Topological Solitons
vs
Topological Defects

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Abstract

Topological particle-like solutions to be found in realistic field theories under nonperturbative approach are divided in 2 classes: topological defects (TD) and topological solitons (TS). We exemplify and compare such solutions in D=2 and D=3. Soliton analog of Abrikosov-Nielsen-Olesen strings-vortices are presented. We note that Weinberg-Salam EW theory allows in principle existence of 3D topological solitons in its bosonic sector

Introduction

- Importance of nonperturbative effects in QCD is widely accepted: Confinement and SSB are essentially nonperturbative effects. Necessity of nonperturbative approaches is due to essential nonlinearity of Yang-Mills field.

- $SU(2)$ Yang-Mills field is an essential ingredient of Weinberg-Salam EW theory. Again, its nonlinearity makes nonperturbative study necessary if one is interested in complete study of physical picture which corresponds to Standard Model Lagrangian. In particular, one can hope to get answer for the old O.Rabi's question: "Who ordered this ?" –(about discovery of muon). Thorough nonperturbative (i.e. lattice) study of EW theory is thus highly desirable before going beyond the Standard Model.

- Localized extended solutions (both defects and solitons) are nonanalytical in coupling constant g; thus their study can provide one with valuable nonperturbative information.

Definitions

- Both topological defects, TD and topological solitons, TS, describe particle-like (extended localized, lumps) distributions of field energy, but they (TDs and TSs) differ in topological properties:

- Solitons are uniform at space infinity, $R \to \infty$, field distributions of *all* fields involved. For TSs topological charge (index) is a mapping degree of the field distribution inside infinite radius ($R = \infty$) sphere, which can be considered as the single point - because of constancy of all fields on it. Space R^D is compactified by adding this infinite point, and thus soliton maps $R^D_{comp} \to S^N$.

- Defects are given by field distributions, which are nonuniform at $R = \infty$. Their topological indices are mapping degrees of the sphere with $R = \infty$ set by the field distribution on this sphere, $S^{D-1} \to S^N$.

- Thus Topological Defects ARE NOT Topological Solitons, and vice versa, Topological Solitons ARE NOT Topological Defects.

Examples of TS end TD

TDs D=1 : sine-Gordon kinks

D=2: Abrikosov-Nielsen-Olesen strings-vortices

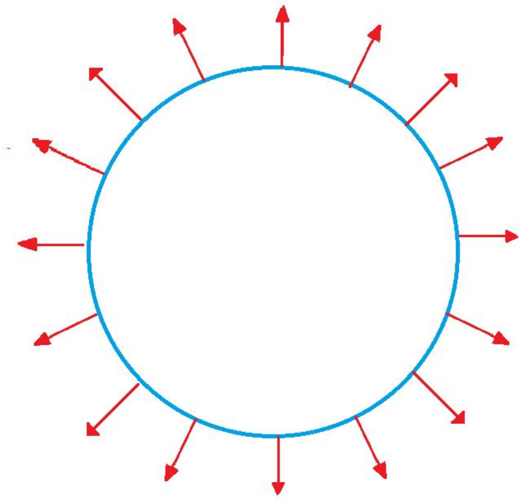
D=3: 't Hooft - Polyakov monopoles - hedgehogs

TSs D=2: Belavin-Polyakov solitons (instantons),

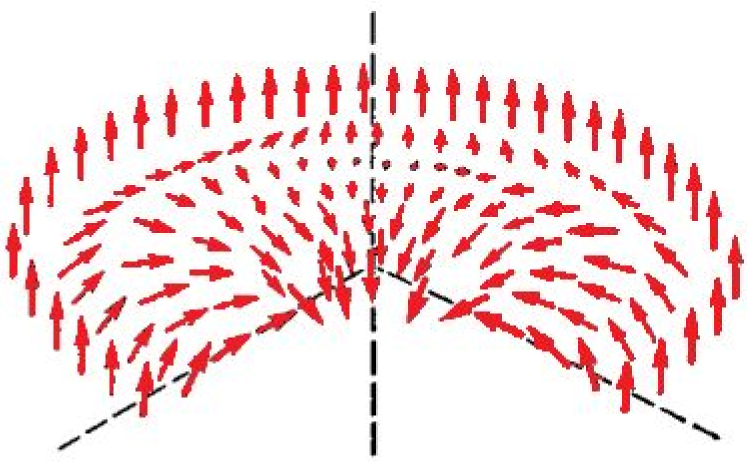
solitons in 2D Heisenberg magnets, "Baby-skytmions"

D=3: Skyrmions, solitons in 3D Heisenberg magnets

Examples ,D=2



Defects: nonuniform distributions at space infinity



Solitons: uniform distributions at space infinity

Examples of Top. Solitons

- $D = 2$, Nonlinear sigma model (NLSM), Heisenberg magnet, isovector scalar field.

  $$\mathcal{L} = (\partial_\mu s^a)^2, \mu = 0, 1, 2, s^a s^a = 1, a = 1, 2, 3,$$

  s^a is a 3-component unit isovector.

  Boundary condition at $R = \infty, R^2 = x^2 + y^2$: $s^a(\infty) = s_0^a$, i.e.$s_0^a = (0, 0, 1)$, or $s_0^a = (0, 0, -1)$. Topological charge Q_{top} is an index of mapping $R^2_{comp} \to S^2$.

  Extended solutions: Belavin-Polyakov 2D topological solitons with $Q_{top} = m$.

- $D = 3$ Skyrme model of baryons, also NLSM, but 4-component one. Scalar SU(2)-valued field $u^a, u^a u^a = 1, a = 1, 2, 3, 4$. Boundary condition at $R = \infty, R^2 = x^2 + y^2 + z^2, u^a(\infty) = u_0^a$, i.e. $u_0^a = (0, 0, 0, 1)$, or $u_0^a = (0, 0, 0, -1)$.

  Topological charge Q_{top} is an index of mapping $R^3_{comp} \to S^3$.

  Extended solutions: Skyrmions,top. solitons with $Q_{top} = m$

Examples of Top. Defects, D=2

- $D = 2$ Abelian Higgs model (NLSM), U(1) gauged complex scalar model

  $$\mathcal{L} = |\mathcal{D}_\mu \phi|^2 - \frac{1}{4} F_{\mu\nu}^2 - V(\phi), \mu = 0, 1, 2,$$

  ϕ is a complex scalar, $V(\phi)$ is a well-known Higgs potential.

- Topological charge Q_{top} is an index of mapping of sphere S^1 of infinite radius , $S^1 \to S^1$.

- Boundary condition for $Q_{top} = 1$ at $R = \infty, R^2 = x^2 + y^2 : (\phi_1 + i\phi_2)(\infty) = x/R + iy/R,$ (needles of Higgs field directed along radius-vector, nonuniformity !)

- Extended ANO solutions (Abrikosov-Nielsen-Olesen strings-vortices) exist for various Q_{top}, they are topological defects, the quasi-Higgs field is nonuniform at spatial infinity. But hamilonian density IS localized.

  Wide applications for cosmic string discussion. However problems with matching 2 and more defects in physically acceptable way (see Fig.)

Examples of Top. Defects, D=3

- $D = 3$ Georgi-Glashow EW model, SO(3) isovector scalar model gauged by SU(2) Yang-Mills field

$$\mathcal{L} = \mathcal{D}_\mu \phi \mathcal{D}^\mu \phi - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} - V(\phi^a \phi^a),$$

$\mu = 0, 1, 2, \quad a = 1, 2, 3,$ ϕ^a is 3-component isovector scalar, $V(\phi^a \phi^a)$ is a well-known Higgs potential.

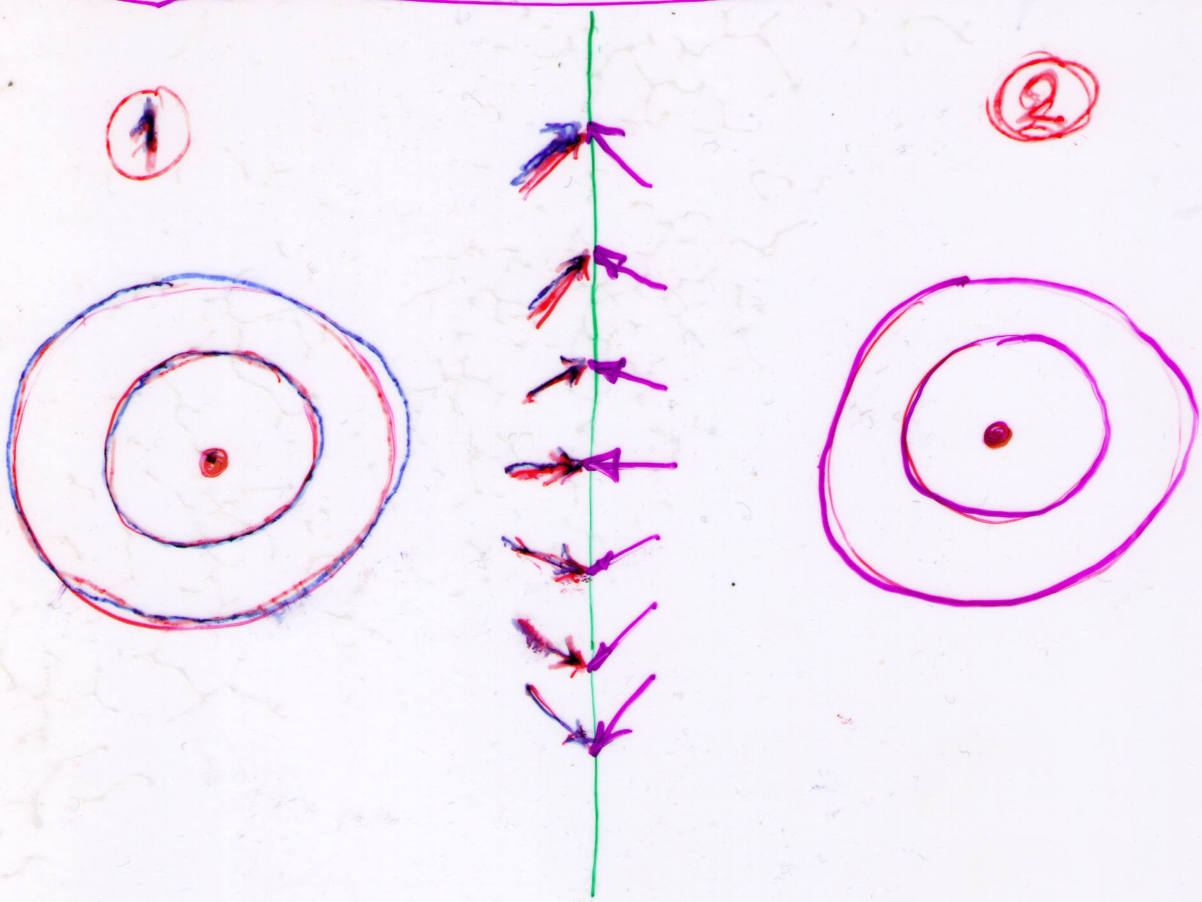
- Topological charge Q_{top} is an index of mapping of sphere S^2 of infinite radius , $S^2 \rightarrow S^2$.

- Boundary condition for $Q_{top} = 1$ at $R = \infty, R^2 = x^2 + y^2 + z^2 : (\phi^1, \phi^2, \phi^3)(\infty) = (x/R, y/R, z/R)$, (needles of Higgs field directed along radius-vector, again nonuniformity !)

- Extended solutions ('t Hooft-Polyakov monopoles-hedgehogs) exist for various Q_{top}, they are topological defects, the quasi-Higgs field is nonuniform at spatial infinity. But hamilonian density IS localized.

  Again problems with matching 2 and more defects in physically acceptable way.

Technical problems with t. defects



Possible ways out

1. To apply local gauge transformations
2. To use "junctions" in between defects
3. To consider multi-defects configurations

There is no such problem with top. solitons

Top. Solitons vs Defects

- Ways of overcoming 'matcing problems' for 2 and more well-separated defects:
  (i) inserting 'junctions' in between defects,
  (ii) setting 'multi-defects' configurations.

  $\Rightarrow$ Weakness of (i) way : it is already another set of initial problem.
  $\Rightarrow$ Weakness of (ii) way : even for infinite spatial separation one obtains correlated defects, it is not what we would like to have (say, as initial data for Cauchy problem).

- Natural question: are there soliton analogs of ANO strings-vortices in $D = 2$ and of 't Hooft-Polyakov monopoles-hedgehogs in $D = 3$?

- The answer in $D = 2$ is positive and is given by $2D$ topological solitons of the 'A3M' model.

- The answer for $D = 3$ case will hopefully be found by thorough nonperturbative investigation of bosonic sector of Weinberg-Salam EW Lagrangian.

Top. Solitons in A3M model (1)

- Instead of complex scalar field in Abelian Higgs model (AHM) we study 3-component isovector scalar field $s^a(x)$ taking values on unit sphere $S^2 : s^a s^a = 1$, having however selfinteraction of so-called 'easy-axis' type (well-known in magnetism theory). Similar to AHM introduce gauge-invariant interaction of this field with Maxwell field, making global $U(1)$ symmetry of easy-axis magnets local one. As a results we arrive at $A3M$ model, first introduced and studied in PLB'97 paper (IB and A.Bogolubskaya)

$$\mathcal{L} = \left(\bar{\mathcal{D}}_\mu s_- \mathcal{D}^\mu s_+ + \partial_\mu s_3 \partial^\mu s_3\right) - V(s_a) - \frac{1}{4} F_{\mu\nu}^2, \tag{1}$$

$$\bar{\mathcal{D}}_\mu = \partial_\mu + igA_\mu, \quad \mathcal{D}_\mu = \partial_\mu - igA_\mu,$$

$$s_+ = s_1 + is_2, \quad s_- = s_1 - is_2,$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad V(s_a) = \beta^2(1 - s_3^2),$$

$\mu, \nu = 0, 1, ..., D,$ This NLSM model is the gauge-invariant extension of classical Heisenberg antiferromagnet model with easy-axis anisotropy. This model supports $D = 2$ topological solitons, which can be found using the following ansatz:
vortex – for the Maxwell field,
hedgehog – for scalar Heisenberg field.

Top. Solitons in A3M model (2)

- Topological charge of $A3M$ solitons is defined as mapping degree of $s^a(x)$ 3-component Heisenberg field distribution inside infinite radius ($R = \infty$) sphere, $R^2_{comp} \rightarrow S^2$.

  $A3M$ solitons exist for integer Q_{top}- similar to Belavin-Polyakov $2D$ solitons in *isotropic* Heisenberg magnet.

- Boundary conditions correspond to uniform distribution of the $s^a(x)$ field at $R = \infty$, and zero value of Maxwell field $A_\mu(x)$ at space infinity.

- Energy of 2 A3M solitons with $Q_{top} = 1$ proves to be greater than energy of 1 soliton with $Q_{top} = 2$. As a result 2 such solitons attract to each other and coalesce into 1 $Q_{top} = 2$ soliton.

- Beautiful, even unique, mathematical properties of the A3M model (2 exact results obtained in computer simulations) can most probably be accounted for its high symmetry ($U(1) \times Z(2)$). In particular, the A3M model is a 2-step generalization of well-known sine-Gordon equation.

Top. Solitons in SU2-Higgs model (1)

- Consider the simplest EW model (reduction of bosonic sector of Salam-Weinberg model), so-called SU2-Higgs model with frozen radial degree of freedom.

$$\mathcal{L} = (\mathcal{D}_\mu \Phi_b)^\dagger (\mathcal{D}^\mu \Phi_b) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$$

$\mathcal{D}_\mu \Phi_b = \partial_\mu \Phi_b + \frac{i}{2} g \tau^a A^a_\mu \Phi_b,\ \mu = 0, 1, 2, 3, \quad a = 1, 2, 3, \quad b = 1, 2,$ Φ_b is 2-component complex doublet, defined by 4 real numbers φ_c, such that $\varphi_c \varphi_c = 1, \quad c = 1, 2, 3, 4.$ Thus SU2-Higgs model describes gauge-invariant interaction of $SU(2)$ Yang-Mills with isospinor unit scalar field, taking values on S^3, this model also belongs to a class of NLSMs.

- Boundary conditions at $R = \infty, R^2 = x^2 + y^2 + z^2 : \varphi^c(\infty) = \varphi^c_0$, i.e.$\varphi^c_0 = (0, 0, 0, 1)$, or $s^a_0 = (0, 0, 0, -1)$. Topological charge Q_{top} is an index of mapping $R^3_{comp} \to S^3$ defined by distribution of isospinor scalar field $\Phi_b(x)$ inside infinite radius sphere S^3.

Top. Solitons in SU2-Higgs model (2)

- Existence of topological solitons with integer topological charge Q_{top} is not excluded. To find TSolitons one has to use
  (i) hedgehog ansatz for isospinor field with chosen Q_{top},
  (ii)Generic 3-term ansatz for $D = 3$ Yang-Mills solitons ($A_0^a = 0$):

$$gA_i^a = \varepsilon_{iak}\frac{x_k}{R^2}s(R)+$$

$$+\frac{b(R)}{R^3}[(\delta_{ia}R^2 - x_i x_a) + \frac{p(R)x_i x_a}{R^4}],$$

$$i, k = 1, 2, 3 \qquad R^2 = x^2 + y^2 + z^2.$$

  Study of TSolitons in SU(2)-Higgs model is in progress.

- Note that SU(2)-Higgs model does not support topological defects.

Instead of Conclusions

- Both TDefects and TSolitons describe localized, particle-like distributions of energy density, however it seems to be the only point of their similarity :-)

- TDefects define nonuniform field distribution at space infinity (at least for one of the fields involved). This cause unavoidable problems with their matching. TSolitons are free of this problems.

- It is not advisable to use the term "solitons" for "defects", because it can lead to misunderstanding and even wrong conclusions on existence/nonexistence.

- Study of solitons within the Standard Model seems to be increasingly important and interesting for obtaining complete physical picture.

Thank you for your time!

$2D$ Solitonic Strings in a toy Electroweak Model

Motivation

- Topological defects with localized energy distributions are widely discussed in condensed matter, cosmology and particle physics models.

- Well-known example in 2D: Abrikosov-Nielsen-Olesen (ANO) strings-vortices. However there are at least technical problems when matching 2 ANO defects.

- There is no problem when matching topological solitons. So it seems interesting to find solitonic analog of ANO defects.

- Do such topological solitons exist?

- If yes, are they stable?

- How do they interact with each other?

- What about $3D$?

Easy-axis Heisenberg Antiferromagnet

- Lagrangian of the easy-axis Heisenberg field is:

$$\mathcal{L} = \partial_\mu s_a \partial^\mu s_a - V(s_a), \quad V(s_a) = \beta^2(1 - s_3^2),$$

$$s_a s_a = 1, \quad \mu, = 0, 1, ..., D, \quad a = 1, 2, 3.$$

- Note Lorentz invariance of this nonlinear σ-model, global $U(1)$ and $Z(2)$ symmetry, $Z(2)$ symmetry is spontaneously broken;
  Generalisation of sine-Gordon equation.
  Possesses $U(1)$ charged kinks for $D = 1$.

- This model can be called the $A3$ model, here A stands for "anisotropic", 3 means 3-component.

The $A3M$ model (1)

$A3M$: M stands for "Maxwell".
Now let's make $U(1)$ symmetry local .

The gauge-invariant Lagrangian density of the A3M model reads:

$$\mathcal{L} = \eta^2 \left(\bar{\mathcal{D}}_\mu s_- \mathcal{D}^\mu s_+ + \partial_\mu s_3 \partial^\mu s_3 \right) - V(s_a) - \frac{1}{4} F_{\mu\nu}^2,$$

$$\bar{\mathcal{D}}_\mu = \partial_\mu + igA_\mu, \quad \mathcal{D}_\mu = \partial_\mu - igA_\mu,$$
$$s_+ = s_1 + is_2, \quad s_- = s_1 - is_2,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad V(s_a) = \beta^2(1 - s_3^2),$$

where β^2, η^2, are constants, $\left[\eta^2\right] = L^{(1-D)}$, $\left[\beta^2\right] = L^{-(1+D)}$, g is a coupling constant, $[g^2] = L^{(D-3)}$, $\mu, \nu = 0, 1, ..., D,$. Equivalently,

$$\mathcal{L} = \eta^2 (\partial_\mu s_a)^2 - V(s_a) - \frac{1}{4} F_{\mu\nu}^2 + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = 2g\eta^2 A_\mu (s_2 \partial^\mu s_1 - s_1 \partial^\mu s_2) + g^2 \eta^2 (s_1^2 + s_2^2) A_\mu A^\mu.$$

The $A3M$ model (2)

Making rescaling $x_\mu \to g^{-1}\eta^{-1}x_\mu$, $A_\mu \to \eta^{-1}A_\mu$, we obtain the Euler-Lagrange equations of the A3M model in dimensionless form. These equations, governing evolution of the fields $s_a(x)$, $A_\mu(x)$, in $(D+1)$-dimensional space-time, take the simplest form if the Lorentz gauge, $\partial_\mu A^\mu = 0$, is chosen:

$$\partial_\mu\partial^\mu s_i + [\partial_\mu s_a \partial^\mu s_a + 2A_\mu j^\mu + p(s_3^2 - \delta_{i3})$$

$$+A_\mu A^\mu(s_1^2 + s_2^2 - \delta_{1i} - \delta_{2i})]s_i - 2A_\mu(\delta_{2i}\partial^\mu s_1 - \delta_{1i}\partial^\mu s_2) = 0,$$

$$j_\mu = s_2\partial_\mu s_1 - s_1\partial_\mu s_2,$$

$$\partial_\mu\partial^\mu A_\nu + 2j_\nu + 2(s_1^2 + s_2^2)A_\nu = 0,$$

$$\mu, \nu = 0, 1, ..., D, \quad i = 1, 2, 3$$

(we denote $p = \beta^2 g^{-2}\eta^{-4}$). Equation (4) can be rewritten using variables $s_\pm = s_1 \pm is_2$ and s_3:

$$\partial_\mu\partial^\mu s_\pm + [\partial_\mu s_a \partial^\mu s_a + 2A_\mu j^\mu + ps_3^2 - A_\mu A^\mu s_3^2]s_\pm$$

$$-2iA_\mu\partial^\mu s_\pm = 0,$$

$$\partial_\mu\partial^\mu s_3 + [\partial_\mu s_a \partial^\mu s_a + 2A_\mu j^\mu - p(1 - s_3^2) + A_\mu A^\mu(1 - s_3^2)]s_3 = 0.$$

The $A3M$ model (3)

It is instructive to present the equations of the A3M model in terms of angular variables θ, ϕ on the unit sphere S^2,

$$s_1 = \sin\theta\cos\phi, \quad s_2 = \sin\theta\sin\phi, \quad s_3 = \cos\theta.$$

As a result the Lagrangian takes the form (in rescaled x_μ, A_μ):

$$g^{-2}\eta^{-4}\mathcal{L} = \partial_\mu\theta\partial^\mu\theta$$

$$+\sin^2\theta\left[\partial_\mu\phi\partial^\mu\phi - 2A_\mu\partial^\mu\phi + A_\mu A^\mu - p\right] - \frac{1}{4}F_{\mu\nu}^2.$$

and the Euler-Lagrange equations become:

$$\partial_\mu\partial^\mu\theta + \frac{1}{2}\sin 2\theta\left[p - \partial_\mu\phi\partial^\mu\phi + 2A_\mu\partial^\mu\phi - A_\mu A^\mu\right] = 0,$$

$$\partial_\mu\left[\sin^2\theta(\partial^\mu\phi - A^\mu)\right] = 0,$$

$$\partial_\mu\partial^\mu A_\nu + 2j_\nu + 2A_\nu\sin^2\theta = 0, \quad j_\nu = -\sin^2\theta\partial_\nu\phi.$$

Now look for stationary solutions, arrive at

$$\partial_k^2\theta - \frac{1}{2}\sin 2\theta\left[p + (\partial_k\phi - A_k)^2\right] = 0,$$

$$\partial_k\left[\sin^2\theta(\partial_k\phi - A_k)\right] = 0,$$

$$\partial_k^2 A_m + 2\sin^2\theta(\partial_m\phi - A_m) = 0.$$

The A3M model(4):ansatz

Let us study localized solutions for $D = 2$ using the "hedgehog" ansatz for the unit isovector field $s_i(\mathbf{x})$, $i = 1, 2, 3$,

$$s_1 = \cos m\chi \sin\theta(r), \quad s_2 = \sin m\chi \sin\theta(r), s_3 = \cos\theta(r),$$

$$\sin\chi = \frac{y}{r}, \quad \cos\chi = \frac{x}{r}, \quad r^2 = x^2 + y^2,$$

where m is an integer number, and the "vortex" ansatz for the Maxwell field $A_\mu(\mathbf{x})$,

$$A_0 = 0, \quad A_1 = A_x = -m\alpha(r)\frac{y}{r^2}, \quad A_2 = A_y = m\alpha(r)\frac{x}{r^2}.$$

As a result we obtain equations for $\theta(r)$ and $\alpha(r)$,

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - \sin\theta\cos\theta\left[\frac{m^2(\alpha-1)^2}{r^2} + p\right] = 0,$$

$$\frac{d^2\alpha}{dr^2} - \frac{1}{r}\frac{d\alpha}{dr} + 2(1-\alpha)\sin^2\theta = 0.$$

with boundary conditions

$$\theta(0) = \pi, \quad \theta(\infty) = 0,$$

$$\alpha(0) = 0, \quad \frac{d\alpha}{dr}(\infty) = 0.$$

The A3M model (5)

Using series expansion of $\theta(r)$ and $\alpha(r)$ at $r \to 0$, we find for $m = 1$

$$\theta(r) = \pi - C_1 r + o(r),$$

$$\alpha(r) = r^2(E_1^2 - \frac{1}{4}C_1^2 r^2) + o(r^4).$$

and for $m = 2$

$$\theta(r) = \pi - C_2 r^2 + o(r^2),$$

$$\alpha(r) = r^2(E_2^2 - \frac{1}{12}C_2^2 r^4) + o(r^6).$$

These equations are helpful when searching for solutions of the problem with $m = 1$ and $m = 2$, respectively, e.g., by the shooting method.

A3M solitons (1)

Numerical investigation shows that for $0 < p < p_{cr}$ there exists a unique soliton solution; the profile functions $\theta(r)$ and $\alpha(r)$ of the $A3M$ solitons with $m = 1$ are presented below in Fig.1. The asymptotic value $\alpha_\infty = \alpha(r = \infty)$ decreases monotonically as p is increased, with $\alpha_\infty \to 1$ when $p \to 0$.

Distributions of the energy density

$$\mathcal{H}(r) = \left(\frac{d\theta}{dr}\right)^2 + \sin^2\theta \left[p + \frac{m^2(\alpha - 1)^2}{r^2}\right] + \frac{m^2}{2}\left(\frac{1}{r}\frac{d\alpha}{dr}\right)^2,$$

and the magnetic field, $-B(r) = (d\alpha/dr)/r$, for solitons with $m = 1$ are plotted below in Figs. 2 and 3. The magnetic flux of the 2D solitons of the $A3M$ model is uniquely determined by values of p and the topological charge $Q_t = m$,

$$\Phi = \int \mathbf{BdS} = \int A_k dx_k = m \int_0^{2\pi} \alpha(\infty) d\varphi = 2\pi m \alpha_\infty(p).$$

A3M solitons (2)

Radial functions $\theta(r)$ and $\alpha(r)$ have been found by numerical solving of boundary value problem for various $0 < p < p_{cr} \approx 0.41$.

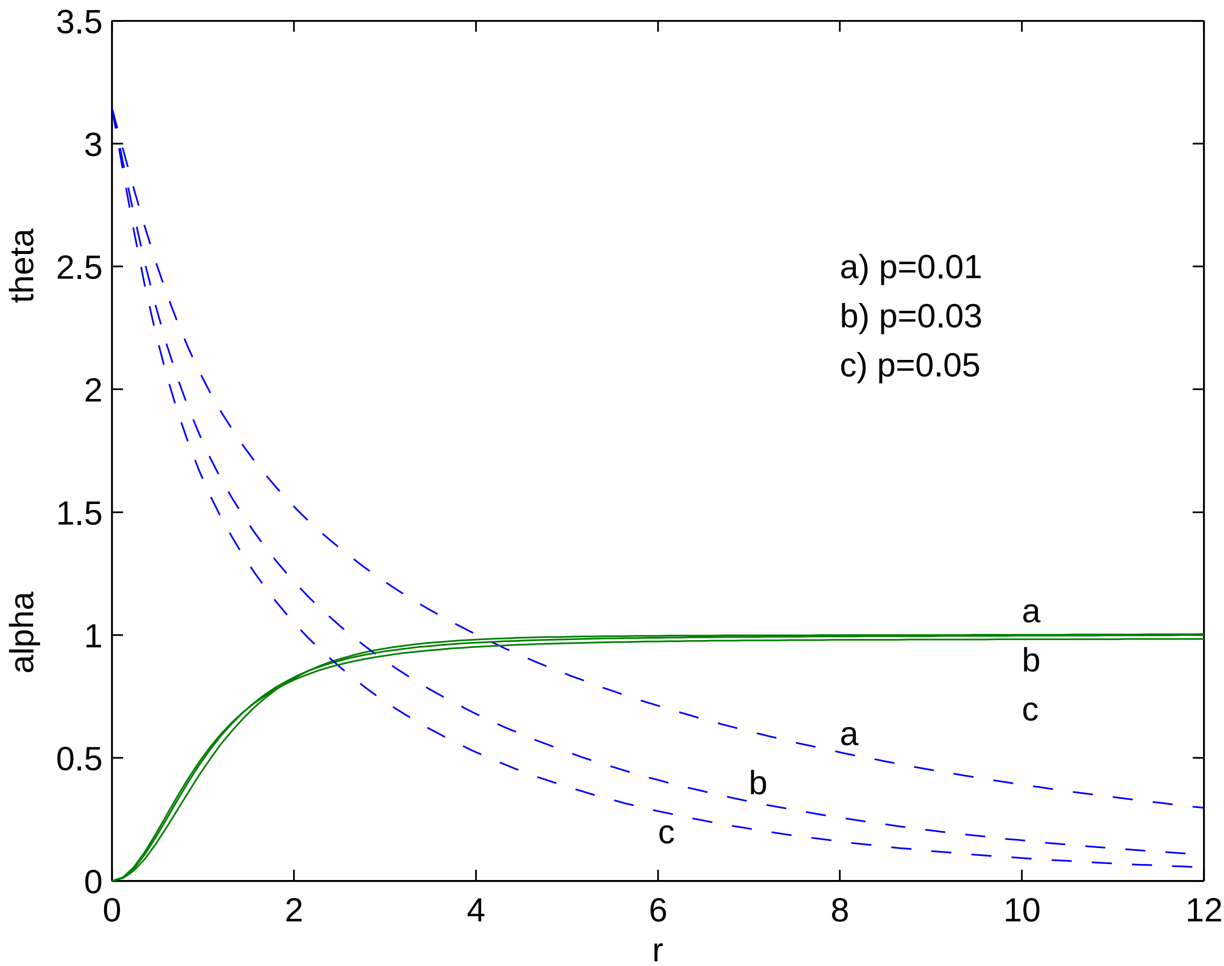


Figure 1: $\theta(r)$ and $\alpha(r)$ of the $A3M$ solitons

A3M solitons (3)

Energy density distributions $\mathcal{H}(r)$ have been found numerically from $\theta(r)$ and $\alpha(r)$ for various $0 < p < p_{cr} \approx 0.41$.

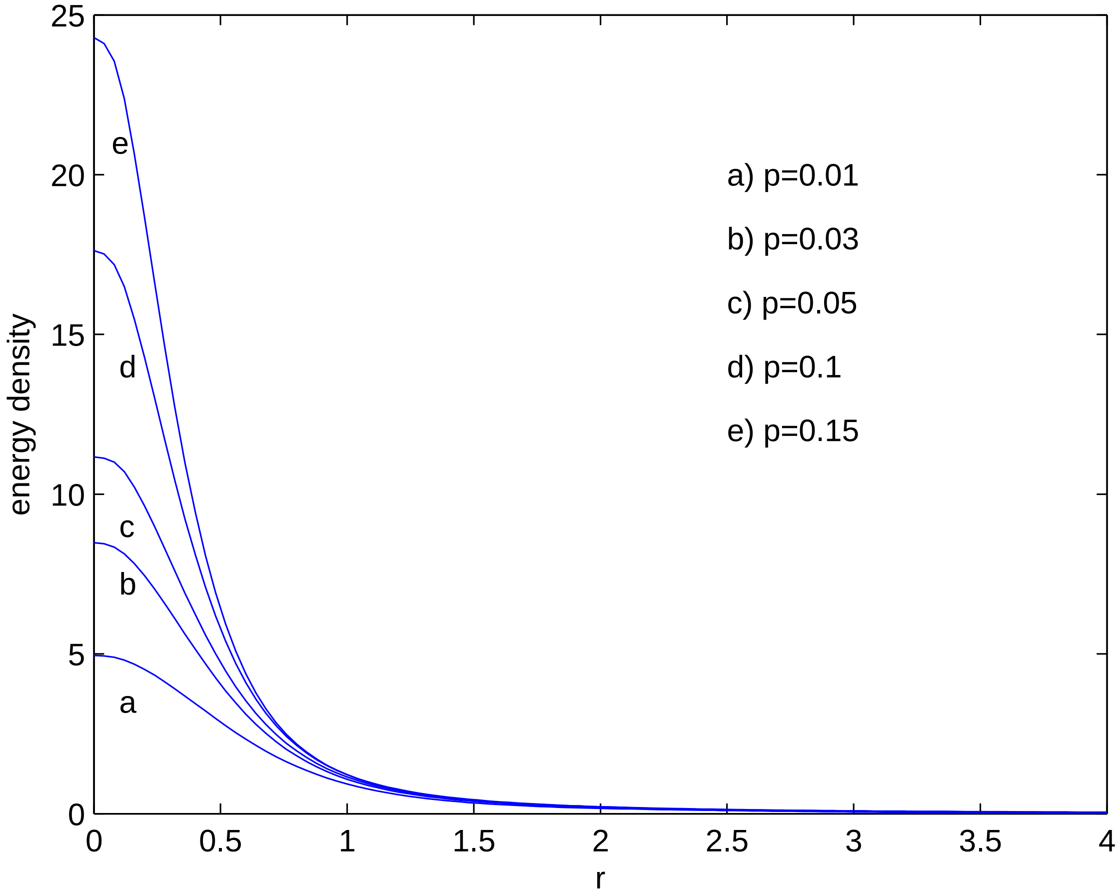


Figure 1: Profiles of energy density $\mathcal{H}(r)$ of the $A3M$ solitons

A3M solitons (4)

Magnetic field distributions $B(r)$ have been found numerically from $\alpha(r)$ for various $0 < p < p_{cr} \approx 0.41$.

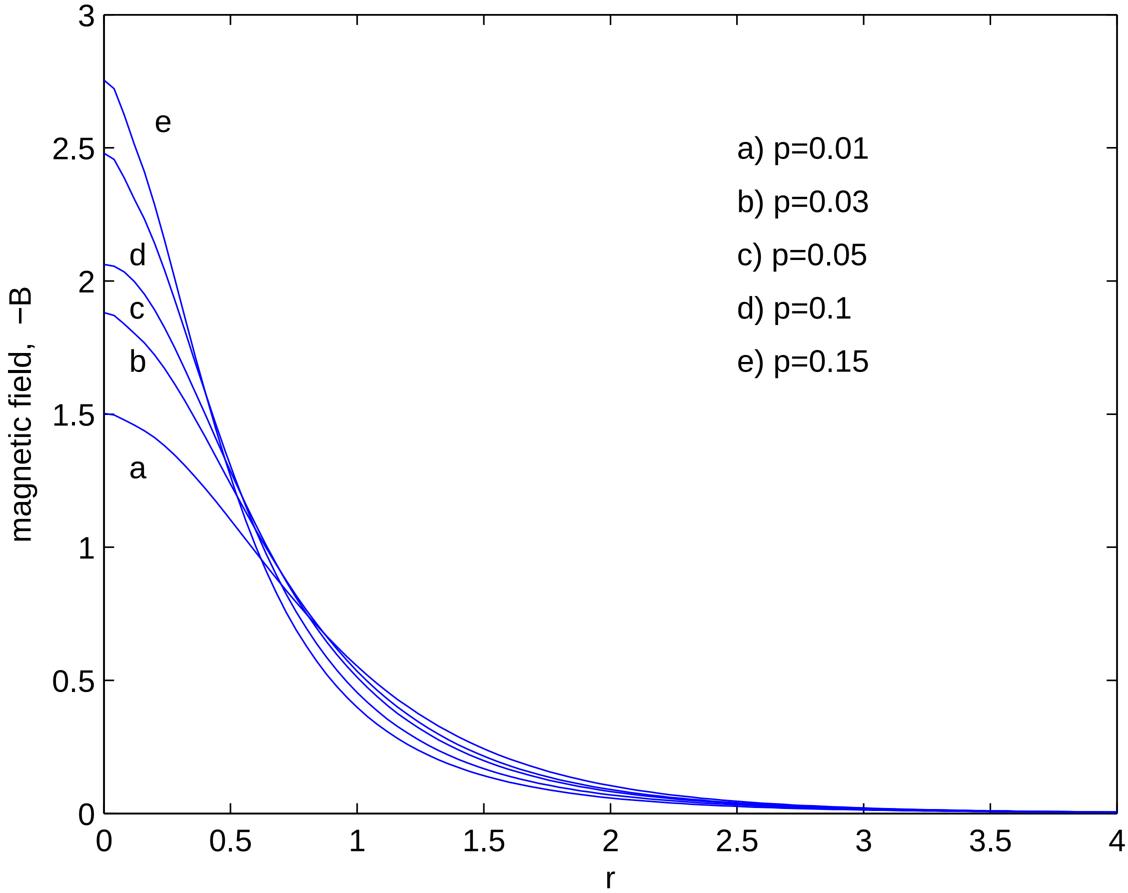


Figure 1: Profiles of magnetic field $B(r)$ of the $A3M$ solitons

A3M solitons(5): surprise

Dependence of α_∞ on p has been found from series of numerical simulations at various p.

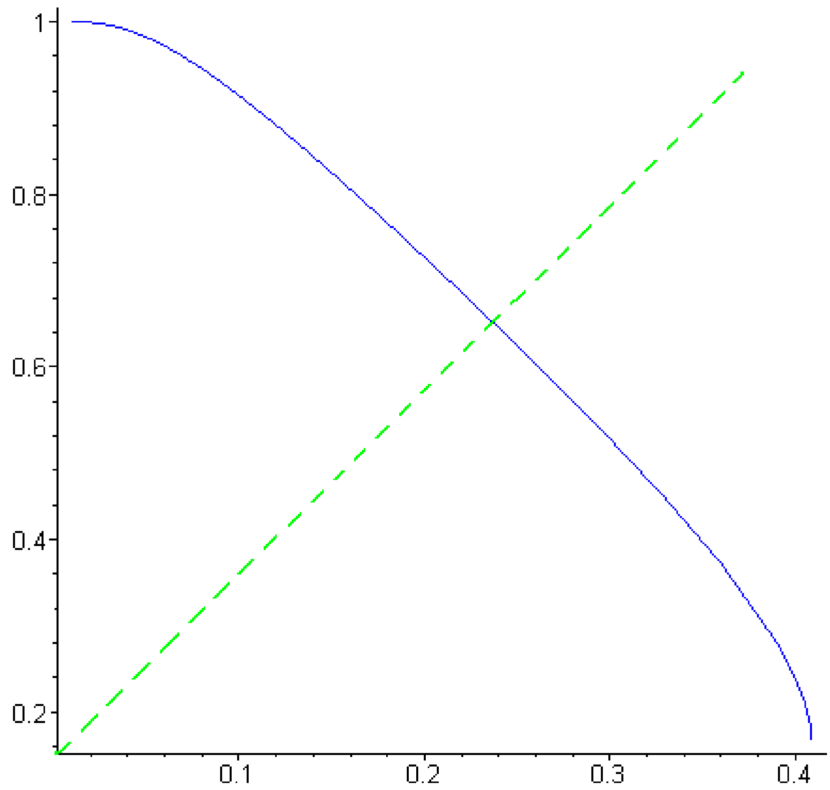


Figure 1: α_∞ versus p for A3M solitons

Why such surprising symmetry of the plot?
The only explanation available at the moment:
"Due to exclusive symmetry of the A3M model! "

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Thus, EXACT result found by numerical simulations!