BRASIL-JINR FORUM

FRONTIERS IN NUCLEAR, ELEMENTARY PARTICLE AND CONDENSED MATTER PHYSICS

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Solution of the Bethe-Salpeter equation in Minkowski space

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OUTLINE

- Context
- LF projection
- Nakanishi PTIR for 3+1 and 2+1 bound states
- Solution of the BS 3+1 & 2+1 in Minkowski space
- Continuum solutions in 3+1 in Minkwoski space
- Prospects

Context

Bethe-Salpeter equation:
non perturbative regime in NN, NNN..., hadron structure, Graphene, 2D materials...;

Solutions for bound-states in Euclidean space:
with free propagators (Dorkin et al. FBS49, 233 (2011)), 3D reduction (Gross & Stadler, FBS49, 91 (2011)), Dyson-Schwinger and BSE - QCD -(Roberts, Prog. Part. Nucl. Phys. 61, 50 (2008)), Dorkin, Kaptari, Kampfer, PRC91, 055201 (2015)**…**

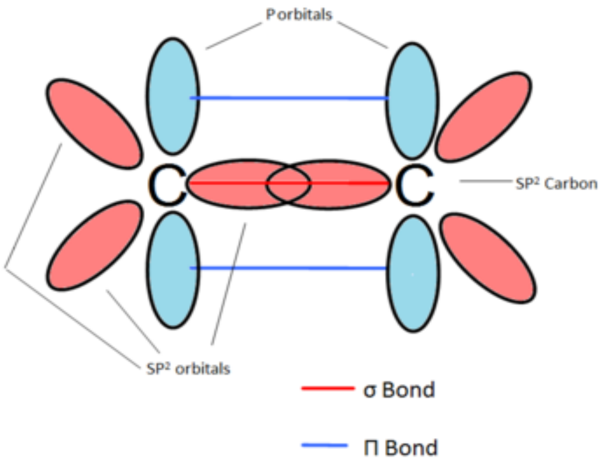
Minkowski space -Light-Front projection & expansion of the BSE kernel:
Weinberg (1966)… Ji, Brodsky, Lepage, Karmanov, Carbonell, Mathiot, Miller, Sales…

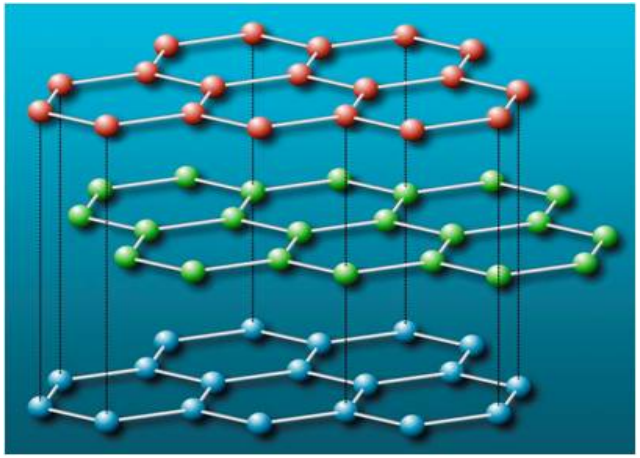
LF methods: reduction to the valence state dynamics & truncation over Fock-states within the kernel: "LF projection with a Quasi-Potential Approach" (Sales et al. PRC 61, 044003 (2000), PRC63, 064003 (2001),…Frederico & Salmè, FBS49, 163 (2011)), Garsevanishvili et al. Phys. Rep. 458 (2008) 247

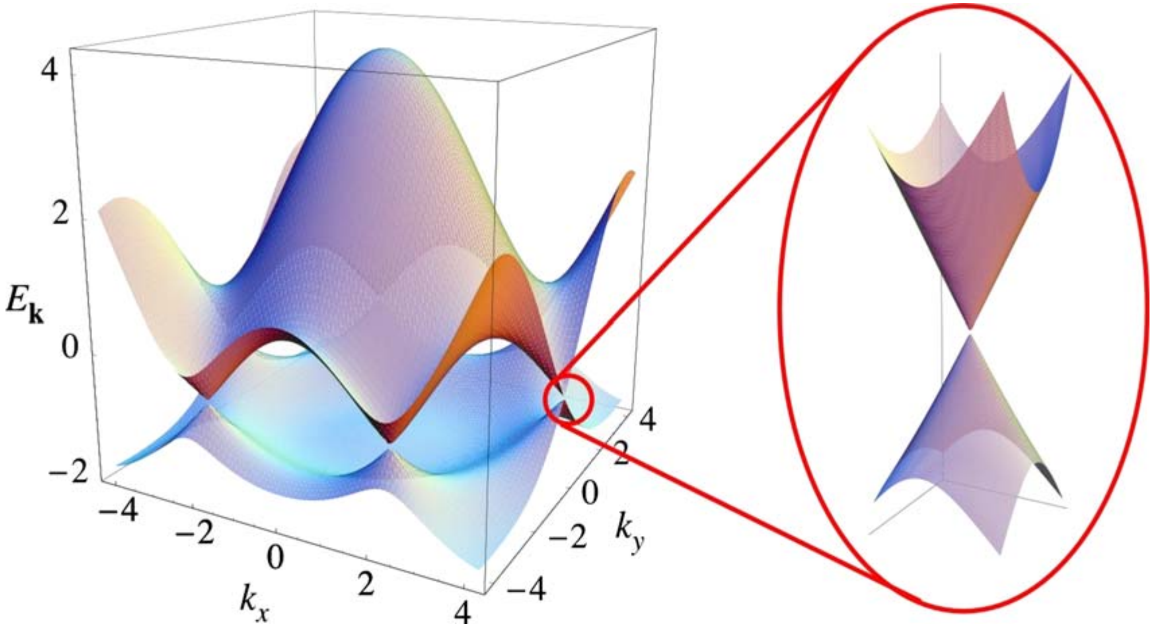
"Iterated resolvent method" within hamiltonian approach - H-C Pauli

- Brodsky, Pauli, Pinsky, PhysRep301, 299 (1998) -
- Revival with AdS/QCD models - de Teramond & Brodsky.

Context Graphene 2D







Castro Neto, Guinea, Peres, Novoselov, Geim, Rev. Mod. Phys. 81, 109 (2009)

Context

Perturbation Theory Integral Representation (PTIR)

Nakanishi PTIR: ”Parametric representation of any Feynman diagram for interacting bosons, with a denominator carrying the overall analytic behavior in Minkowski space.” (PR 127, 1380 (1962); PR 130, 1230 (1963); PR 133, B214 (1964); PR 135, B1224 (1964).)

Uniqueness theorem for the PTIR multi-leg transition amplitudes for bosonic systems - ”Graph Theory and Feynman Integrals” (Gordon and Breach, NY, 1971).

Solution of the Bethe-Salpeter equation in Minkowski space with Nakanishi PTIR for bound-state bosons:

K. Kusaka and A. G. Williams, Phys. Rev. D **51**, 7026 (1995); K. Kusaka, K. Simpson, and A. G. Williams, Phys. Rev. D **56**, 5071 (1997).

Context

Solution of the Bethe-Salpeter equation in Minkowski space:

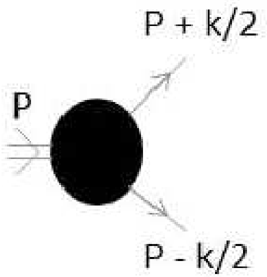
With Nakanishi PTIR for bound-state bosons:

PTIR & LF projection bound state of bosons and fermions:
Karmanov & Carbonell, EPJ A 27, 1 (2006); & 11 (2006) (X-ladder);
39 (2009) 53 (EMFF); 46 (2010) 387 (2F),

PTIR & LF projection bound and scattering states Bosons + uniqueness:
Frederico, Salmè, Viviani, PRD 85, 036009 (2012); PRD89, 016010 (2014);
arXiv:1504.01624 [hep-ph]

Direct Solution in Minkowski space for bound and scattering states:
Carbonell & Karmanov, bs: PLB727 (2013)319, scatt: PRD90 (2014) 056002,
Transition ff: PRD91 (2015) 076010

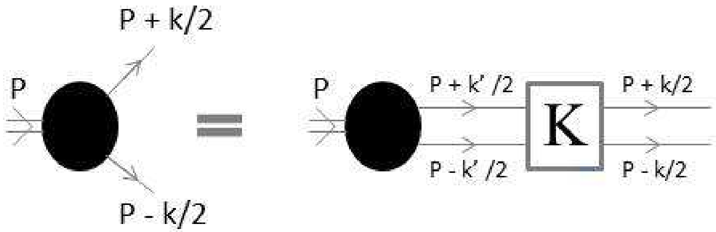
Nakanishi method for bound states



N. Nakanishi, Phys. Rev. **127**, 1380 (1962); **130**, 123
(1963)*Graph Theory and Feynman Integrals* (Gordon an
Breach, New York, 1971).

$$\Phi_b(k,p) = -i \int_{-1}^{1} dz' \int_0^{\infty} d\gamma' \frac{g_b(\gamma', z'; \kappa^2)}{[\gamma' + m^2 - \frac{1}{4}p^2 - k^2 - p\cdot kz' - i\epsilon]^{2+n}}$$

- Uniqueness of the weight function: perturbation theory $\frac{\partial}{\partial\gamma} g_{n+1}(\gamma, z) = (n+2) g_n(\gamma, z)$.



Solution of the bound state for bosons (ladder):

K. Kusaka and A. G. Williams, Phys. Rev. D **51**, 7026 (1995); K. Kusaka, K. Simpson, and A. G. Williams, Phys. Rev. D **56**, 5071 (1997).

- Sophisticated algebraic manipulations.

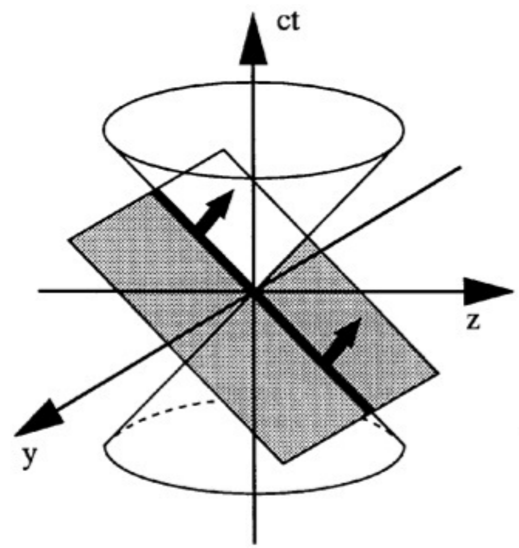
- Simplification with Light-Front projection:
  Karmanov, Carbonell, Eur. Phys. J. A 27, 1 (2006)

- Scattering: Frederico, Salmè, Viviani, Phys. Rev. D 85, 036009 (2012)

Light-Front Time Evolution

$$\tilde{\Phi}(x,p) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\Phi(k,p)$$

$$p^\mu = p_1^\mu + p_2^\mu \qquad k^\mu = \frac{p_1^\mu - p_2^\mu}{2}$$



$$\begin{aligned}\tilde{\Phi}(x,p) &= \langle 0|T\{\varphi_H(x^\mu/2)\varphi_H(-x^\mu/2)\}|p\rangle \\ &= \theta(x^+)\langle 0|\varphi(\tilde{x}/2)e^{-iP^-x^+/2}\varphi(-\tilde{x}/2)|p\rangle e^{ip^-x^+/4} + \bullet\bullet\bullet \\ &= \theta(x^+)\sum_{n,n'} e^{ip^-x^+/4}\langle 0|\varphi(\tilde{x}/2)|n'\rangle\langle n'|e^{-iP^-x^+/2}|n\rangle\langle n|\varphi(-\tilde{x}/2)|p\rangle + \bullet\bullet\bullet\end{aligned}$$

$x^+ = 0$ only valence state remains! How to rebuilt the full BS amplitude?

Iterated Resolvents: Brodsky, Pauli, and Pinsky, Phys. Rep. 301, 299 (1998)

BS amplitude from the valence LF wave function: sketch

- Quasi-Potential approach for the LF projection (3D equations);
- Derivation of an effective Mass-squared operator acting on the valence wave function;
- The effective interaction is expanded perturbatively in correspondence with the Fock-content of the intermediate states;
- $\Pi(p)$ reverse LF-time operator: computed perturbatively

Reverse operation: valence wave function $\Rightarrow$ BS amplitude

$$|\Psi\rangle = \Pi(p)\, |\phi_{LF}\rangle$$

Sales, et al. PRC61, 044003 (2000); PRC63, 064003 (2001); Frederico et al. NPA737, 260c (2004); Marinho et al., PRD 76, 096001 (2007); Marinho et al. PRD77, 116010 (2008); Frederico and Salmè, FBS49, 163 (2011).

<BS Ampl.| 4d operator |BS Ampl> = <val.|3d operator |val.>

LF 2-body operators

EM current:

QP Sales et al PRC 61, 044003 (2000),

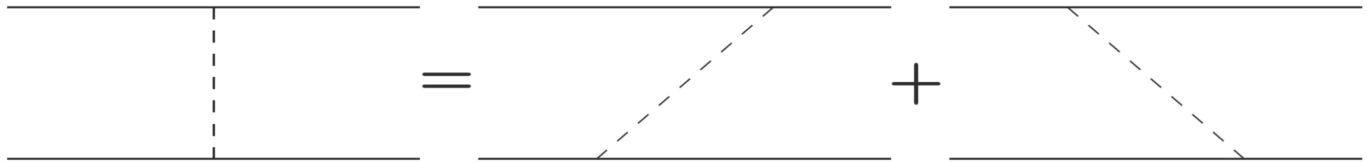
LF WTI Kvinikhidze & Blankleider PRD68(03)02581

QP WTI two-boson/two-fermion
Marinho et al PRD76(07)096001;PRD77(08)116010

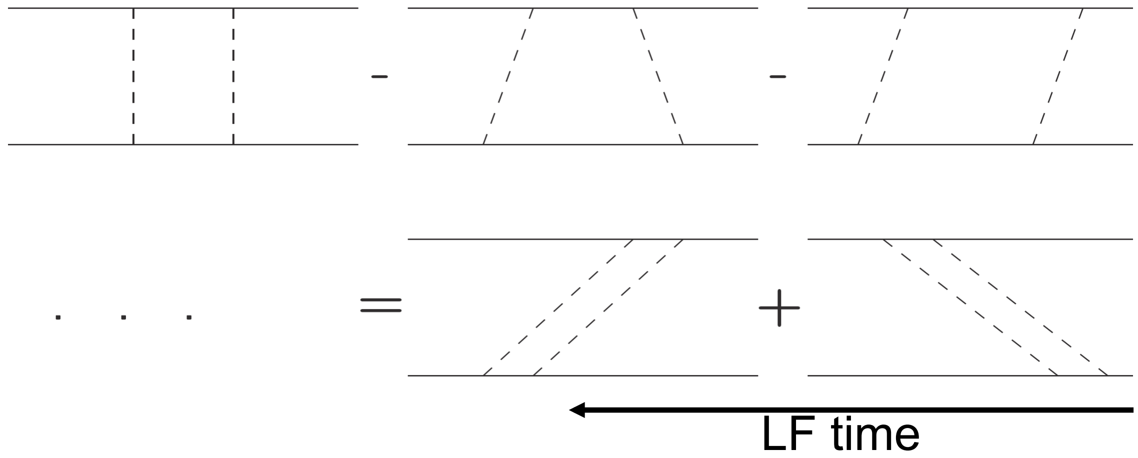
Example:Bosonic Yukawa model

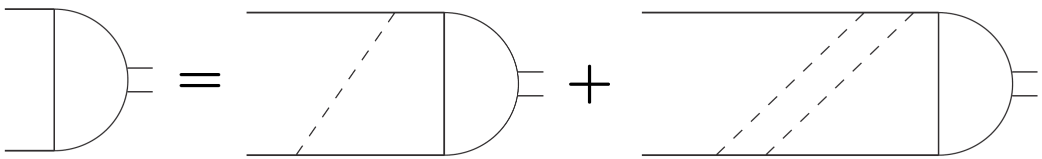
$$\mathcal{L}_I = g_S\phi_1^\dagger\phi_1\sigma + g_S\phi_2^\dagger\phi_2\sigma$$

$$w^{(1)} =$$



$$w^{(2)} =$$





$$\left[g_0^{-1} - w\right]$$

Mass^2 eigenvalue eq. & valence wf:

$$g(K_\lambda)^{-1}\left|\phi_\lambda\right\rangle = 0$$

LF QP 3-particles

Marinho & Frederico PoS(LC2008)036; Marinho (PhD thesis ITA 2007)

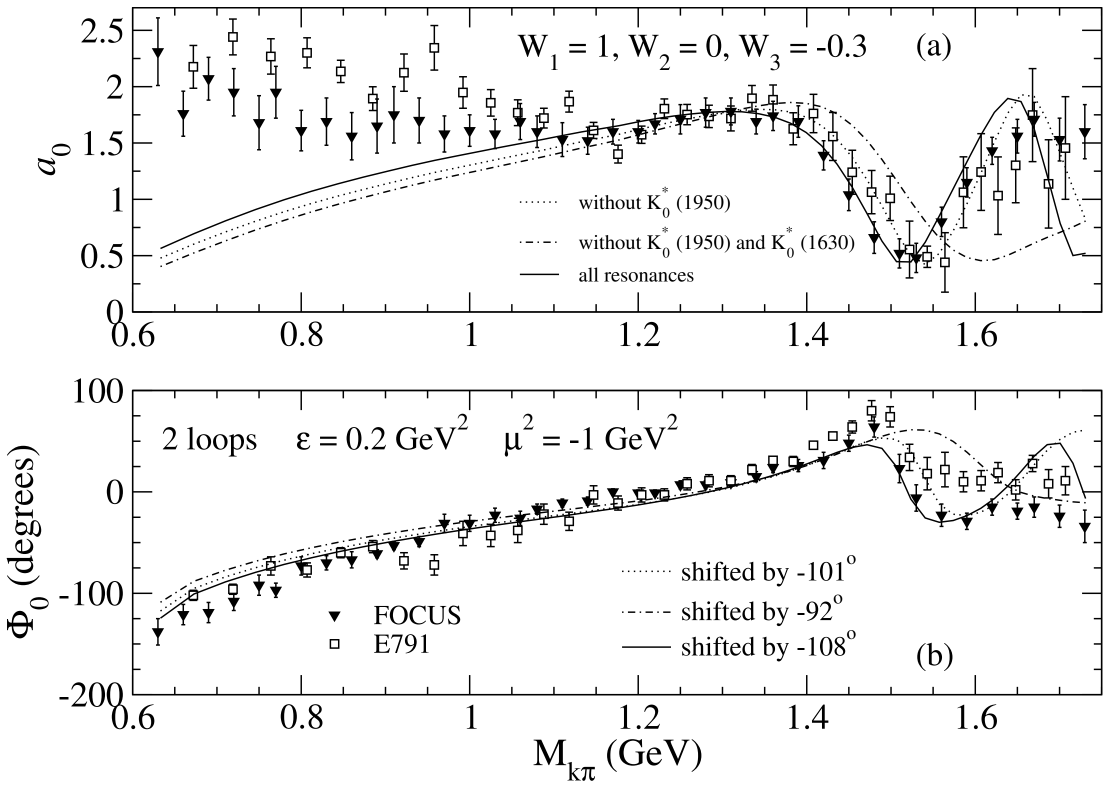
Karmanov & Maris PoS LC2008, 037 (2008), FBS 46, 95 (2009)

FSI in heavy-meson decay:

Magalhães et al, PRD 84, 094001 (2011) $D^{\pm} \to K^{\mp}\pi^{\pm}\pi^{\pm}$

Guimarães et al. JHEP 08, 135 (2014)

LF projection



Nakanishi PTIR for Bound State 3+1

Karmanov, Carbonell, Eur. Phys. J. A 27, 1 (2006)

$$\int \frac{dk^-}{2\pi}\Phi_b(k,p) = \int \frac{dk^-}{2\pi} G_0^{(12)}(k,p) \int \frac{d^4k'}{(2\pi)^4} i\mathcal{K}(k,k',p)\Phi_b(k',p).$$

Pole dislocation method: de Melo et al Nucl.Phys. A631 (1998) 574C

$$\int_0^\infty d\gamma' \frac{g_b(\gamma',z;\kappa^2)}{[\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2-i\epsilon]^2} = \int_0^\infty d\gamma' \int_{-1}^1 dz' V_b^{\rm LF}(\gamma,z;\gamma',z') g_b(\gamma',z';\kappa^2)$$

$$\kappa^2 = \frac{M^2}{4} - m^2$$

$$V_b^{\rm LF}(\gamma,z;\gamma',z') = ip^+ \int_{-\infty}^\infty \frac{dk^-}{2\pi} G_0^{(12)}(k,p) \int \frac{d^4k'}{(2\pi)^4} \frac{i\mathcal{K}(k,k',p)}{[k'^2+p\cdot k'z'-\gamma'-\kappa^2+i\epsilon]^3}$$

→ Applying Uniqueness →

$$g_b(\gamma,z;\kappa^2) = \int_0^\infty d\gamma' \int_{-1}^1 dz' \mathcal{V}_b(\gamma,z;\gamma',z';\kappa^2) g_b(\gamma',z';\kappa^2)$$

Frederico, Salmè, Viviani PRD89, 016010 (2014)

$$V_b^{(Ld)}(\gamma, z; \gamma', z') = -g^2 p^+ \int \frac{d^4k''}{(2\pi)^4} \frac{1}{[k''^2 + p \cdot k'' z' - \gamma' - \kappa^2 + i\epsilon]^3}$$
$$\times \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \frac{1}{[(\frac{p}{2} + k)^2 - m^2 + i\epsilon]} \frac{1}{[(\frac{p}{2} - k)^2 - m^2 + i\epsilon]} \frac{1}{(k - k'')^2 - \mu^2 + i\epsilon}$$
$$= -\frac{g^2}{2(4\pi)^2} \int_{-\infty}^{\infty} d\gamma'' \frac{\theta(\gamma'')}{[\gamma + \gamma'' + z^2 m^2 + \kappa^2(1 - z^2) - i\epsilon]^2}$$
$$\times \left[\frac{(1+z)}{(1+\zeta')} \theta(\zeta' - z) h'(\gamma'', z; \gamma', \zeta', \mu^2) + \frac{(1-z)}{(1-\zeta')} \theta(z - \zeta') h'(\gamma'', -z; \gamma', -\zeta', \mu^2) \right]$$

$$h' = \lim_{\lambda \to 0} \frac{\partial}{\partial \lambda} h$$

$$h(\gamma'', z; \gamma', \zeta', \mu^2, \lambda) = \frac{(1+z)}{(1+\zeta')} \int_0^1 \frac{dv}{(1-v)^2} \int_0^1 d\xi \delta[\gamma'' - \xi \Gamma(v, z, \zeta', \gamma') - \xi \lambda]$$

$$\Gamma(v, z, \zeta', \gamma') = \frac{(1+z)}{(1+\zeta')} \left\{ \frac{v}{(1-v)} \left[\zeta'^2 \frac{M^2}{4} + \kappa^2(1 + \zeta^2) + \gamma' \right] + \frac{\mu^2}{v} + \gamma' \right\}$$

Wick-Cutkosky model ($\mu = 0$)

$$g_b^{LW}(\gamma, z) = \frac{g^2}{2(4\pi)^2}\theta(\gamma) \int_0^\infty \frac{d\gamma'}{\gamma'} \int_{-1}^{+1} dz' \frac{g_b^{LW}(\gamma', z')}{\left[z'^2 \frac{M^2}{4} + \kappa^2 + \gamma'\right]} \times$$
$$\left[\theta(z'-z)\ \theta\left(\gamma' - \frac{(1+z')}{(1+z)}\gamma\right) + \theta(z-z')\ \theta\left(\gamma' - \frac{(1-z')}{(1-z)}\gamma\right)\right]$$

Note: If $\gamma \to \infty$ with $g_b^{LW}(\gamma', z') \to const. \Rightarrow g_b^{LW}(\gamma, z) \to 1/\gamma$.
By a continuously iterating, one sees that $g_b^{LW}(\gamma, z)$ decreases faster than any power of $1/\gamma$!!!!

Factorized form $g_b^{LW}(\gamma', z') = f_b^{LW}(z')\delta(\gamma' - \epsilon)$ ($\epsilon \geq 0$)

$$g_b^{LW}(\gamma, z) = \frac{g^2}{2(4\pi)^2}\ \delta(\gamma - \epsilon) \int_{-1}^{+1} dz' \frac{f_b^{LW}(z')}{\left[z'^2 \frac{M^2}{4} + \kappa^2\right]} \left[\frac{(1+z)}{(1+z')}\ \theta\left(z' - z\right) + \frac{(1-z)}{(1-z')}\ \theta\left(z - z'\right)\right]$$

Nakanishi PTIR for Bound State 2+1

V. Gigante, TF, C. Guitierrez, L. Tomio, FBS (2015)

$$\int_0^\infty d\gamma' \frac{g_B\left(\gamma', z, \kappa^2\right)}{\left[\gamma' + \gamma + z^2 m^2 + (1-z^2)\,\kappa^2\right]^2} = \int_{-1}^{1} dz' \int_0^\infty d\gamma' V(z, z', \gamma, \gamma') g_B(\gamma', z', \kappa^2).$$

$$V(z, z', \gamma, \gamma') = \frac{g^2}{24\pi^{\frac{3}{2}}} \Gamma\left(\frac{5}{2}\right) \frac{1}{[\gamma + (1-z^2)\kappa^2 + z^2 m^2 - i\epsilon]} \frac{1}{\left[\gamma' + z'^2 m^2 + (1-z'^2)\kappa^2\right]^{\frac{3}{2}}}$$
$$\times \left[\frac{1+z}{1+z'}\theta(z'-z)F(z, z', \gamma, \gamma') + \frac{1-z}{1-z'}\theta(z-z')F(-z, -z', \gamma, \gamma')\right]$$

$$F(z, z', \gamma, \gamma') = \frac{2\frac{1+z}{1+z'}\left(z'^2 \frac{M^2}{4} + \kappa^2 + \frac{3}{2}\gamma'\right) + \gamma + m^2 z^2 + \kappa^2(1-z^2)}{\left[\gamma + \frac{1+z}{1+z'}\gamma' + z^2 m^2 + (1-z^2)\kappa^2\right]^2}$$

LADDER APPROX.

Uniqueness 2+1

$$g_B^{(1)}\left(\gamma,z;\kappa^2\right)=\frac{g^2}{6\pi^{\frac{5}{2}}}\Gamma\left(\frac{5}{2}\right)\int_0^\infty d\gamma^{'}\int_{-1}^{1}dz^{'}g_B^{(1)}\left(\gamma^{'},z^{'};\kappa^2\right)\left[\theta(z^{'}-z)\Lambda(z,z^{'},\gamma,\gamma')+\theta(z-z^{'})\Lambda(-z,-z^{'},\gamma,\gamma')\right]$$

$$\Lambda(z,z^{'},\gamma,\gamma^{'})=\left(\frac{1+z}{1+z^{'}}\right)^{\frac{5}{2}}\frac{d^2}{d\gamma^2}\int_0^1 dv\,\frac{v^2}{[v(1-v)]^{\frac{5}{2}}}\int_0^\infty dw\int_0^1 d\eta\;\eta^2\;\delta\left[\gamma-\eta w^2-\eta\Gamma(v,z,z^{'},\gamma^{'})\right]$$

$$\Gamma(v,z,z^{'},\gamma^{'})=\frac{1+z}{1+z^{'}}\left[\frac{v}{1-v}\left(z^{'2}\frac{M^2}{4}+\kappa^2+\gamma^{'}\right)+\frac{\mu^2}{v}+\gamma^{'}\right]$$

Numerical method

$$g_b^{(Ld)}(\gamma, z; \kappa^2) = \sum_{\ell=0}^{N_z} \sum_{j=0}^{N_g} A_{\ell j} G_\ell(z) \mathcal{L}_j(\gamma).$$

$$G_\ell(z) = 4(1-z^2)\Gamma(5/2)\sqrt{\frac{(2\ell+5/2)(2\ell)!}{\pi\Gamma(2\ell+5)}} C_{2\ell}^{(5/2)}(z).$$

even Gegenbauer polynomials

$$\mathcal{L}_j(\gamma) = \sqrt{a}\, L_j(a\gamma) e^{-a\gamma/2}$$

Laguerre polynomials

Solution of the eigenvalue problem for g^2 for each given B

$B=2m-M$ binding energy

Coupling constant (3+1) vs. Binding

Testing Uniqueness [Frederico, Salmè, Viviani PRD89, 016010 (2014)]

TABLE II. Values of $\alpha = g^2/(16\pi m^2)$ obtained by solving the eigenequations (32) (i.e., with the application of the uniqueness theorem) and (29). Results correspond to $\mu/m = 0.50$ varying the binding energies, B/m. The second column shows the values obtained in Ref. [3], where the uniqueness theorem was exploited and an iterative method was adopted; the third column corresponds to the solution of Eq. (32) by using our basis [cf. Eqs. (38)–(40)]; the fourth column contains our results from Eq. (29).

$\mu/m = 0.50$			
B/m	α[3]	α Eq. (32)	α Eq. (29)
0.002	1.211	1.216	1.216
0.02	1.624	1.623	1.623
0.20	3.252	3.251	3.251
0.40	4.416	4.415	4.416
0.80	6.096	6.094	6.094
1.20	7.206	7.204	7.204
1.60	7.850	7.849	7.849
2.00	8.062	8.061	8.061

Ladder approx.

[3] K. Kusaka, K. Simpson, and A. G. Williams, Phys. Rev. D **56**, 5071 (1997).

Valence Probability 3+1

$\mu/m = 0.50$

B/m	α	P_{val}
0.001	1.167	0.98
0.01	1.440	0.96
0.10	2.498	0.87
0.20	3.251	0.83
0.50	4.900	0.77
1.00	6.711	0.74
2.00	8.061	0.72

Coupling constant (2+1) vs. Binding

B/m	$\mu = 0.1$	Eucl.	$\mu = 0.5$	Eucl.
0.01	0.82	0.79	5.33	5.31
0.1	4.26	4.26/4.268†	14.88	14.88
0.2	8.07	8.06	22.67	22.67
0.5	19.50	19.51	42.33	42.33
1	36.05	36.03/36.052 †	67.38	67.39

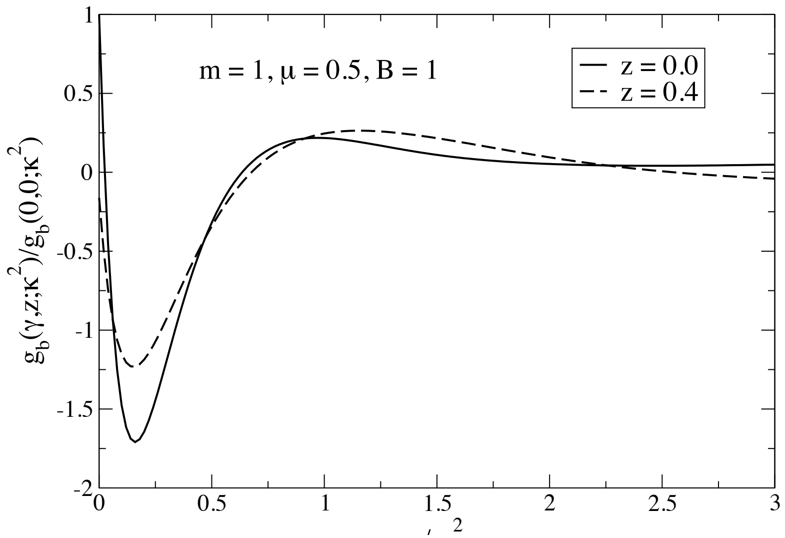
Ladder approx.

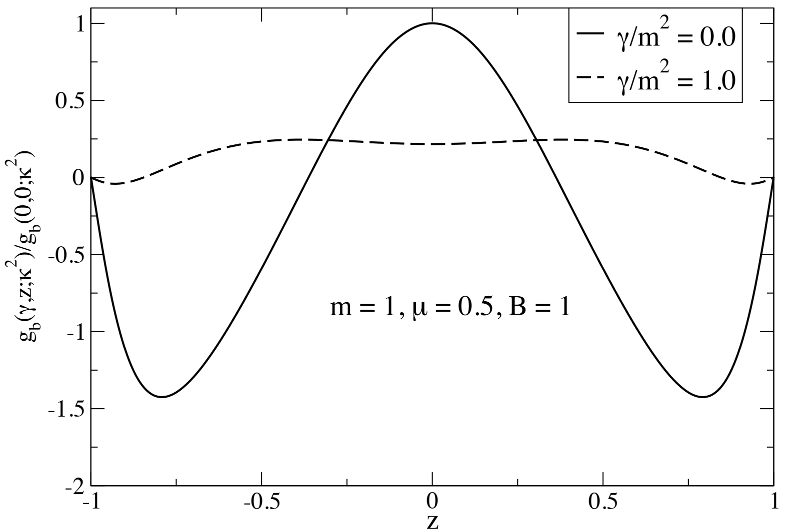
n=1

TABLE I. Values of g^2/m^3 calculated with ladder approximationfor different binding energies B and exchanged boson masses μ. Comparison with Euclidean space calculations including from Nieuwenhuis and Tjon, Few-Body Syst. 21, 167 (1996) (†).

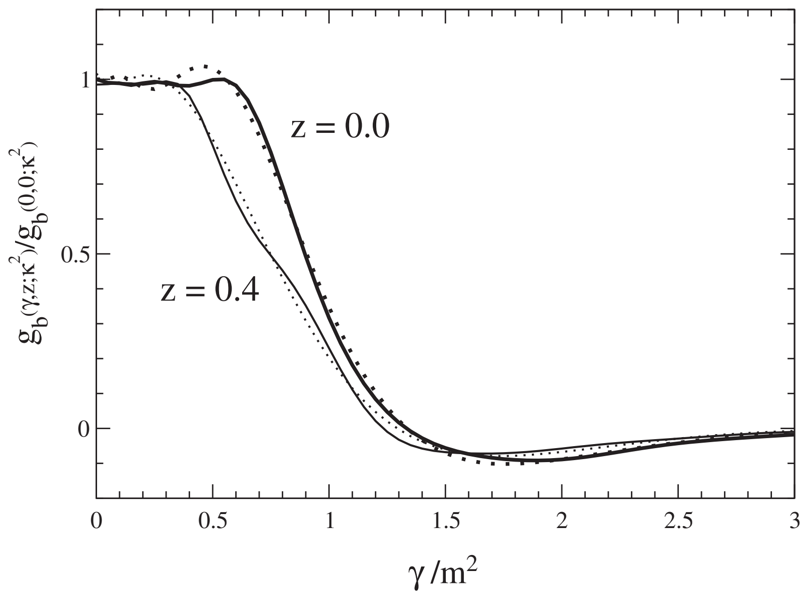
Nakanishi weight function

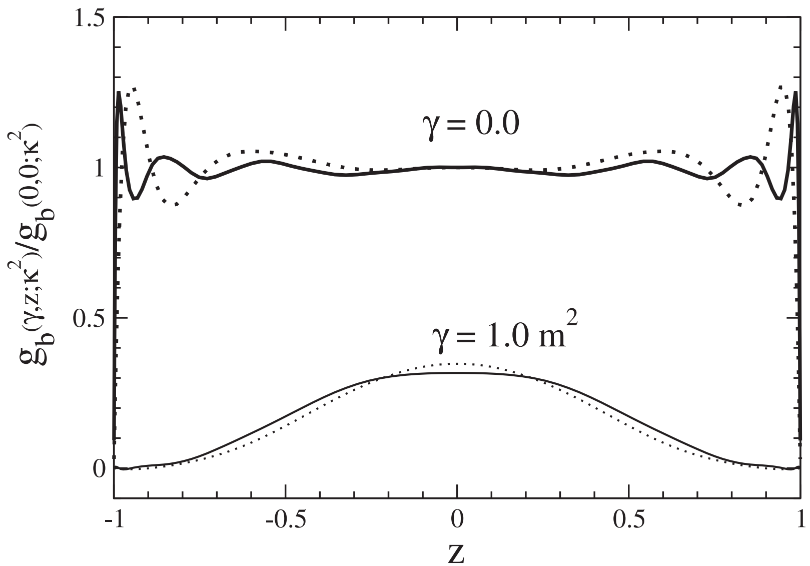
2+1 n=1





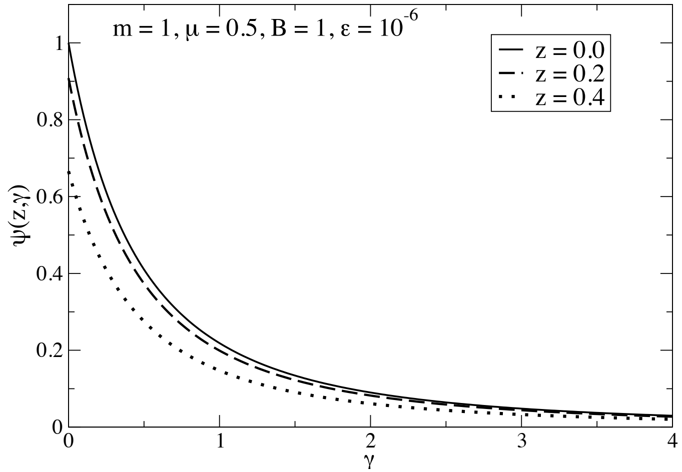
3+1 n=1

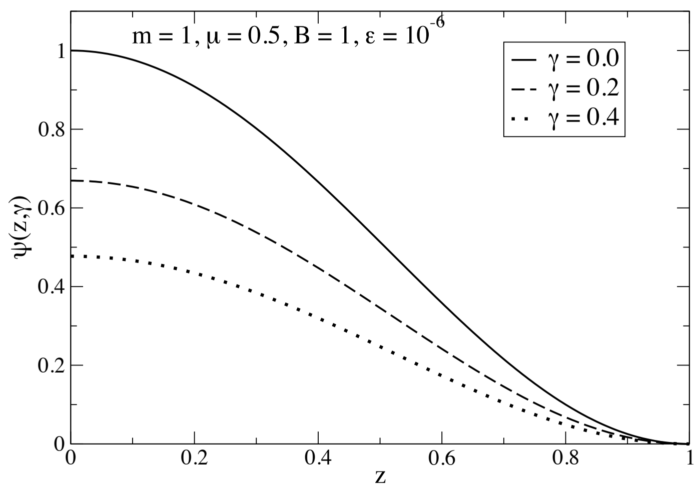




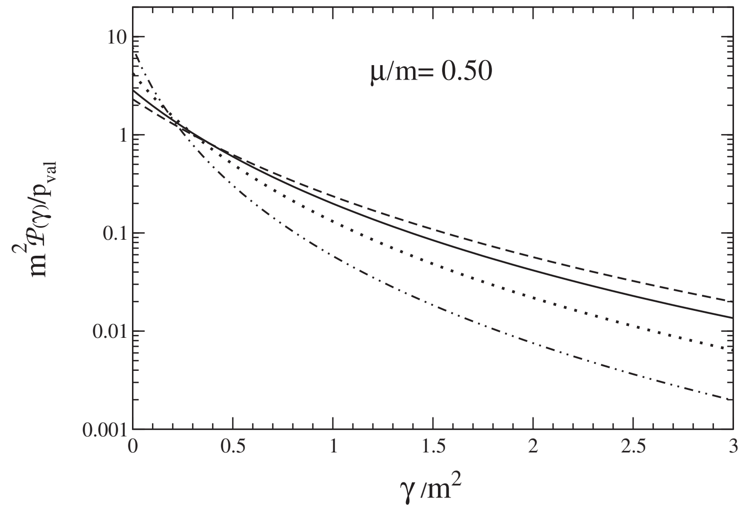
Valence wave function

2+1 n=1





3+1 n=1



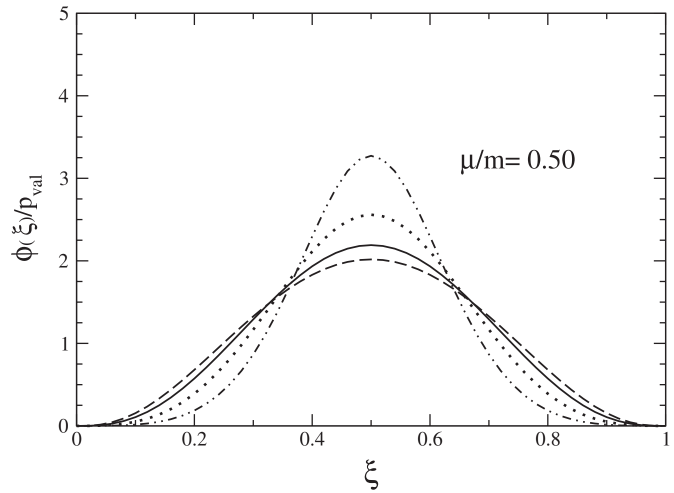


FIG. 3. The longitudinal LF distribution $\phi(\xi)$ for the valence component Eq. (34) vs the longitudinal-momentum fraction ξ for $\mu/m = 0.05$, 0.15, 0.50. Dash-double-dotted line: $B/m = 0.20$. Dotted line: $B/m = 0.50$. Solid line: $B/m = 1.0$. Dashed line: $B/m = 2.0$. Recall that $\int_0^1 d\xi \phi(\xi) = P_{\rm val}$ (cf. Table III).

The states in the continuum:
projection onto LF of the scattered part of the BS amplitude

$$\Phi^{(+)}(k,p) = (2\pi)^4\delta^{(4)}(k-k_i) + G_0^{(12)}(k,p)\int \frac{d^4k'}{(2\pi)^4}\, i\,\mathcal{K}(k,k',p)\Phi^{(+)}(k',p),$$

Scattered part of the valence wave function :

$$\int \frac{dk^-}{2\pi}\left[\Phi^{(+)}(k,p) - (2\pi)^4\delta^{(4)}(k-k_i)\right] = \int \frac{dk^-}{2\pi} G_0^{(12)}(k,p)\, i\mathcal{K}(k,k_i,p) +$$

$$+\int \frac{dk^-}{2\pi} G_0^{(12)}(k,p)\int \frac{d^4k'}{(2\pi)^4}\, i\,\mathcal{K}(k,k',p)\left[\Phi^{(+)}(k',p) - (2\pi)^4\delta^{(4)}(k'-k_i)\right]$$

Scattering Eq. for the Nakanishi weight function:

$$\int_{-1}^{1} dz' \int_{-\infty}^{\infty} d\gamma' \frac{g^{(+)}(\gamma',z',z;\gamma_i,z_i)}{[\gamma'+\gamma+z^2m^2+(1-z^2)\kappa^2+\frac{M}{2}z\,z'(\frac{M}{2}z_i+k_i^-)+2z'\cos\theta\sqrt{\gamma\gamma_i}-i\epsilon]^2} =$$

$$= \mathcal{I}^{LF}(\gamma,z;\gamma_i,z_i,\cos\theta) + \int_{-\infty}^{\infty} d\gamma' \int_{-1}^{1} d\zeta \int_{-1}^{1} d\zeta'\, V_S^{LF}(\gamma,z;\gamma_i,z_i,\gamma',\zeta,\zeta',\cos\theta)\, g^{(+)}(\gamma',\zeta,\zeta';\gamma_i,z_i).$$

$\Rightarrow$ full scattering amplitude directly from the valence wave function!

($\gamma_i = |\mathbf{k}_{i\perp}|^2$ is the incident transverse momentum.)

The scattering amplitude

$$f(s,\theta) = -\frac{\boldsymbol{i}}{M\,8\pi} \lim_{\tilde{k}'\to\tilde{k}_f} \langle \tilde{k}' | g_0^{-1}(p) | \phi_{LF}^{(+)}; p, \tilde{k}_i \rangle$$

is explicitly given by:

$$f(s,\theta) = \frac{\textbf{-1}}{M\,8\pi} \lim_{(\gamma,z)\to(\gamma_f,z_f)} \frac{p^+}{4}(1-z^2)\left(M^2 - 4\frac{m^2+\gamma}{1-z^2}\right) \phi_{LF}^{(+)}(z,\gamma,cos\theta) =$$

$$= \frac{\textbf{-1}}{M\,8\pi} \lim_{(\gamma,z)\to(\gamma_f,z_f)} \left(M^2 - 4\frac{m^2+\gamma}{1-z^2}\right) \psi^{(+)}(z,\gamma,cos\theta) \;,$$

The scattering amplitude

$$f(s,\theta) = +\frac{\textbf{1}}{M\,8\,\pi} \lim_{(\gamma,z)\to(\gamma_f,z_f)} \left[\gamma + (1-z^2)\kappa^2 + z^2 m^2\right] \Bigg[\;\mathcal{I}^{LF}(\gamma,z;\gamma_i,z_i) +$$

$$+ \int_{-\infty}^{\infty} d\gamma' \int_{-1}^{1} d\zeta \int_{-1}^{1} d\zeta' \; V_s^{LF}(\gamma,z;\gamma_i,z_i,\gamma',\zeta,\zeta',cos\theta) g^{(+)}(\gamma',\zeta,\zeta';\gamma_i,z_i) \Bigg] \;,$$

where $\gamma_f = \gamma_i$ and $z_f = z_i$.

- Notice that the factor $\gamma + (1-z^2)\kappa^2 + z^2 m^2$ vanishing for $(\gamma, z) \to (\gamma_f, z_f)$, is canceled out by the corresponding one in $\mathcal{I}^{LF}$ and V_s^{LF}.

Support of $g^{(+)}(\gamma', ...)$

For $M \to 2m$ only $g^{(+)}(\gamma' > 0, ...)$ survives in the ladder approximation

Zero-energy scattering ($\kappa^2 = 0$ & $M = 2m$): Ladder approx.

$$
\begin{aligned}
g^{(+)}(\gamma, z) \quad = \quad & \frac{g^2}{\mu^2}\,\theta(\gamma)\left[\theta(z)\,\theta(1 - z - \gamma/\mu^2) + \theta(-z)\,\theta(1 + z - \gamma''/\mu^2)\right] \\
+ \quad & \frac{g^2}{2(4\pi)^2}\int_0^\infty d\gamma' \int_{-1}^{1} dz' \left[\frac{(1+z)}{(1+z')}\,\theta(z' - z)\,h(\gamma, z; \gamma', z'; \mu^2)|_{\kappa^2=0}\right. \\
+ \quad & \left.(z \to -z, z' \to -z')\right]\, g^{(+)}(\gamma', z')
\end{aligned}
$$

$\Rightarrow$ scattering length can be obtained.

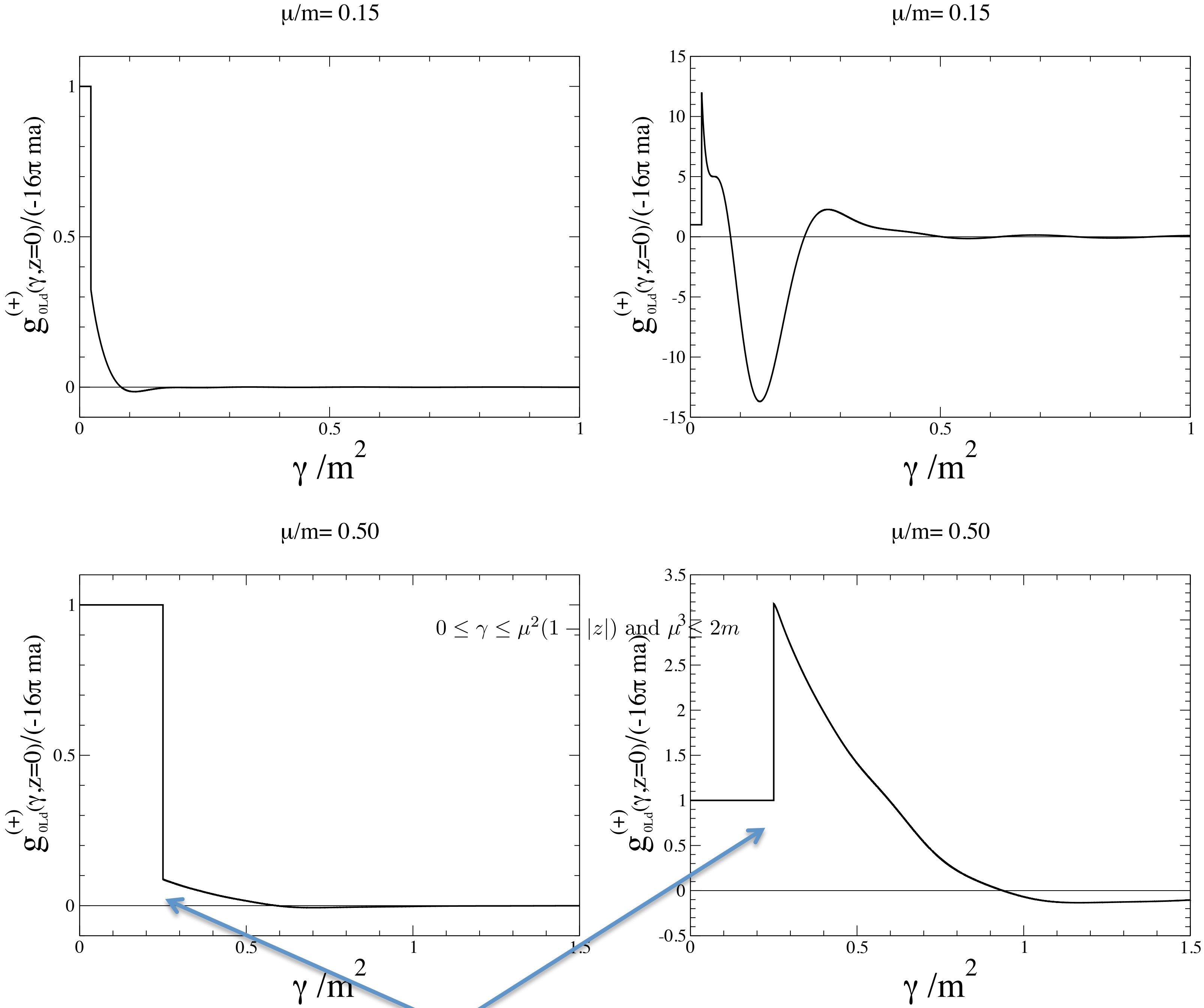
Frederico, Salmè, Viviani, arXiv:1504.01624 [hep-ph]

Carbonell & Karmanov PLB727 (2013) 319 (CK)

$\mu = 0.5$ and $m = 1$

α	a_{CK} [13]	a_{FSV}	a_{UNI}	a^{BA}
0.01	-0.0403	-0.0403	-0.0403	-0.04
0.05	-0.209	-0.209	-0.209	-0.20
0.10	-0.438	-0.438	-0.438	-0.40
0.20	-0.971	-0.971	-0.971	-0.80
0.30	-1.64	-1.64	-1.64	-1.20
0.40	-2.50	-2.50	-2.50	-1.60
0.50	-3.66	-3.66	-3.66	-2.00
0.60	-5.34	-5.34	-5.34	-3.60
0.70	-7.98	-7.99	-7.98	-2.80
0.80	-12.8	-12.8	-12.8	-3.20
0.90	-24.7	-24.7	-24.8	-3.60
1.00	-103.0	-103.2	-103.0	-4.00
1.10	62.0	61.9	61.8	-4.40
1.50	11.0	11.0	11.0	-6.00
2.00	6.34	6.34	6.34	-8.00
2.50	4.54	4.53	4.53	-10.00

$\alpha = 0.1$ $\qquad$ $\alpha = 2.5$



$$0 \leq \gamma \leq \mu^2(1 - |z|) \text{ and } \mu \leq 2m$$

Prospects in 3+1

- Nakanishi PTIR for mesons and barions: higher Fock-states
- Quark self-energy, SD and qq-scattering
- Form-Factors, GPD's…
- Relativistic few-nucleon systems $nn\,,\, nnn\cdots$
- Relativistic few-meson systems $\pi\pi\,,\, K\pi\,,\, \pi\pi$
- Heavy meson decay & FSI
- CP violation in $B^{\pm}\rightarrow\pi^{\pm}\pi^{+}\pi^{-}\,,\; K^{\pm}\pi^{+}\pi^{-}\cdots$

(Bediaga, TF, Lourenço PRD89 (2014) 094013)

Prospects in 2+1

- Scattering e-h→e-h and effect on the conductivity
- Electron Self-energy Schwinger-Dyson - nanoribbons
- Light-front Bethe-Salpeter equation excitons
- Curved surfaces and excitons + Nakanishi
- Raman spectroscopy with relativistic models …
- ….

THANK YOU!