mathrm{out})\hat{Q}a^{\dagger}(\mathrm{out})|0,\mathrm{out}\rangle = -e,\\ \langle 0,\mathrm{in}|b(\mathrm{in})\hat{Q}b^{\dagger}(\mathrm{in})|0,\mathrm{in}\rangle &= \langle 0,\mathrm{out}|b(\mathrm{out})\hat{Q}b^{\dagger}(\mathrm{out})|0,\mathrm{out}\rangle = e.\end{aligned}$$

Kinetic energies of all one-particle states are positive.

In and out particles near the step

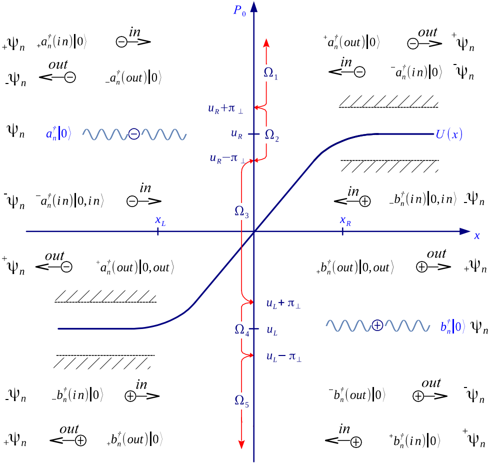
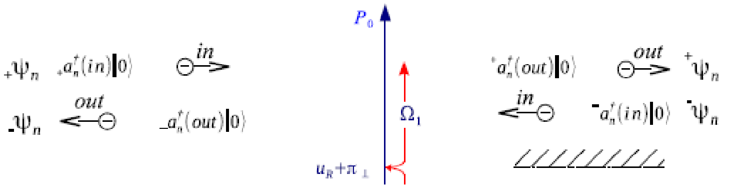


Figure: In and out particles near the step

Transmission and reflection in the first range



Relative amplitudes R of an electron reflection, and relative amplitudes T of an electron transmission are

$$R_{+,n} = \left\langle 0 \left| \, _-a_n(\text{out}) \, _+a_n^\dagger(\text{in}) \right| 0 \right\rangle, \quad R_{-,n} = \left\langle 0 \left| \, ^+a_n(\text{out}) \, ^-a_n^\dagger(\text{in}) \right| 0 \right\rangle,$$

$$T_{+,n} = \left\langle 0 \left| ^+a_n(\text{out}) \, _+a_n^\dagger(\text{in}) \right| 0 \right\rangle, \quad T_{-,n} = \left\langle 0 \left| _-a_n(\text{out}) \, ^-a_n^\dagger(\text{in}) \right| 0 \right\rangle, \quad n$$

The corresponding probabilities satisfy the unitarity relations

$$|R_{+,n}|^2 = |R_{-,n}|^2, \quad |T_{+,n}|^2 = |T_{-,n}|^2, \quad |R_{\zeta,n}|^2 + |T_{\zeta,n}|^2 = 1.$$

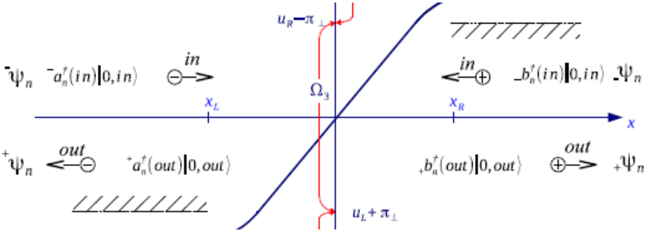
Consistency with potential scattering theory

Let us consider the evolution of the in-state ${}_{+}a^{\dagger}_{n_1}(\mathrm{in})\,|0\rangle$:
From the point of view of the time evolution this state can be reflected, with the probability $|R_{+,n}|^2$ and can be transmitted, with the probability $|T_{+,n}|^2$.
From the point of view of the time independent potential scattering theory, we have to calculate two mean currents in our in-state, one J_R of out-particles ${}_{-}a^{\dagger}_{n_1}(\mathrm{out})\,|0\rangle$, and another one J_T of out-particles ${}^{+}a^{\dagger}_{n_1}(\mathrm{out})\,|0\rangle$. Both currents are proportional (equal) to the mean numbers of the corresponding out-particles in our in-state,

$$\begin{aligned} J_R &= \left\langle 0\right|\, {}_{+}a_{n_1}(\mathrm{in})\left[\, {}_{-}a^{\dagger}_{n_1}(\mathrm{out})\; {}_{-}a_{n_1}(\mathrm{out})\right]\, {}_{+}a^{\dagger}_{n_1}(\mathrm{in})\left|\, 0\right\rangle \\ &= \left|g\left({}_{+}\left|{}^{+}\right.\right)\right|^{-2}\left|g\left({}_{-}\left|{}^{+}\right.\right)\right|^{2} = \left|R_{+,n}\right|^2, \\ J_T &= \left\langle 0\right|\, {}_{+}a_{n_1}(\mathrm{in})\left[\, {}^{+}a^{\dagger}_{n_1}(\mathrm{out})\; {}^{+}a_{n_1}(\mathrm{out})\right]\, {}_{+}a^{\dagger}_{n_1}(\mathrm{in})\left|\, 0\right\rangle \\ &= \left|g\left({}_{+}\left|{}^{+}\right.\right)\right|^{-2} = \left|T_{+,n}\right|^2. \end{aligned}$$

Thus, in the range Ω_1 realization of rules of the potential scattering theory in the framework of QFT allows one to obtain the correct result $J_R + J_T = 1$.

In- and out-particles in the Klein zone



in- and out-electrons are situated on the left of the step, and in- and out-positrons are situated on the right of the step. The vacuum is unstable, and processes of pair creation are possible. **in**-electrons that are moving to the step from the left are subjected to the complete reflection. **in**-positrons that are moving to the step from the right are subjected to the complete reflection. Our identification of states in the Klein zone **coincides with the one proposed by Nikishov** in the framework of RQM, Proc. Lebedev Inst.**111** (1979); It differs from an identification given by Hansen and Ravndal in Phys. Scrip.**23** (1981) and repeated in various publications.

Vacuum instability and pair creation in the Klein zone

The operator V_{Ω_3} relates in- and out-vacua, $|0,\mathrm{in}\rangle = V_{\Omega_3}|0,\mathrm{out}\rangle$,

$$c_v = \langle 0,\mathrm{out}|0,\mathrm{in}\rangle = \langle 0,\mathrm{out}|V|0,\mathrm{out}\rangle = \prod_{n\in\Omega_3} g\left({}^{-}|{}_{+}\right)^{-1} g\left({}^{-}|{}_{-}\right)$$

$$= \prod_{n\in\Omega_3} g\left({}_{-}|{}^{+}\right)^{-1} g\left({}_{+}|{}^{+}\right), \quad P_v = |c_v| = \prod_{n\in\Omega_3} p_v^n,$$

$$p_v^n = \left|g\left({}^{-}|{}_{+}\right)\right|^{-2}\left|g\left({}^{-}|{}_{-}\right)\right|^{2} = \left|g\left({}_{-}|{}^{+}\right)\right|^{-2}\left|g\left({}_{+}|{}^{+}\right)\right|^{2}.$$

Relative amplitudes of a pair creation and a pair annihilation

$$w\left(+-|0\right)_{n'n} = c_v^{-1}\langle 0,\mathrm{out}\left|\,{}^{+}a_{n'}(\mathrm{out})\;{}_{+}b_n(\mathrm{out})\right|0,\mathrm{in}\rangle$$

$$= \delta_{n,n'} w_n\left(+-|0\right), \quad w_n\left(+-|0\right) = g\left({}_{+}|{}^{+}\right)^{-1},$$

$$w\left(0|-+\right)_{nn'} = c_v^{-1}\langle 0,\mathrm{out}\left|\,{}_{-}b_n^{\dagger}(\mathrm{in})\;{}^{-}a_{n'}^{\dagger}(\mathrm{in})\right|0,\mathrm{in}\rangle$$

$$= \delta_{n,n'} w_n\left(0|-+\right), \quad w_n\left(0|-+\right) = -g\left({}^{-}|{}_{-}\right)^{-1}.$$

Pair creation in the Klein zone

Differential mean numbers of out-particles in the vacuum $|0, \mathrm{in}\rangle$ are:

$$N_n^a(\mathrm{out}) = \left\langle 0, \mathrm{in} \left| \, {}^+a_n^\dagger(\mathrm{out}) \, {}^+a_n(\mathrm{out}) \right| 0, \mathrm{in} \right\rangle = \left| g\left({}_-|{}^+\right) \right|^{-2},$$
$$N_n^b(\mathrm{out}) = \left\langle 0, \mathrm{in} \left| \, {}_+b_n^\dagger(\mathrm{out}) \, {}_+b_n(\mathrm{out}) \right| 0, \mathrm{in} \right\rangle = \left| g\left({}_+|{}^-\right) \right|^{2},$$
$$N_n^{\mathrm{cr}} = N_n^a(\mathrm{out}) = N_n^b(\mathrm{out}) = \left| g\left({}_-|{}^+\right) \right|^{-2} = \left| g\left({}_+|{}^-\right) \right|^{2},$$
$$N = \sum_{n\in\Omega_3} N_n^{\mathrm{cr}} = \sum_{n\in\Omega_3} \left| g\left({}_+|{}^-\right) \right|^{-2} = \sum_{n\in\Omega_3} \left| g\left({}_-|{}^+\right) \right|^{-2} .$$

Vacuum-to-vacuum differential transition probability p_v^n,

$$\left.\begin{array}{l} p_v^n = \left| g\left({}_-|{}^+\right) \right|^{-2} \left| g\left({}_+|{}^+\right) \right|^{2} = \left| g\left({}_+|{}^+\right) \right|^{2} \left(N_n^{\mathrm{cr}}\right)^{-1} \\ \left| g\left({}_+|{}^-\right) \right|^{2} = \left| g\left({}_+|{}^+\right) \right|^{2} + 1 \Longrightarrow \left| g\left({}_+|{}^+\right) \right|^{2} = N_n^{\mathrm{cr}} - 1, \end{array}\right\}$$
$$\Longrightarrow p_v^n = \left(1 - N_n^{\mathrm{cr}}\right) \Longrightarrow P_v = \prod_n \left(1 - N_n^{\mathrm{cr}}\right) .$$

Reflection of particles in the Klein zone

Relative scattering amplitudes

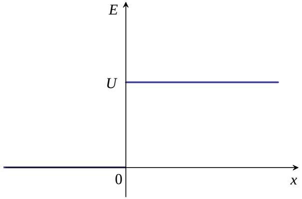
$$
\begin{aligned}
& w\left(+|+\right)_{n'n} = c_v^{-1}\langle 0,\mathrm{out}\left|\ {}^{+}a_{n'}\left(\mathrm{out}\right)\ {}^{-}a_n^{\dagger}(\mathrm{in})\right|0,\mathrm{in}\rangle = \delta_{n,n'} w_n\left(+|+\right), \\
& w_n\left(+|+\right) = g\left({}^{+}|{}_{-}\right) g\left({}^{-}|{}_{-}\right)^{-1} = g\left({}_{+}|{}^{-}\right) g\left({}_{+}|{}^{+}\right)^{-1}, \\
& w\left(-|-\right)_{n'n} = c_v^{-1}\langle 0,\mathrm{out}\left|\ {}_{+}b_{n'}\left(\mathrm{out}\right)\ {}_{-}b_n^{\dagger}(\mathrm{in})\right|0,\mathrm{in}\rangle = \delta_{n,n'} w_n\left(-|-\right), \\
& w_n\left(-|-\right) = g\left({}^{-}|{}_{+}\right) g\left({}^{-}|{}_{-}\right)^{-1} = g\left({}_{-}|{}^{+}\right) g\left({}_{+}|{}^{+}\right)^{-1}, \\
& \left|w_n\left(-|-\right)\right|^2 = \left(1 - N_n^{\mathrm{cr}}\right)^{-1} = \left|w_n\left(+|+\right)\right|^2 .
\end{aligned}
$$

Then the total probability of reflection of an electron and a positron on the x-electric potential step is

$$
\left|w_n\left(-|-\right)\right|^2 p_v^n = 1!, \qquad \left|w_n\left(+|+\right)\right|^2 p_v^n = 1!
$$

Regularized Klein step

Sauter potential with small $\mathbb{U}\alpha \ll 1$ imitates the Klein step sufficiently well, and coincides with the latter as $\alpha \to 0$,



In the range Ω_1 where $p_0 > U$

$$\left|g\left({}_+|^-\right)\right|^{-2} \approx \frac{4k}{(1-k)^2}, \quad \left|g\left({}_+|^+\right)\right|^2 = \left|g\left({}_+|^-\right)\right|^2 + 1,$$

$$k = \begin{cases} k_f = k_b \frac{\pi_0(\mathrm{L})+\pi_\perp}{\pi_0(\mathrm{R})+\pi_\perp}, & \text{fermions} \\ k_b = \frac{|p^{\mathrm{R}}|}{|p^{\mathrm{L}}|}, & \text{bosons} \end{cases}, \quad k \text{ kinematic factor} .$$

Transmission and reflection above the step

$$\left|T_{\zeta,n}\right|^2 = \left|g\left({}_+\left|{}^+\right.\right)\right|^{-2} = \frac{4k}{(1+k)^2},$$

$$\left|R_{\zeta,n}\right|^2 = \left|g\left({}_+\left|{}^-\right.\right)\right|^2 \left|g\left({}_+\left|{}^+\right.\right)\right|^{-2} = \frac{(1-k)^2}{(1+k)^2}.$$

For bosons, $k_b = \left|p^{\mathrm{R}}\right| / \left|p^{\mathrm{L}}\right|$ and there is a complete coincidence with the non relativistic result, see Landau and Lifshitz.
For fermions: Let $p_\perp = 0$ then $\pi_\perp = m$, and
$\pi_0\left(\mathrm{L}\right) = p_0 = m + E, \quad \pi_0\left(\mathrm{R}\right) = p_0 - \mathbb{U} = m + E - \mathbb{U},$

$$k_f = \mu k_b,\ \mu = \frac{\pi_0\left(\mathrm{L}\right) + m}{\pi_0\left(\mathrm{R}\right) + m} = \left[1 - \mathbb{U}/\left(E + 2m\right)\right]^{-1}.$$

In the nonrelativistic limit $\mathbb{U} \ll E + 2m$, $\mu \approx 1 + \mathbb{U}/\left(E + 2m\right)$.

Reflection and pair creation on the step

Relative probability amplitudes of the reflection and of the electron-positron pair creation in the Klein zone are:

$$|w_n(+-|0)|^2 = |g(_+|^+)|^{-2} = \frac{4|k|}{(1+k)^2},$$

$$p_n^{(v)} = |w_n(-|-)|^2 = |g(_+|^-)|^2 |g(_+|^+)|^{-2} = \frac{(1-k)^2}{(1+k)^2}.$$

These expressions for $|w_n(+-|0)|^2$ and $|w_n(-|-)|^2$ are similar to expressions for transmission and reflection probabilities in the ranges Ω_1 and Ω_5.

However, the interpretation of these quantities in the range Ω_3 differs essentially from their interpretation in the ranges Ω_1 and Ω_5.

Moreover, here, in case of fermions, $k_f < 0$. This formal similarity without a correct interpretation was the reason for a systematic misunderstanding in treating quantum processes in the Klein zone.