Gravitational Instability in General Relativity and in $F(R)$-theories

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based on common works with A.D. Dolgov and L. Reverberi

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Outline

- Introduction
- Jeans instability in Newtonian theory with time dependent background
- Evolution of fluctuations in General Relativity
- Gravitational instability in $\mathbf{F(R)}$-modified theories
- Jeans-like instability in quickly oscillating background
- Conclusions

The instability of self-gravitating systems:

- First investigated by Jeans in non-relativistic Newtonian gravity
  J.H. Jeans, *Philosophical Transactions of the Royal Society A 199 (1902) 1.*
- It was extended to General Relativity (GR) by Lifshitz
  E.M. Lifshitz, *ZhETF*, **16** *(1946) 587.*
- Nowadays it is widely used in cosmology to study the rise of density perturbations in the expanding universe
  Ya.B. Zeldovich, I.D. Novikov, *Relativistic Astrophysics. Vol. 2. The Structure And Evolution Of The Universe, 1983.* **V. Mukhanov**, *Physical Foundations of Cosmology, 2005.* **S. Weinberg**, *Cosmology, 2008.* **D.S. Gorbunov, V.A. Rubakov**, *Introduction to the theory of the early universe: Cosmological perturbations and inflationary theory, 2011.*

The comparison between theoretical calculations and astronomical data is a very powerful tool for testing the Standard Model of cosmology.

Non-relativistic Jeans problem in Newtonian Gravity

The original Jeans approach is based on the well known Poisson equation:

$$\mathbf{\Delta\Phi = \frac{4\pi}{m_{Pl}^2}\varrho = 4\pi G\varrho}$$

The evolution of the matter density ϱ, the pressure $\mathbf{P}$, and the velocity $\vec{\mathbf{v}}$ is governed by two hydrodynamic equations:

$$\partial_t(\varrho\vec{v}) + \varrho(\vec{v}\nabla)\vec{v} + \nabla P + \varrho\nabla\Phi = 0$$
$$\partial_t\varrho + \nabla(\varrho\vec{v}) = 0$$

Usually this system of equations is solved perturbatively under assumption of infinitesimally small fluctuations over the known background quantities:

$$\varrho = \varrho_b + \delta\varrho\,, \quad \vec{v} = \vec{v}_b + \delta\vec{v}\,, \quad P = P_b + \delta P\,, \quad \Phi = \Phi_b + \delta\Phi$$

To close the system, one usually imposes the “acoustic” equation of state

$$\delta P = c_s^2\delta\varrho\,,$$

where c_s is the speed of sound.

Non-relativistic Jeans problem in Newtonian Gravity

In the Jeans theory it is assumed:

- the background mass density is homogeneous and time independent
- the background pressure and the background velocity vanish, namely, $\mathbf{P_b = 0}$ and $\vec{\mathbf{v}}_\mathbf{b} = 0$.

The problem: these assumptions are not self-consistent. Time independent ϱ is not a solution of equations.

- From Euler equation it follows that the background potential must be spatially constant: $\boldsymbol{\nabla \Phi_b = 0}$.
- It contradicts the Poisson equation $\boldsymbol{\Delta \Phi = 4\pi G \varrho}$ in zeroth order, i.e. for the background quantities, since the matter density $\varrho = \varrho_\mathbf{b} + \delta\varrho$.

$\Delta\Phi = 4\pi G \varrho$

Ya.B.Zeldovich, I.D.Novikov:

- The problem is absent in cosmology: zeroth order background equations are satisfied.

V. Mukhanov:

- addition an antigravitating substance, such as e.g. vacuum-like energy, would counterbalance the gravitational attraction of the background, so that Poisson equation would be satisfied at zeroth order.

V. Zhuk:

- the background density is zero, so Poisson equation becomes a relation between first order terms.

EA, AD, LR:

- Poisson equation is valid in particular for zero order terms, so the solution of equations of motion leads to time dependent background energy density and gravitational potential.

Development of Jeans instability goes faster than in the standard theory

Time Dependent Problem

Initial values:

- homogeneous distribution $\varrho_0 = \mathbf{const}$ inside a sphere $\mathbf{r} < \mathbf{r_m}$
- particle velocities $\mathbf{v_0} = \mathbf{0}$ and pressure $\mathbf{P_0} = \mathbf{0}$

The potential Φ is a solution of the Poisson equation:

$$\Phi_0(\mathbf{r} > \mathbf{r_m}) = -\mathbf{MG}/\mathbf{r}, \;\; \Phi_0(\mathbf{r} < \mathbf{r_m}) = 2\pi \mathbf{G} \varrho_0 \mathbf{r}^2/3 + \mathbf{C_0}$$

where $\mathbf{C_0} = -2\pi \mathbf{G} \varrho_0 \mathbf{r_m^2}$ and $\mathbf{M} = 4\pi \varrho_0 \mathbf{r_m^3}/3$.

From Euler equation:

$$\mathbf{v_1}(\mathbf{r}, \mathbf{t}) = -\nabla \Phi_0 \mathbf{t} = -4\pi \mathbf{G} \varrho_0 \mathbf{r} \mathbf{t}/3$$

From the continuity equation:

$$\varrho_1 = \frac{2\pi}{3} \mathbf{G} \varrho_0^2 \mathbf{t}^2 \;\; \text{or} \;\; \varrho_b(\mathbf{t}, \mathbf{r}) = \varrho_0 \left(1 + \frac{2\pi}{3} \mathbf{G} \varrho_0 \mathbf{t}^2 \right)$$

The time variation of the potential:

$$\Phi_b(\mathbf{r}, \mathbf{t}) = \Phi_0 + \Phi_1 = \frac{2\pi}{3} \mathbf{G} \mathbf{r}^2 \varrho_0 \left(1 + \frac{2\pi}{3} \mathbf{G} \varrho_0 \mathbf{t}^2 \right)$$

Evolution of Perturbations over Time-dependent Background

$\varrho = \varrho_b(r,t) + \delta\varrho$, $\Phi = \Phi_b(r,t) + \delta\Phi$, $v = v_1(r,t) + \delta v$, $\delta P = c_s^2 \delta\varrho$

The usual first order expansion:

$$\begin{aligned} \Delta(\delta\Phi) &= 4\pi G \delta\varrho \\ \partial_t \delta v + \nabla\delta\Phi + \delta\varrho/\varrho_0 \nabla\Phi_b + \nabla\delta P/\varrho_0 &= 0 \\ \partial_t \delta\varrho + \varrho_0 \nabla(\delta v) &= 0 \end{aligned}$$

Making Fourier transformation $\sim \exp[-i\gamma t + ik_j x^j]$ and neglecting **r**-dependent term $\delta\varrho/\varrho_0 \nabla\Phi_b$ we obtain the eigenvalue equation:

$$k^2(\gamma^2 - k^2 c_s^2 + 4\pi G\varrho_0) = 0 \rightarrow \quad \gamma = \pm\sqrt{k^2 c_s^2 - 4\pi G\varrho_0}$$

For small **k** we find the usual exponential Jeans instability:

$$\delta\varrho_J/\varrho \sim \exp[t\sqrt{4\pi G\varrho_0 - k^2 c_s^2}]$$

Jeans Characteristics

The *Jeans wave vector:*

$$\mathbf{k_J} = \frac{\sqrt{4\pi G\varrho_b}}{c_s} = \frac{\sqrt{4\pi\varrho_b}}{m_{Pl}c_s}$$

- the boundary value separating acoustic oscillations and rising perturbations.

The *Jeans wavelength:* $\lambda_J = 2\pi/k_J$.

The *Jeans mass:*

$$M_J = \frac{4\pi\varrho_b\lambda_J^3}{3} = \frac{4\pi^{5/2}c_s^3 m_{Pl}^3}{3\varrho_b^{1/2}}$$

- Objects with $M > M_J$ continue collapsing until, and if, the equation of state becomes more rigid. If this never happens, they would turn into black holes.

Effect of Time Variation of the Background

The impact of the rising background energy density on the rise of perturbations in an adiabatic approximation:

$$\frac{\delta\varrho_{J1}}{\varrho_0} \sim \exp\left\{\int_0^t dt\sqrt{4\pi G\varrho_b(t,r) - k^2c_s^2}\right\}$$

where $\varrho_b(t,r) = \varrho_0\left(1 + 2\pi/3G\varrho_0 t^2\right)$.

The enhancement factor:

- $\delta\varrho_{J1}/\delta\varrho_J = 1.027$ after a time $t = t_{grav} = 1/\sqrt{4\pi G\varrho_0}$;
- $\delta\varrho_{J1}/\delta\varrho_J = 1.23$ for $t = 2\,t_{grav}$;
- $\delta\varrho_{J1}/\delta\varrho_J = 1.89$ for $t = 3\,t_{grav}$,
- $\delta\varrho_{J1}/\delta\varrho_J = 11.9$ for $t = 5\,t_{grav}$.

NB: To derive equations we assumed $t < t_{grav}$, so we should not treat these factors as numerically accurate. However, we can interpret them as an indication that the rise of fluctuations is indeed faster then in the usual Jeans scenario.

Jeans Instability in General Relativity

Basic equations are the usual GR equations:

$$\mathbf{G}_{\mu\nu} \equiv \mathbf{R}_{\mu\nu} - \frac{1}{2}\mathbf{g}_{\mu\nu}\mathbf{R} = 8\pi\mathbf{G}\mathbf{T}_{\mu\nu} \equiv \tilde{\mathbf{T}}_{\mu\nu}$$

- Equations of motion of matter (the continuity and Euler equations) are automatically included
- Equations of motion of matter can be obtained from the covariant conservation condition of the energy-momentum tensor

$$\mathbf{D}_\mu \mathbf{T}^\mu_\nu = 0$$

NB: In the first case one has to include the terms proportional to the square of Christoffel symbols in the expression for the Ricci tensor.

If we confine ourselves to the first order in Γ in $\mathbf{R}_{\mu\nu}$ we do not obtain self-consistent equations.

Metric and Christoffel symbols

We take the Newtonian gauge, in which the metric has the form

$$ds^2 = A dt^2 - B\delta_{ij}dx^i dx^j$$

The corresponding Christoffel symbols are:

$$\Gamma^t_{tt} = \frac{\dot{A}}{2A}, \quad \Gamma^t_{jt} = \frac{\partial_j A}{2A}, \quad \Gamma^j_{tt} = \frac{\delta^{jk}\partial_k A}{2B}, \quad \Gamma^t_{jk} = \frac{\delta_{jk}\dot{B}}{2A},$$
$$\Gamma^k_{jt} = \frac{\delta^k_j \dot{B}}{2B}, \quad \Gamma^k_{lj} = \frac{1}{2B}(\delta^k_l \partial_j B + \delta^k_j \partial_l B - \delta_{lj}\delta^{kn}\partial_n B)$$

Ricci Tensor and Curvature Scalar

For the Ricci tensor with an account of the quadratic in Γ terms we obtain:

$$\begin{aligned}
R_{tt} &= \frac{\Delta A}{2B} - \frac{3\ddot{B}}{2B} + \frac{3\dot{B}^2}{4B^2} + \frac{3\dot{A}\dot{B}}{4AB} + \frac{\partial^j A \partial_j B}{4B^2} - \frac{\partial^j A \partial_j A}{4AB} \\
R_{tj} &= -\frac{\partial_j \dot{B}}{B} + \frac{\dot{B}\partial_j B}{B^2} + \frac{\dot{B}\partial_j A}{2AB} \\
R_{ij} &= \delta_{ij}\left(\frac{\ddot{B}}{2A} - \frac{\Delta B}{2B} + \frac{\dot{B}^2}{4AB} - \frac{\dot{A}\dot{B}}{4A^2} - \frac{\partial^k A \partial_k B}{4AB} + \frac{\partial^k B \partial_k B}{4B^2}\right) \\
&- \frac{\partial_i \partial_j A}{2A} - \frac{\partial_i \partial_j B}{2B} + \frac{\partial_i A \partial_j A}{4A^2} + \frac{3\partial_i B \partial_j B}{4B^2} + \frac{\partial_i A \partial_j B + \partial_j A \partial_i B}{4AB}
\end{aligned}$$

The corresponding curvature scalar is:

$$R = \frac{\Delta A}{AB} - \frac{3\ddot{B}}{AB} + \frac{2\Delta B}{B^2} + \frac{3\dot{A}\dot{B}}{2A^2 B} - \frac{\partial^j A \partial_j A}{2A^2 B} - \frac{3\partial^j B \partial_j B}{2B^3} + \frac{\partial^j A \partial_j B}{2AB^2}$$

Einstein Tensor

Expressions for the Einstein tensor $\mathbf{G}_{\mu\nu} = \mathbf{R}_{\mu\nu} - \mathbf{1/2}\,\mathbf{g}_{\mu\nu}\mathbf{R}$:

$$
\begin{aligned}
G_{tt} &= -\frac{A\Delta B}{B^2} + \frac{3\dot{B}^2}{4B^2} + \frac{3A\partial^j B\partial_j B}{4B^3} \\
G_{tj} &= R_{tj} \\
G_{ij} &= \delta_{ij}\left(\frac{\Delta A}{2A} + \frac{\Delta B}{2B} - \frac{\ddot{B}}{A} + \frac{\dot{B}^2}{4AB} + \frac{\dot{A}\dot{B}}{2A^2} - \frac{\partial^k A\partial_k A}{4A^2} - \frac{\partial^k B\partial_k B}{2B^2}\right) \\
&- \frac{\partial_i\partial_j A}{2A} - \frac{\partial_i\partial_j B}{2B} + \frac{\partial_i A\partial_j A}{4A^2} + \frac{3\partial_i B\partial_j B}{4B^2} + \frac{\partial_i A\partial_j B + \partial_j A\partial_i B}{4AB}
\end{aligned}
$$

Energy-momentum Tensor

The energy-momentum tensor is taken in the ideal liquid form without dissipative corrections:

$$\mathbf{T}_{\mu\nu} = (\varrho + \mathbf{P})\mathbf{U}_\mu \mathbf{U}_\nu - \mathbf{P}\mathbf{g}_{\mu\nu}$$

- ϱ and $\mathbf{P}$ are respectively the energy and pressure densities of the liquid
- the four-velocity is: $\mathbf{U}^\mu = \mathbf{dx}^\mu/\mathbf{ds}$ and $\mathbf{U}_\mu = \mathbf{g}_{\mu\alpha}\mathbf{U}^\alpha$

We assume that three-velocity $\mathbf{v}^\mathbf{j} = \mathbf{dx}^\mathbf{j}/\mathbf{dt}$ is small and neglect quadratic in $\mathbf{v}$ terms. Correspondingly:

$$\mathbf{U_j} = -\frac{\mathbf{B v_j}}{\sqrt{\mathbf{A}}\sqrt{\mathbf{1 - (B/A) v_j v^j}}} \approx -\frac{\mathbf{B v_j}}{\sqrt{\mathbf{A}}}$$

Now we can write:

$$\begin{aligned}
\mathbf{T_{tt}} &= (\varrho + \mathbf{P})\mathbf{U_t^2} - \mathbf{PA} \approx \varrho\mathbf{A} \\
\mathbf{T_{jt}} &= (\varrho + \mathbf{P})\mathbf{U_t U_j} \approx -(\varrho + \mathbf{P})\mathbf{B v_j} \\
\mathbf{T_{ij}} &= (\varrho + \mathbf{P})\mathbf{U_i U_j} - \mathbf{P g_{ij}} \approx \mathbf{PB}\delta_\mathbf{ij}
\end{aligned}$$

Equations of Motion

The equations for $\mathbf{G_{tt}}$ and for $\partial_\mathbf{i}\partial_\mathbf{j}$-component of equation for $\mathbf{G_{ij}}$ are:

$$-\Delta\mathbf{B} = \tilde{\varrho}$$
$$\partial_\mathbf{i}\partial_\mathbf{j}(\mathbf{A}+\mathbf{B}) = \mathbf{0}$$

The continuity and Euler equations are respectively:

$$\dot{\varrho} + \partial_\mathbf{j}[(\varrho+\mathbf{P})\mathbf{v}^\mathbf{j}] + \frac{3}{2}\varrho\dot{B} = \mathbf{0}$$
$$\varrho\dot{\mathbf{v}}_\mathbf{j} + \partial_\mathbf{j}\mathbf{P} + \frac{1}{2}\varrho\,\partial_\mathbf{j}\mathbf{A} = \mathbf{0}$$

We assume that the background metric slowly changes as a function of space and time and study small fluctuations around background quantities:

$$\varrho = \varrho_\mathbf{b} + \delta\varrho,\ \delta\mathbf{P} = \mathbf{c_s^2}\delta\varrho,\ \mathbf{v} = \delta\mathbf{v},\ \mathbf{A} = \mathbf{A_b} + \delta\mathbf{A},\ \mathbf{B} = \mathbf{B_b} + \delta\mathbf{B}$$

Evolution of Fluctuations in GR

The corresponding linear equations for infinitesimal quantities:

$$-\Delta\delta B = \delta\tilde{\varrho}$$
$$\partial_i\partial_j(\delta A + \delta B) = 0$$
$$\dot{\delta}\varrho + \varrho\partial_j\delta v^j + \frac{3}{2}\varrho\delta\dot{B} = 0$$
$$\varrho\,\delta\dot{v}_j + \partial_j\delta P + \frac{1}{2}\varrho\,\partial_j\delta A = 0$$

We look for the solution in the form $\sim \exp[-i\gamma t + ik_j x^j]$ and obtain:

$$\gamma^2 = \frac{c_s^2 k^2 - \tilde{\varrho}/2}{1 + 3\tilde{\varrho}/(2k^2)}$$

This result almost coincides with the Newtonian one. An extra term in the denominator is small when $k \sim k_J = \sqrt{4\pi G\varrho_0}/c_s$

Gravity Modification

Action in $\mathbf{f(R)}$ theories:

$$\mathbf{A_{grav}} = -\frac{\mathbf{m_{Pl}^2}}{16\pi}\int \mathbf{d^4x}\sqrt{-\mathbf{g}}[\mathbf{R} + \mathbf{F(R)}] + \mathbf{S_m}$$

Here $\mathbf{m_{Pl} = 1.22 \cdot 10^{19} GeV}$ is the Planck mass and $\mathbf{S_m}$ is the matter action.

Non linear $\mathbf{F(R)}$ - function:

- changes gravity at large distances and is responsible for cosmological acceleration
- the modified GR equations have a solution $\mathbf{R = const}$ in the absence of matter.

Pioneering Works

The pioneering suggestion:

- *S. Capozziello, S. Carloni, A. Troisi,* Recent Res. Develop. Astron. Astrophys.1(2003)625; astro-ph/0303041.
- *S.M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner,* Phys. Rev. D70 (2004) 043528, astro-ph/0306438.

$$\mathbf{F(R) = -\mu^4/R}$$

$\mu^2 \sim |R_c| \sim 1/t_u^2$, is a small parameter with dimension of mass squared; $t_U \approx 14$ Gyr is the universe age.

- Agreement with Newtonian limit for sufficiently small μ.
- Strong instability in presence of matter

Can modified gravity explain accelerated cosmic expansion?
A.D. Dolgov, M. Kawasaki, Phys.Lett. B573 (2003) 1.

Modified modified gravity: free from exponential instability

W.Hu, I. Sawicki, Phys. Rev. D **76**, 064004 (2007).

$$F_{HS}(R) = -\frac{R_{vac}}{2}\frac{c\left(\frac{R}{R_{vac}}\right)^{2n}}{1+c\left(\frac{R}{R_{vac}}\right)^{2n}},$$

A.Appleby, R. Battye, Phys. Lett. B **654**, 7 (2007).

$$F_{AB}(R) = \frac{\epsilon}{2}\log\left[\frac{\cosh\left(\frac{R}{\epsilon}-b\right)}{\cosh b}\right] - \frac{R}{2},$$

A.A. Starobinsky, JETP Lett. **86**, 157 (2007).

$$F_S(R) = \lambda R_0\left[\left(1+\frac{R^2}{R_0^2}\right)^{-n} - 1\right].$$

Guiding Example

The specific $\mathbf{F(R)}$:

$$\mathbf{F(R) = -\lambda R_c \left[1 - \left(1 + \frac{R^2}{R_c^2} \right)^{-n} \right]}$$

where $\mathbf{n}$ is an integer, $\boldsymbol{\lambda > 0}$, $\mathbf{|R_c| \sim 1/t_U^2}$.

Further problems:

- the solution of EoM for the gravitational field must be singular with $\mathbf{R} \to \infty$ in the past to produce a reasonable late-time cosmology.

S.A. Appleby, R.A. Battye, A.A. Starobinsky, JCAP 1006 (2010) 005.

- systems with rising mass/energy density will evolve to a $\mathbf{R} \to \infty$ singularity in the future.

A.V. Frolov, Phys. Rev. Lett. **101** *(2008) 061103; E.V. Arbuzova, A.D. Dolgov, Phys. Lett. B* **700** *(2011) 289; L. Reverberi, Phys. Rev. D* **87***, 084005 (2013).*

$\mathbf{R^2}$-term: Curvature Oscillations

The singularity can be avoided if one adds an extra term:

$$\mathbf{F(R) \to F(R) - \frac{R^2}{6m^2}}$$

- Small $\mathbf{R}$: the system tends to evolve to higher values of curvature;
- as $|\mathbf{R}|$ grows the $\mathbf{R^2}$-term eventually becomes dominant and pushes the system back to lower values of $|\mathbf{R}|$.

This results in oscillating solutions $\mathbf{R(t)}$, possibly with very large amplitude.

- *E. V. Arbuzova, A. D. Dolgov, L. Reverberi*, Eur.Phys.J.**C**(2012) 72:2247, arXiv:1211.5011; Phys.Rev.**D** 88, 024035 (2013), arXiv:1305.5668.

An oscillating behavior of $\mathbf{R}$ is a generic feature of modified gravity theories.
We assume that the form of the oscillations is arbitrary, keeping their amplitude and frequency as free parameters.

Gravitational Instability in Modified Gravity

Starobinsky model with $\mathbf{R^2}$ term:

$$F(R) = -\lambda R_c \left[1 - \left(1 + \frac{R^2}{R_c^2} \right)^{-n} \right] - \frac{R^2}{6m^2}$$

- $\mathbf{m \gtrsim 10^5}$ GeV to preserve successful predictions of BBN.
- $\mathbf{R^2}$ prevents from hitting infinity but still the maximum amplitude of $\mathbf{R}$ reaches a value much larger than in GR.

Gravitational instability in modified gravity:

- *H. Motohashi, A.A. Starobinsky, J. Yokoyama*, Int.J.Mod.Phys. D **18** (2009)1731; *S. Capozziello, M. De Laurentis, S. D. Odintsov and A. Stabile*, Phys.Rev. D **83** (2011) 064004; *S. Capozziello, M. De Laurentis, I. De Martino, M. Formisano and S. D. Odintsov*, Phys.Rev. D **85** (2012) 044022.

Jeans instability in classical and modified gravity in background with rising energy density with background metric slowly changing as a function of space and time:

- *E.V. Arbuzova, A.D. Dolgov and L. Reverberi*, Phys.Lett.B **739** (2014) 279.

Modified Equations of Motion

Below we investigate gravitational instability in quickly oscillating curvature background.

GR equations are modified as:

$$(1+F_{,R})\,R_{\mu\nu}-\frac{1}{2}\,(R+F)\,g_{\mu\nu}+\left(g_{\mu\nu}D_\alpha D^\alpha-D_\mu D_\nu\right)F_{,R}=\frac{8\pi T_{\mu\nu}}{m_{Pl}^2}$$

where $F_{,R}=dF/dR$.

- *A.Zhuk, S.Capozzillo et al., J. Matsumoto*: Perturbative expansion of $F(R)$ was performed either around $R=0$ or $R=R_c$, where R_c is the cosmological curvature scalar.
- *ADR*: we expand $F(R)$ around curvature of the background metric R_m, which is typically much larger than R_c.

$F(R)$-function has very different values for $R\ll R_c$, $R\sim R_c$, $R\gg R_c$

Assumptions

We assume:

- the background spacetime weakly deviates from the Minkowski metric;

NB: The corrections due to gravity modifications may be significantly different from those of **GR**: $\mathbf{R}$ may be very different from $\mathbf{R_{GR} = -\tilde{T}}$ (ADR).

- $|\mathbf{R_c}| \ll |\mathbf{R}| \ll \mathbf{m^2}$.
  Both limits are natural for relatively dense systems, more dense than the average cosmological background but much less dense than $\mathbf{m_{Pl}^2 m^2}$.

It is expected that in this limit:

$$|\mathbf{F(R)}| \ll |\mathbf{R}|\,, \ |\mathbf{F'(R)}| \ll \mathbf{1}\,.$$

This is surely fulfilled for the Starobinsky model, for which at $\mathbf{R \gg R_c}$:

$$\mathbf{F(R) \simeq -\lambda R_0 \left[1 - \left(\frac{R_0}{R}\right)^{2n}\right] - \frac{R^2}{6m^2}}\,.$$

Evolution of Fluctuations in $\mathbf{F(R)}$ - theories

New equations for the gravitational field:

$$\mathbf{G_{\mu\nu} + \frac{1}{3\omega^2}(D_\mu D_\nu - g_{\mu\nu} D^2) R = \tilde{T}_{\mu\nu}\,,}$$

where $\mathbf{G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu} R/2}$ is still the usual Einstein tensor and

$$\omega^{-2} = -3 F''_{RR}\,.$$

- Once written in this form, the equation is largely independent of the specific model considered except for the value of ω.
- Along the background solution $\omega \simeq$ **const**.

As usually, the metric and the curvature tensor are expanded around their background values to the first order in infinitesimal perturbations:

$$\mathbf{A = A_b + \delta A\,, \quad B = B_b + \delta B\,, \quad R = R_b + \delta R\,.}$$

Internal Solution

Here we use the internal solution:

- *E.V. Arbuzova, A.D. Dolgov, L. Reverberi*, Spherically Symmetric Solutions in F(R) Gravity and Gravitational Repulsion. Astropart.Phys.54(2014)44-47.

$$\mathbf{B_b(r,t) = 1 + \frac{2M(r,t)}{m_{Pl}^2 r} \equiv 1 + B_1^{(Sch)}}$$

$$\mathbf{A_b(r,t) = 1 + \frac{R(t)\, r^2}{6} + A_1^{(Sch)}(r,t)}$$

where

$$\mathbf{M(r,t) = \int_0^r d^3r\, T_{00}(r,t) = 4\pi \int_0^r dr\, r^2\, T_{00}(r,t)},$$

$$\mathbf{A_1^{(Sch)}(r,t) = \frac{r_g r^2}{2r_m^3} - \frac{3r_g}{2r_m} + \frac{\pi \ddot{\varrho}_m}{3m_{Pl}^2}(r_m^2 - r^2)^2}$$

and $\mathbf{r_g = 2M/m_{Pl}^2}$ with **M** being the total mass of the object under scrutiny.

Evolution of Fluctuations in $F(R)$ - theories

In what follows:

- We assume that $\omega =$ **const** and study the development of instabilities described by the fourth order differential equation, which governs evolution of perturbations in this model.
- In this case, the evolution of instabilities is quite different from the standard situation described by the second order equation of **GR**.
- We will not dwell on a particular choice of the $\mathbf{F(R)}$-function, but assume that the high frequency oscillations of the curvature are a generic phenomenon in such models.
- All known $\mathbf{F(R)}$-scenarios would lead to a singularity with $\mathbf{R} \to +\infty$, if not $\mathbf{R^2/m^2}$-term specially added. This term creates a repulsive effective potential for the evolution of $\mathbf{R}$ and so leads to an oscillatory behavior.

4th order Equation

The fourth order equation for the function $\delta \mathbf{B}$:

$$
\begin{aligned}
&\delta \ddddot{\mathbf{B}} - \delta \dddot{\mathbf{B}} \left(1 + \frac{2k^2}{3\omega^2}\right) \frac{\dot{R}_b}{2k^2} \\
&+\delta \ddot{B} \left[\omega^2 - \frac{\tilde{\varrho}_b \omega^2}{2k^2} \left(1 + \frac{8k^2}{3\omega^2}\right) + k^2(1 + c_s^2) - \ddot{A}_b \right. \\
&\qquad \left. - \frac{1}{k^2} \left(1 + \frac{2k^2}{3\omega^2}\right) \left(\ddot{R}_b + \frac{\dot{A}_b \dot{R}_b}{4}\right) - \frac{\dot{A}_b^2}{4} \right] \\
&+\delta \dot{\mathbf{B}} \left[-\frac{\dddot{A}_b}{2} - \frac{1}{4k^2} \left(1 + \frac{2k^2}{3\omega^2}\right) \left(2\dddot{R}_b + \dot{A}_b \ddot{R}_b + 2\dot{R}_b c_s^2 k^2\right) - \frac{\ddot{A}_b \dot{A}_b}{4} \right. \\
&\qquad \left. + \frac{\dot{A}_b}{2} \left(\omega^2 + k^2(1 - c_s^2) + \frac{2\tilde{\varrho}_b}{3} - \frac{\tilde{\varrho}_b \omega^2}{2k^2}\right) \right] \\
&+\delta \mathbf{B} \left[c_s^2 k^2 (k^2 + \omega^2) - 2c_s^2 \tilde{\varrho}_b \omega^2 \left(1 + \frac{2k^2}{3\omega^2}\right) - \frac{\tilde{\varrho}_b \omega^2}{2} \left(1 + \frac{4k^2}{3\omega^2}\right) \right] = 0 .
\end{aligned}
$$

Simplification and Parametrization

In a wide parameter range $\dot{A}_b^2 \ll \ddot{A}_b$, $\dot{A}_b \dot{R}_b \ll \ddot{R}_b$, $\dot{A}_b \ddot{A}_b \ll \dddot{A}_b$, and $\dot{A}_b \ddot{R}_b \ll \dddot{R}_b$, so the corresponding terms can be neglected, and we take $\mathbf{A_b = 1 + R_b\, r^2/6}$.

The factor $(\mathbf{kr})^2$ near its Jeans value $\mathbf{k = k_J} = \sqrt{\tilde{\varrho}/(\mathbf{2c_s^2})}$:

$$(\mathbf{rk_J})^2 = \frac{\mathbf{3}}{\mathbf{2c_s^2}} \frac{\mathbf{r_g r^2}}{\mathbf{r_m^3}} \ll \mathbf{1}\,, \quad \textbf{if} \quad \mathbf{c_s^2 \gg r_g/r_m}\,,$$

where $\mathbf{r_g = 2M/m_{Pl}^2} = \tilde{\varrho}\mathbf{r_m^2/3}$.

Now we introduce the dimensionless time $\boldsymbol{\tau = \omega t}$, the dimensionless parameters

$$\mathbf{a} \equiv \frac{\tilde{\varrho}_\mathbf{b}}{\mathbf{k^2}}\,, \quad \mathbf{b} \equiv \frac{\mathbf{k^2}}{\boldsymbol{\omega}^2}\,, \quad \mathbf{c \equiv c_s^2}\,,$$

and taking the limit $(\mathbf{kr})^2 \ll \mathbf{1}$.

4th order Equation: Dimensionless Form

Further simplification of the notation by introducing the parameters:

$$\alpha = \frac{a}{2}\left(1 + \frac{2b}{3}\right),$$

$$\Omega^2 = 1 - \frac{a}{2}\left(1 + \frac{8b}{3}\right) + b(1+c),$$

$$\mu = b\left[c(1+b) - \frac{a}{2}\left(1 + \frac{4b}{3}\right) - 2ac\left(1 + \frac{2b}{3}\right)\right].$$

Denoting $\delta \mathbf{B} \equiv \mathbf{z}$, $\mathbf{R_b} = -\tilde{\varrho}_\mathbf{b}\mathbf{y}$, we obtain:

$$z'''' + \alpha y' z''' + (\Omega^2 + 2\alpha y'')z'' + \alpha(y''' + bcy')z' + \mu z = 0\,.$$

Modified Jeans Instability

Usual Jeans equation ($\tilde{\varrho} = \mathbf{8\pi G\varrho}$):

$$\delta\ddot{\varrho} + \left(\mathbf{c_s^2 k^2} - \frac{\tilde{\varrho}_\mathbf{b}}{\mathbf{2}}\right)\delta\varrho = \mathbf{0}$$

- $\mathbf{c_s^2 k^2} - \tilde{\varrho}_\mathbf{b}/\mathbf{2} > \mathbf{0}$: sound waves
- $\mathbf{c_s^2 k^2} - \tilde{\varrho}_\mathbf{b}/\mathbf{2} < \mathbf{0}$: unstable modes

The unstable modes appear at the Jeans scale:

$$\mathbf{k^2} \leq (\mathbf{k_J^{GR}})^\mathbf{2} \equiv \frac{\tilde{\varrho}_\mathbf{b}}{\mathbf{2c_s^2}}$$

The condition of stability is determined by the sign of $\boldsymbol{\mu}$.

- In the limit of small amplitude of curvature oscillations we neglect $\mathbf{y}(\boldsymbol{\tau})$
- Equation with constant coefficients is solved by the substitution $\mathbf{z} = \mathbf{e}^{\gamma\boldsymbol{\tau}}$.

The eigenvalue γ is thus determined by the algebraic equation:

$$\gamma^\mathbf{4} + \boldsymbol{\Omega}^\mathbf{2}\gamma^\mathbf{2} + \boldsymbol{\mu} = \mathbf{0}$$

Modified Jeans Instability

The eigenvalues γ^2 are:

$$\gamma^2 = -\frac{\Omega^2}{2} \pm \sqrt{\frac{\Omega^4}{4} - \mu}.$$

- $\mu < 0$: one of the roots $\gamma^2 > 0$, so one of the eigenvalues is positive.

It corresponds to the usual exponential Jeans instability with

$$(k_J^{MG})^2 = (k_J^{GR})^2 \left[1 + \frac{(k_J^{GR})^2}{3\omega^2} \right],$$

which recovers the **GR** result in the limit $\omega \to \infty$.

- $0 < \mu < \Omega^4/4$: both possible values of γ^2 are real and negative, so γ is purely imaginary which corresponds to acoustic oscillations.

These two cases are in a one-to-one correspondence to the usual Jeans analysis.

- $\mu \geq \Omega^4/4$: there would exist a new type of unstable oscillating solution with exponentially rising amplitude.

New Effects

If $\mathbf{y}(\tau)$ is non-negligible, it enters in the 4th order equation as an oscillating function of the "time" τ.

It can induce:

- An analogue of the parametric resonance instability: very fast rise of perturbations at a certain set of frequencies.
- "Anti-friction" effect: an explosive rise of $\mathbf{z}$ in a wide range of frequencies. It appears at sufficiently large amplitudes of oscillations of $\mathbf{y}$ such that the coefficients in front of the odd derivative terms become periodically negative.

Both effects do not exist in the standard General Relativity and, if discovered, would be a proof of modified gravity.

On the contrary, the non-observation of these effects would allow to put stringent restrictions on the parameters of $\mathbf{F(R)}$-theories.

Forms of $y(\tau)$

We assume that $\mathbf{y}(\tau)$ is a periodic function describing curvature oscillations and take two possible forms of them:
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1. purely harmonic ones:

$$\mathbf{y}_{\mathbf{harm}}(\tau) = \mathbf{y}_{\mathbf{eq}}(\tau) + \mathbf{y}_0 \cos\left(\Omega_1 \tau + \theta\right),$$

   where $\mathbf{y}_{\mathbf{eq}}$ is the equilibrium point around which the curvature oscillates.
   In this case $\Omega_1 = \Omega$ and $\mathbf{y}_0 < \mathbf{y}_{\mathbf{eq}}$.

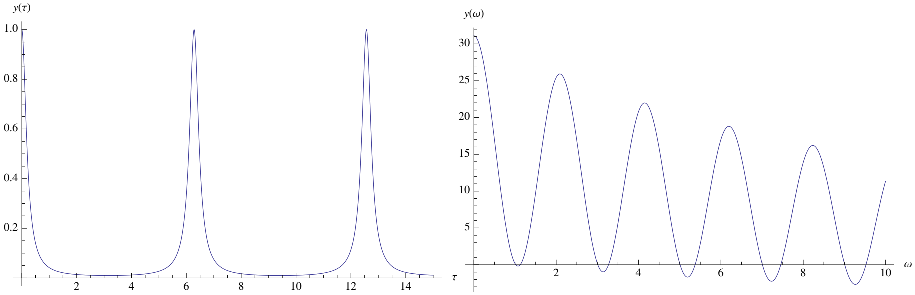
2. spiky solutions approximated as:

$$\mathbf{y}_{\mathbf{sp}}(\tau) = \frac{\mathbf{y}_0\, \mathbf{d}^2}{\mathbf{d}^2 + \sin^2(\Omega_2 \tau + \theta)},$$

   where $\mathbf{d} \ll 1$, so we have narrow peaks with large separation between them.

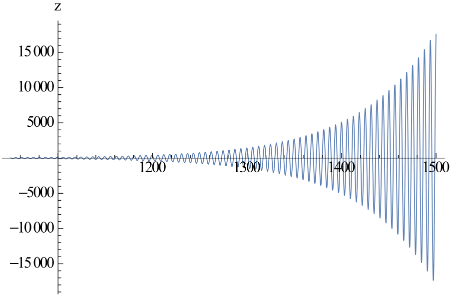
The Fourier transform of $\mathbf{y}_{\mathbf{sp}}(\tau)$ contains modes with much higher frequencies than Ω_2. The dominant mode is exited at $2\Omega_2$, where $\Omega_2 = \Omega/2$.

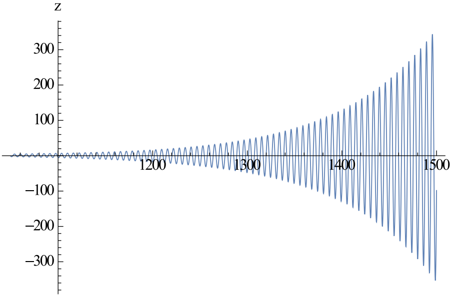
Spike-like Oscillations of Curvature



Left panel: spike-like $\mathbf{y(\tau)}$ with $\mathbf{y_0 = 1}$, $\mathbf{\Omega_2 = 0.5}$, and $\mathbf{d = 0.1}$.
Right panel: Fourier transform of $\mathbf{y(\tau)}$.

Note that the amplitudes of the even harmonics are much larger than the amplitudes of the odd ones.

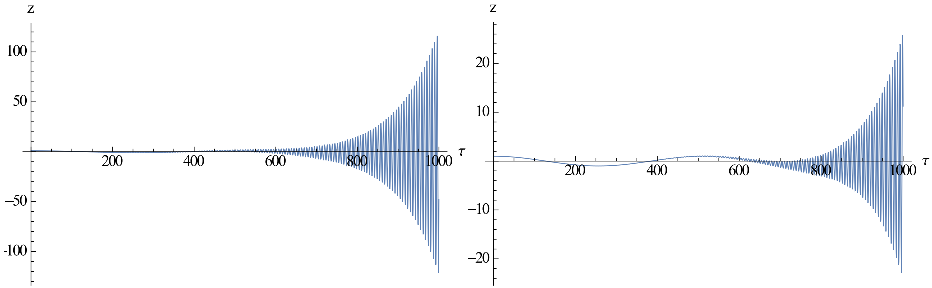




Left panel: Parametric resonance excitation of $\mathbf{z}(\boldsymbol{\tau})$ for harmonic curvature oscillations with $\mathbf{y_0 = 5}$ and $\mathbf{\Omega_1/\Omega = 2}$; $\mathbf{a = b = 0.01}$, $\mathbf{c = 0.02}$.
Right panel: Results for $\mathbf{\Omega_1/\Omega = 2.015}$, other parameters unvaried.

Clearly in the second case the resonant behavior is much weaker.

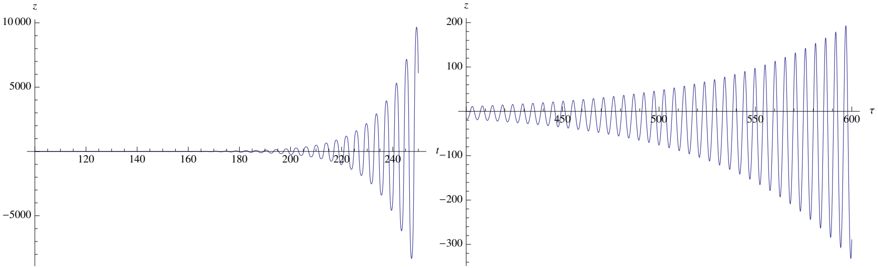
Numerical Results: Parametric Resonance for Spike-like Oscillations



Left panel: Parametric resonance excitation of $\mathbf{z}(\boldsymbol{\tau})$ for spike-like curvature oscillations with $\mathbf{y_0} = \mathbf{30}$ and $\boldsymbol{\Omega_2/\Omega} = \mathbf{0.5}$; $\mathbf{a} = \mathbf{b} = \mathbf{0.01}$, $\mathbf{c} = \mathbf{0.02}$.
Right panel: Results for $\boldsymbol{\Omega_2/\Omega} = \mathbf{1}$, other parameters unvaried.

The main mode of parametric resonance should be at $\boldsymbol{\Omega_2 = \Omega}$ (right panel).
However, it is subdominant with respect to the mode at $\boldsymbol{\Omega_2/\Omega} = \mathbf{1/2}$ due to the suppression of the odd Fourier amplitudes of $\mathbf{y}(\boldsymbol{\tau})$.

Numerical Results: Antifriction Amplification



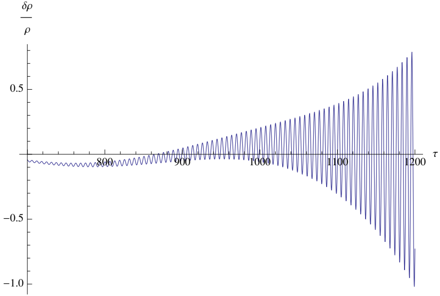
Left panel: Antifriction effect in evolution of $\mathbf{z(\tau)}$ for harmonic curvature oscillations with $\mathbf{y_0 = 169}$, $\mathbf{\Omega_1/\Omega = 3.2}$.
Right panel: Antifriction effect in evolution of $\mathbf{z(\tau)}$ for spike-type curvature oscillations with $\mathbf{y_0 = 400}$, $\mathbf{\Omega_2/\Omega = 0.6}$.

The antfriction amplification is observed at the frequencies away from the resonance values, if $\mathbf{y_0}$ exceeds certain threshold value, $\mathbf{y_{th}}$. The farther away the frequency is from the resonance, the larger is the threshold.

Density Perturbations

The magnitude of the density perturbations, $\delta\varrho/\varrho_b$, expressed through $\mathbf{z} \equiv \delta\mathbf{B}$:

$$\frac{\delta\varrho}{\varrho_b} = z\left[\frac{1+b}{a(1+2b/3)} - 2\right] + \frac{1}{2}z'\,y' + \frac{z''}{a(1+2b/3)}\,.$$



Evolution of $\delta\varrho/\varrho_b$ in parametric resonance region induced by the spike-type action with $\mathbf{y_0 = 30}$, $\Omega_2/\Omega = 0.5$, $\delta B(0) \equiv z(0) = 10^{-3}$.

In general, exponentially growing solutions for $\mathbf{z}$ will lead to a similar behaviour for the density perturbation $\delta\varrho$.

Conclusions

- A general feature of $\mathbf{F(R)}$ modified gravity is high frequency oscillations of curvature and metric in contracting matter systems.
- We study a rather general equation without restriction to any specific $\mathbf{F(R)}$.
- The fourth order equations, governing evolution of the density perturbations, demonstrate very rich pattern of different types of instabilities.
- There is a close analogue of parametric resonance, which is easy to describe theoretically, almost in the same way as the usual parametric resonance.
- A new kind of instability induced by negative signs of the coefficients in front of the odd derivatives in the equation is found: the antifriction instability.
- Density perturbations would reach unity in a time which can be much shorter than the gravitational time typical for the Jeans-type rise, if the initial metric perturbations are about $\mathbf{10^{-4}}$ or even $\mathbf{10^{-5}}$.
- Though the density contrast is an oscillating function of time, its impact on structure formation may be non-negligible and should lead to constraints on the parameters of the underlying $\mathbf{F(R)}$-theory.

THE END

THANK YOU FOR YOUR ATTENTION!