Few-Nucleon Systems in the Bethe-Salpeter Approach

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We use the relativistic covariant Bethe-Salpeter approach with separable kernel for the nucleon-nucleon interactions to investigate systems of nucleons and reactions with them at high energies: such as static properties of the deuteron, phase shifts and inelasticity parameters of the np-pair, reactions of the elastic electron-deuteron scattering, inelastic electro- and photobreakup of the deuteron with final state interaction etc.

Recently we start to investigate three-nucleon systems using Bethe-Salpeter-Fadeev equations. Our first aim is the $3N$ bound states - $^3He, T$. Our next plans are to study the interacting pd and $3N$ systems in the reactions.

- two-particles BS equation: formalism, separable kernel, solution
- three-particle BSF equation: formalism
- three-particle BSF equation: one-rank separable kernel and partial-states with L=0, solution, results
- three-particle BSF equation: multirank separable kernel and partial-states with $L > 0$, solution, results
- conclusion

Bethe-Salpeter equation for the nucleon-nucleon T matrix (np state)

$$T(p',p;P) = V(p',p;P) + \frac{i}{4\pi^3}\int d^4k\; V(p',k;P)\,G(k;P)\,T(k,p;P)$$

p', p - the relative four-momenta
P - the total four-momentum

$V(p',p;P)$ - the interaction kernel

$$G(k;P) = 1/((P/2+k)^2 - m_N^2 + i\epsilon)/((P/2-k)^2 - m_N^2 + i\epsilon)$$

free two-particle Green function

The partial-wave decomposed equation for T matrix

$$T_{L'L}(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = V_{L'L}(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|) + \frac{i}{4\pi^3} \sum_{L''} \int\limits_{-\infty}^{+\infty} dk_0 \int\limits_{0}^{\infty} \mathbf{k}^2 d|\mathbf{k}|$$

$$V_{L'L''}(p_0', |\mathbf{p}'|; k_0, |\mathbf{k}|)\, S(k_0, |\mathbf{k}|; s)\, T_{L''L}(k_0, |\mathbf{k}|; p_0, |\mathbf{p}|; s)$$

with scalar propagators

$$S(k_0, |\mathbf{k}|; s) = [(s/4 - k_0^2 + E_{\mathbf{k}}^2)^2 - sE_{\mathbf{k}}^2]^{-1}$$

and

$L =^1 S_0$ or $L =^3 S_1 -^3 D_1$

(scalar-vector spin-isospin) (vector-scalar spin-isospin)

Separable Ansatz for the interaction kernel

$$V_{L'L}(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \sum_{i,j=1}^{N} \lambda_{ij}(s) g_i^{L'}(p_0', |\mathbf{p}'|) g_j^{L}(p_0, |\mathbf{p}|)$$

$g_j^{[L]}$ - the model functions and $\lambda_{ij}(s)$ - a matrix of model parameters of the chosen channel

Solution for the T matrix

$$T_{L'L}(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \sum_{i,j=1} \tau_{ij}(s)\, g_i^{[L']}(p_0', |\mathbf{p}'|)\, g_j^{[L]}(p_0, |\mathbf{p}|)$$

where

$$1/\tau_{ij}(s) = 1/\lambda_{ij} + h_{ij}(s),$$

and

$$h_{ij}(s) = -\frac{i}{4\pi^3} \sum_L \int dk_0 \int \mathbf{k}^2 d|\mathbf{k}|\, g_i^{[L]}(k_0, |\mathbf{k}|) g_j^{[L]}(k_0, |\mathbf{k}|) S(k_0, |\mathbf{k}|; s)$$

The relativistic three-particle equation for T matrix

is considered in the Fadeev form with the following assumptions:

- no three-particles interaction $V_{123} = \sum_{i \neq j} V_{ij}$
- two-particles interaction is separable
- nucleon propagators are chosen in a scalar form
- the only strong interactions are considered (not EM), so ${}^3He \equiv T$

Bethe-Salpeter-Fadeev-type equation

$$\begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} - \begin{bmatrix} 0 & T_1G_1 & T_1G_1 \\ T_2G_2 & 0 & T_2G_2 \\ T_3G_3 & T_3G_3 & 0 \end{bmatrix} \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix},$$

where full three-particles T matrix $T = \sum_i T^{(i)}$, G_i is the free two-particles (j and n) Green function (ijn is cyclic permutation of (1,2,3)):

$$G_i(k_j, k_n) = 1/(k_j^2 - m_N^2 + i\epsilon)/(k_n^2 - m_N^2 + i\epsilon),$$

and T_i is the two-particles T matrix which can be written as following

$$T_i(k_1, k_2, k_3; k_1', k_2', k_3') = (2\pi)^4 \delta^{(4)}(k_i - k_i') T_i(k_j, k_n; k_j', k_n').$$

with $s_i = (k_j + k_n)^2 = (k_j' + k_n')^2$.

Introducing the equal-mass Jacobi momenta

$$p_i = \frac{1}{2}(k_j - k_n), \quad q_i = \frac{1}{3}K - k_i, \quad K = k_1 + k_2 + k_3.$$

one can separate the conserved total momentum

$$T^{(i)}(k_1, k_2, k_3; k_1', k_2', k_3') = (2\pi)^4 \delta^{(4)}(K - K') T^{(i)}(p_i, q_i; p_i', q_i'; s),$$

with $s = K^2$

Amplitude of three-particle state as a projection of T matrix to the bound state:

$$\Psi^{(i)}(p_i, q_i; s) = \langle p_i, q_i | T^{(i)} | M_B \rangle,$$

with $\sqrt{s} = M_B = 3m_N - E_t$.

Take into account then all three masses are equal $m_1 = m_2 = m_3 = m_N$ then

$$\Psi^{(i)}(p_i, q_i; s) \equiv \Psi(p, q; s)$$

Rank-I separable kernel and $L = 0$ partial-states

If one uses the separable Ansatz rank I for the interaction kernel V:

$$V^a(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \lambda^a g^a(p_0', |\mathbf{p}'|) g^a(p_0, |\mathbf{p}|)$$

then the two-body T matrix is:

$$T^a(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \tau^a(s) g^a(p_0', |\mathbf{p}'|) g^a(p_0, |\mathbf{p}|)$$

and the amplitude can be presented in the form

$$\Psi^a(p, q) = g^a(p) \tau^a[(\frac{3}{2}K + q)^2] \Phi^a(q).$$

with

$$(a = 1) \equiv (L =^1 S_0), \qquad (a = 2) \equiv (L =^3 S_1).$$

Function $\Phi^a(q)$ obeys the equation

$$\Phi^a(q) = 2i \int \frac{d^4q'}{(2\pi)^4} \sum_{b=1}^{2} Z^{ab}(q,q';s) S(\frac{1}{3}K - q') \tau^b(s_2) \Phi^b(q')$$

with

$$Z^{ab}(q,q';s) = C^{ab} g^a(-\frac{1}{2}q - q') S(\frac{1}{3}K + q + q') g^b(q + \frac{1}{2}q')$$

and

$$C^{ab} = \left[\begin{array}{rr} 1/4 & -3/4 \\ -3/4 & 1/4 \end{array} \right]$$

to take into account the $[({}^1S_0, {}^3S_1) \Leftrightarrow ({}^1S_0, {}^3S_1)]$ transitions and spin- and isospin-one-half nature of nucleons.

Partial-wave decomposition

$$\vec{L} = \vec{l} + \vec{\lambda}$$

where L is full momenta, l is momenta of the two-nucleon pair, λ is relative momentum of the 3rd particle.
Since $l = 0$ for ${}^1S_0$ and ${}^3S_1$ states and $L = 0$ for ground state then $\lambda = 0$.
Therefore

$$\Phi^a(q_4, q) = \sum_{b=1}^{2} \frac{i}{4\pi^3} \int dq_0' \int {q'}^2 dq' \, Z^{ab}(q_0, q, q_0', q'; s)$$
$$\times \frac{\tau^b((\frac{2}{3}\sqrt{s} + q_0')^2 - {q'}^2)}{(\frac{1}{3}\sqrt{s} - q_0)^2 - {q'}^2 - m_N^2 + i\epsilon} \Phi^b(q_0', q'; s)$$

with

$$Z^{ab}(q_0, q, q_0', q'; s) = C^{ab} \int d\cos\theta_{qq'}$$
$$\times \frac{g^a(-\frac{1}{2}q_0 - q_0', |-\frac{1}{2}\mathbf{q} - \mathbf{q}'|) g^b m'(q_0 + \frac{1}{2}q_0', |\mathbf{q} + \frac{1}{2}\mathbf{q}'|)}{(\frac{1}{3}\sqrt{s} + q_0 + q_0')^2 - (\mathbf{q} + \mathbf{q}')^2 - m_N^2 + i\epsilon}$$

Singularities

Poles from one-particle propagator

$$q^{0\prime}_{1,2} = \frac{1}{3}\sqrt{s} \mp [E_{|\mathbf{q}'|} - i\epsilon]$$

Poles from propagator in Z-function

$$q^{0\prime}_{3,4} = -\frac{1}{3}\sqrt{s} - q^0 \pm [E_{|\mathbf{q}'+\mathbf{q}|} - i\epsilon]$$

Poles from Yamaguchi-functions

$$q^{0\prime}_{5,6} = -2q^0 \pm 2[E_{|\frac{1}{2}\mathbf{q}'+\mathbf{q}|,\beta} - i\epsilon]$$

and

$$q^{0\prime}_{7,8} = -\frac{1}{2}q^0 \pm \frac{1}{2}[E_{|\mathbf{q}'+\frac{1}{2}\mathbf{q}|,\beta} - i\epsilon]$$

Cuts from two-particle propagator τ

$$q^{0\prime}_{9,10} = \pm\sqrt{q'^2 + 4m^2} - \frac{2}{3}\sqrt{s} \qquad \text{and} \qquad \pm\infty$$

Poles from two-particle propagator τ

$$q^{0\prime}_{11,12} = \pm\sqrt{q'^2 + 4M_d^2} - \frac{2}{3}\sqrt{s}$$

Wick-rotation procedure

If $\sqrt{s} = 3m_N - E_t < 3m_N$ then there are no singularities in the Ist and IIIrd quadrant of q_0 and therefore the Wick-rotation procedure can be applied to the equation. Then

$$\Phi^a(q_4, q) = \sum_{b=1}^{2} \frac{1}{4\pi^3} \int dq_4' \int {q'}^2 dq' Z^{ab}(q_4, q, q_4', q'; s)$$

$$\times \frac{\tau^{ab}((\frac{2}{3}\sqrt{s} + iq_4')^2 - {q'}^2)}{(\frac{1}{3}\sqrt{s} - iq_4)^2 - {q'}^2 - m_N^2} \Phi^b(q_4', q')$$

with

$$Z^{ab}(q_4, q, q_4', q'; s) = C^{ab} \int d\cos\theta_{qq'}$$

$$\times \frac{g_i(-\frac{1}{2}iq_4 - iq_4', |-\frac{1}{2}\mathbf{q} - \mathbf{q}'|) g_{m'}(iq_4 + \frac{1}{2}iq_4', |\mathbf{q} + \frac{1}{2}\mathbf{q}'|)}{(\frac{1}{3}\sqrt{s} + iq_4 + iq_4')^2 - (\mathbf{q} + \mathbf{q}')^2 - m_N^2}$$

Solving the equation

The calculate the integrals the Gauss quadratures method is used:

$$\int A(x,y)f(y) = \sum_{j=1,n} A(x_i,y_j)w_j f(y_j)$$

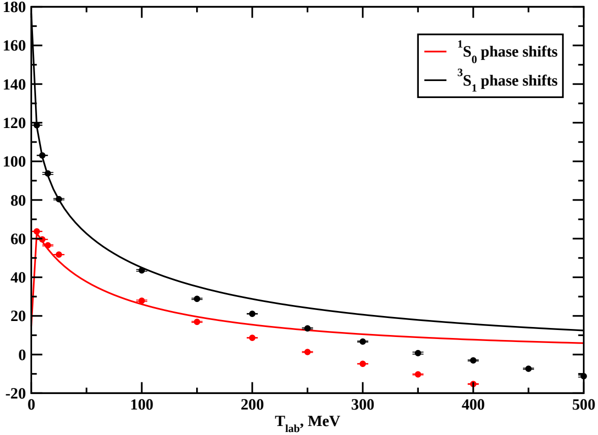
with x_i, y_j and w_j are the Gauss points and weights, resp. and n is the number of points
The homogeneous system of linear equations is solving using the fact that

$$\left[\Phi^{(n+1)}(q_4,q;s)/\Phi^{(n)}(q_4,q;s)\right]_{s=M_B^2,n\to\infty} \to 1$$

where $\Phi^{(n)}$ is nth iteration of the solution.

Rank-one covariant Yamaguchi-function

The proton-neutron scattering phase shifts



Rank-one covariant Yamaguchi-function

Properties of the proton-neutron scattering and deuteron

	$^3S_1$	exp.	$^1S_0$	exp.
a (fm)	5.424	5.424(4)	-23.748	-23.748(10)
r_0 (fm)	1.775	1.759(5)	2.75	2.75(5)
E_d (MeV)	2.2246	2.224644(46)		

Triton binding energy: E_t=11.09 MeV (exp. 8.48 MeV)

To improve E_t one needs to extend the method:

- to increase the rank of kernels
- to add the $^3D_1$ partial-state of np-pair

Multirank separable kernel and $^3D_1$ partial-state

If one uses the *separable Ansatz* rank N for the interaction kernel V:

$$V^a(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \sum_{i,j=1}^{N} \lambda_{ij}^a(s) g_i^a(p_0', |\mathbf{p}'|) g_j^a(p_0, |\mathbf{p}|)$$

then the two-body T matrix is:

$$T^a(p_0', |\mathbf{p}'|; p_0, |\mathbf{p}|; s) = \sum_{i,j=1}^{N} \tau_{ij}^a(s) g_i^a(p_0', |\mathbf{p}'|) g_j^a(p_0, |\mathbf{p}|)$$

and the amplitude can be presented in the form

$$\Psi^a(p,q) = \sum_{i,j=1}^{N} g_i^a(p) \tau_{ij}^a[(\frac{3}{2}K + q)^2] \Phi_j^a(q).$$

with

$$(a=1) \equiv (L =^1 S_0), \qquad (a=2) \equiv (L =^3 S_1), \qquad (a=3) \equiv (L =^3 D_1)$$

Partial-wave decomposition

$$\vec{L} = \vec{l} + \vec{\lambda}$$

where L is full momenta, l is momenta of the two-nucleon pair, λ is relative momenta of the 3rd particle.

- if $l = 0$ for ${}^1S_0$ and ${}^3S_1$ states and $L = 0$ for ground state then $\lambda = 0$
- if $l = 2$ for ${}^3D_1$ state and $L = 0$ for ground state $\lambda = 2$

Therefore

$$\Phi_j^a(q_4, q) = \frac{-1}{4\pi^3} \int dq_4' \int q'^2 dq'$$

$$\sum_{b=1}^{3} \sum_{k,l=1}^{N_b} Z_{jk}^{ab}(q_4, q, q_4', q'; s) \frac{\tau_{kl}^b[(\frac{2}{3}\sqrt{s} + iq_4')^2 - q'^2)]}{(\frac{1}{3}\sqrt{s} - iq_4')^2 - q'^2 - m_N^2} \Phi_l^b(q_4', q')$$

with

$$Z_{ij}^{ab}(q_4, q, q_4', q'; s) = C^{ab} \int d\cos\theta_{qq'} \; K^{ab}(\cos\theta_{qq'})$$

$$\times \frac{g_i^a(-\frac{1}{2}iq_4 - iq_4', |-\frac{1}{2}\mathbf{q} - \mathbf{q}'|) g_j^b(iq_4 + \frac{1}{2}iq_4', |\mathbf{q} + \frac{1}{2}\mathbf{q}'|)}{(\frac{1}{3}\sqrt{s} + iq_4 + iq_4')^2 - (\mathbf{q} + \mathbf{q}')^2 - m_N^2}.$$

$$C^{ab} = \begin{bmatrix} 1/4 & -3/4 & -3/4 \\ -3/4 & 1/4 & 1/4 \\ -3/4 & 1/4 & 1/4 \end{bmatrix}, \quad K^{ab} = \begin{bmatrix} K_{00} & K_{00} & K_{02} \\ K_{00} & K_{00} & K_{02} \\ K_{20} & K_{20} & K_{22} \end{bmatrix}$$

where

$$\begin{aligned}
K_{00} &= 1 \\
K_{02} &= \frac{4\pi}{\sqrt{5}}\, \mathrm{Y}_{20}(\vartheta', 0)\mathrm{Y}_{20}(\theta, 0) \\
K_{20} &= \sqrt{4\pi}\, \mathrm{Y}^*_{20}(\vartheta, 0) \\
K_{22} &= \frac{\sqrt{(4\pi)^3}}{\sqrt{5}}\, \mathrm{Y}^*_{20}(\vartheta, 0)\mathrm{Y}_{20}(\vartheta', 0)\mathrm{Y}_{20}(\theta, 0) \\
\cos\theta &= \cos\theta_{qq'} \\
\cos\vartheta &= (q/2 + q'\cos\theta_{qq'})/|\mathbf{q}/2 + \mathbf{q}'| \\
\cos\vartheta' &= (q + q'/2\cos\theta_{qq'})/|\mathbf{q} + \mathbf{q}'/2|
\end{aligned}$$

Multirank covariant Graz-II kernels

4-channels system of integral equations:

$^1S_0$ – rank-II: $\Phi_{1,2}^{^1S_0}$
$^3S_1 -^3 D_1$ – rank-III: $\Phi_{1,2}^{^3S_1}$

5-channels system of integral equations:

$^1S_0$ – rank-II: $\Phi_{1,2}^{^1S_0}$
$^3S_1 -^3 D_1$ – rank-III: $\Phi_{1,2}^{^3S_1}$, $\Phi_3^{^3D_1}$

Multirank covariant Graz-II kernels

Properties of the proton-neutron scattering and deuteron

$^1S_0$ – rank-II

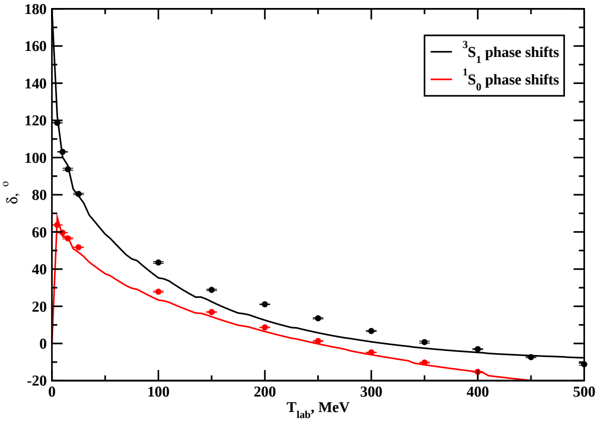
	a (fm)	r_0 (fm)
BS	-23.77	2.78
Experiment	-23.748(10)	2.75(5)

$^3S_1 -^3 D_1$ – rank-III

$P_d, \%$	a (fm)	r_0 (fm)	E_d (MeV)
4	5.419	1.780	2.2254
5	5.420	1.779	2.2254
6	5.421	1.778	2.2254
Experiment	5.424	1.759	2.2246

Multirank covariant Graz-II kernels

The proton-neutron scattering phase shifts



Preliminary results

Triton binding energy, E_t (MeV)

$P_d, \%$	4-channels $^1S_0, ^3S_1$	5-channels $^1S_0, ^3S_1$-$^3D_1$	$\delta, \%$
4	8.70	8.72	0.23
5	8.29	8.31	0.24
6	7.89	7.91	0.25
Experiment	8.48		

Conclusions

- The Bethe-Salpeter-Fedeev equations for the three-nucleons system are solved for the rank-one separable kernel of NN interaction. The states with with orbital momentum $L = 0$ ($^1S_0$ and $^3S_1$) are taken into account. The binding energy and amplitudes for the accounting states are obtained.
- The partial-wave decomposition of the BSF equations is extended to multirank separable kernels and state with orbital momentum $L = 2$ ($^3D_1$). The binding energy and amplitudes for all three partial-states ($^1S_0$,$^3S_1$-$^3D_1$) are obtained.
- The contribution of the $^3D_1$ partial-state to the tritium bound energy is investigated. It is relatively small - about 0.2% but its contribution to the elastic electromagnetic form factors will be estimated further.