Holography, Matrix Factorizations and K-stability

Alessandro Tomasiello

SQS 2019, Yerevan

{based on work with **M. Fazzi**}

Introduction

Particularly simple solutions in string theory:

$$\text{Mink}_4 \times \text{Calabi–Yau}_6$$

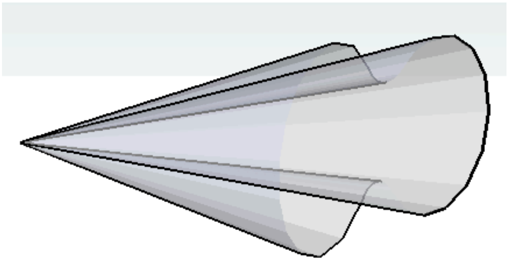
- Kähler
- Ricci-flat

If CY is conical:

$$ds^2_{\text{CY}} = dr^2 + r^2 ds^2_{\text{SE}}$$

$$\text{Calabi–Yau}_6 \equiv \text{Cone}(\text{Sasaki–Einstein}_5)$$

[by definition]



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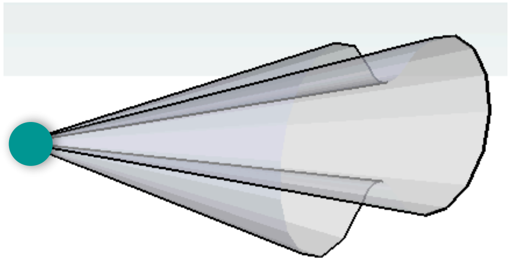

near-horizon limit of D3-branes

$\text{AdS}_5 \times \text{SE}_5 \longleftrightarrow \text{CFT}_4$ with $\mathcal{N} = 1$ susy

$$\mathrm{AdS}_5 \times \mathrm{SE}_5 \quad \longleftrightarrow \quad \mathrm{CFT}_4$$

No general method to find such pairs [or even the two sides]

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Recent progress:

- K-stability: gives a way to check that a cone is CY

[Chen, Donaldson, Sun '12; ...; Collins, Szekelyhidi '15]

- Matrix factorizations: give a way to associate a quiver to a cone

[Van den Bergh '04; ...; Aspinwall, Morrison '12]

This talk: we'll put these two methods to find new holographic pairs

[Fazzi, AT '19]

Plan

I. Review K-stability

II. Matrix factorizations

III. Holographic pairs

I. Sasaki–Einstein

- Kähler manifold: complex & symplectic

  easy to obtain: holomorphic equations in $\mathbb{CP}^N$ or $\mathbb{C}^N$

- When is a Kähler manifold also Ricci-flat?

  compact case: if and only if $c_1 = 0$ [Yau '77]

  noncompact case: this theorem doesn't apply

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example:

"conifold" $\{x^2 + y^2 + z^2 + t^2 = 0\} \subset \mathbb{C}^4$ $\exists$ Ricci-flat metric ✓

[Romans '85; Candelas, de la Ossa '90; Klebanov, Witten '98]

Brieskorn–Pham $\{x^2 + y^2 + z^2 + t^k = 0\} \subset \mathbb{C}^4$ $\nexists$ Ricci-flat metric ×

$k \geq 3$

[Gauntlett, Martelli, Sparks, Yau '06]

So how do we find conical CY?

- Explicit metrics
  - Cosets [eg. conifold]
  - 'Cohomogeneity one' ⇨ system of ODEs

    for ex. $Y^{p,q}$ [Gauntlett, Martelli, Sparks, Waldram '04]

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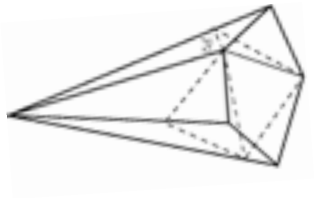
- Toric case: isometry $\supset \mathrm{U}(1)^3$
  - geometry reduces to 3d real cone



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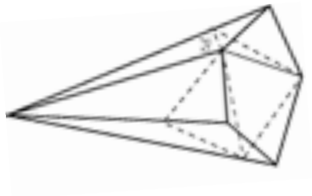

  - easy combinatorial requirement ⇨ Ricci-flat metric

    [Futaki, Ono, Wang '09]

- New method: K-stability
  - 'Stability': Use complexified gauge group to look for solutions of some real equation

[for ex. for self-duality equations on bundles] [Donaldson '85; Uhlenbeck, Yau '86]

developed for Kähler–Einstein
and then for Sasaki–Einstein

[Chen, Donaldson, Sun '12; ...;
Collins, Szekelyhidi '15]

Idea:

- If we know Ricci-flat metric exists, it is the one that minimizes volume

[Martelli, Sparks, Yau '05, 06]

volume of Kähler metric:
function of vector field

$$\xi = \underset{\substack{\uparrow\\ \text{complex}\\ \text{structure}}}{I} \cdot \underset{\substack{\uparrow\\ \text{radial}\\ \text{rescaling}}}{r\partial_r} \quad \in U(1)^r \text{ isometry torus}$$

[$r = 3$: toric case]

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{$r = 3$: toric case}

- To show that Ricci-flat metric does exist:

check volume minimization including degenerations

trick: enough to compute a 'Futaki invariant', derivative on extra parameters

Examples:

- conifold $\{x^2 + y^2 + z^2 + t^2 = 0\} \subset \mathbb{C}^4$

  we know Ricci-flat metric exists: minimize

  $$\mathrm{Vol}(\xi = \textstyle\sum_i a_i v_i) \quad \Rightarrow \quad a_1 = a_2 = 0$$

  and this determines metric.

$\mathrm{U}(1)^3$ isom. torus

$$v_1 = x\partial_y - y\partial_x,\, v_2 = z\partial_t - t\partial_z$$

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$$v_1 = x\partial_y - y\partial_x$$

$$v_2 = k(x\partial_x + y\partial_y + z\partial_z) + 2t\partial_t$$

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But we need to check degenerations

consider for example

$$\{x^2+y^2+z^2+\lambda t^k=0\}$$

- $\lambda \neq 0$: same manifold
- $\lambda = 0$: degeneration, with new isometry $v' = t\partial_t$

minimize with resp. to $\xi = a_1 v_1 + a_2 v_2 + a' v'$ ⇨

- $k < 3$: minimum has $a' = 0$ ✓
- $k \geq 3$: minimum has $a' \neq 0$

⇩

$$x^2 + y^2 + z^2 = 0 \quad (\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C})$$

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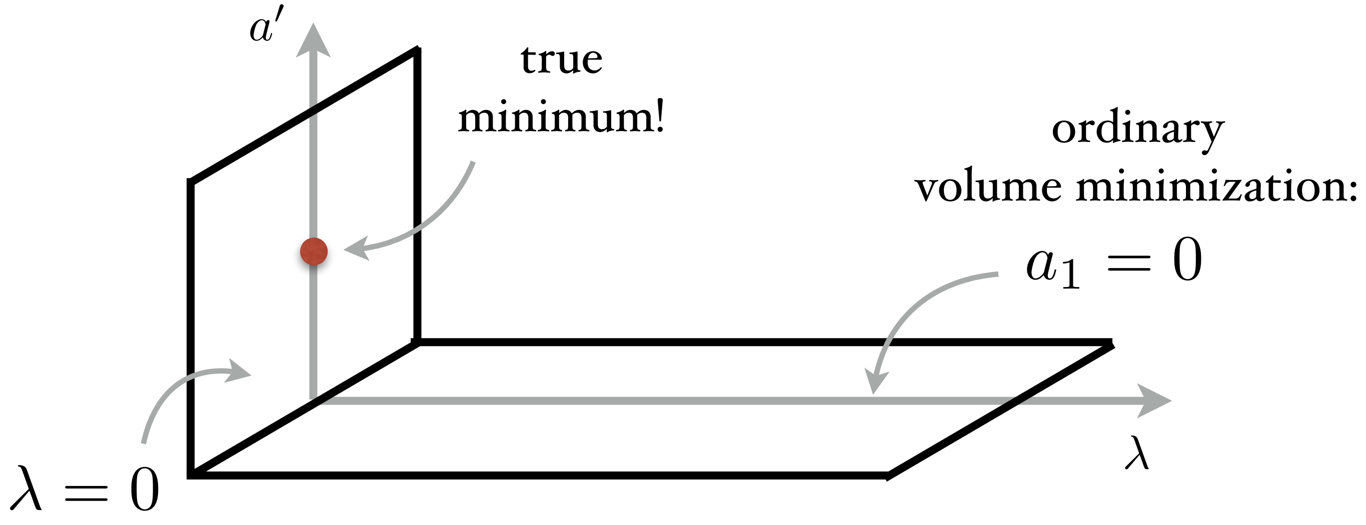

CFT point of view:
emergent symmetry in IR

[Collins, Xie, Yau '16]

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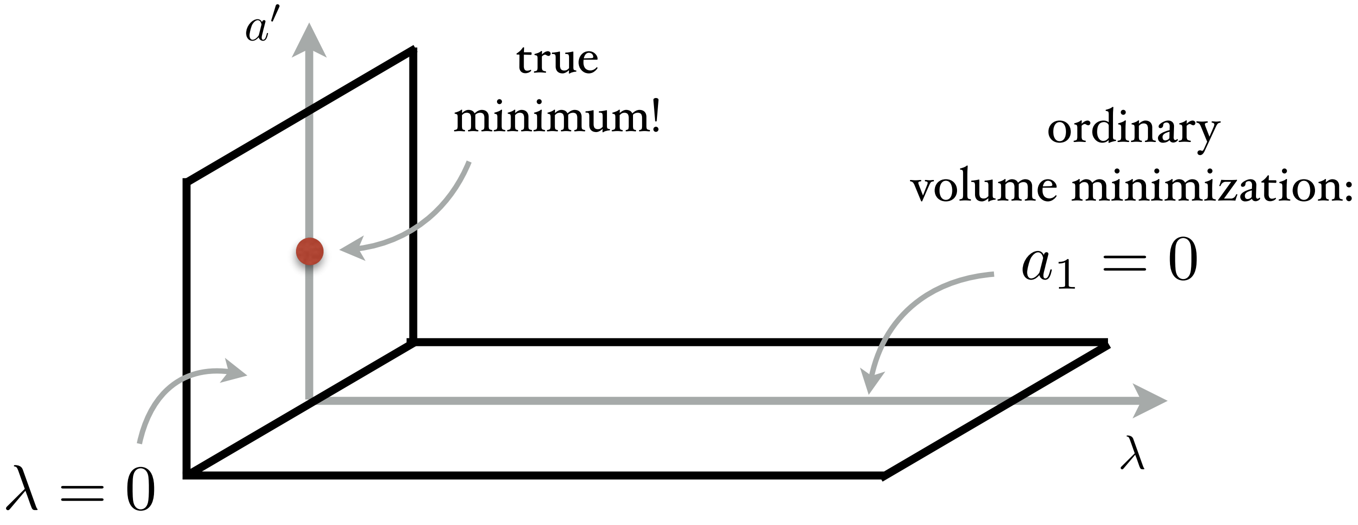

CFT point of view:
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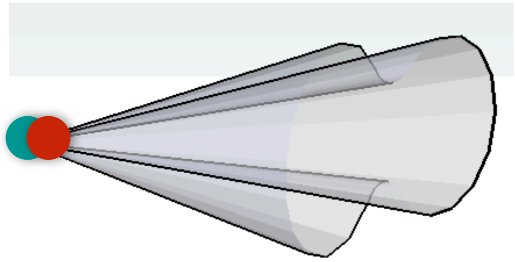
In practice, it's enough to check $\partial_{a'}\mathrm{Vol} \equiv$ Futaki invariant

Slogan: Futaki of all degenerations should be positive

II. Dual CFTs

On a stack of D3-branes ⇨ $\mathcal{N} = 4$ SYM

the singularity at the tip of the cone makes things more interesting



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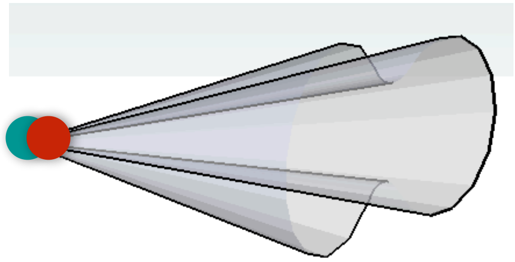

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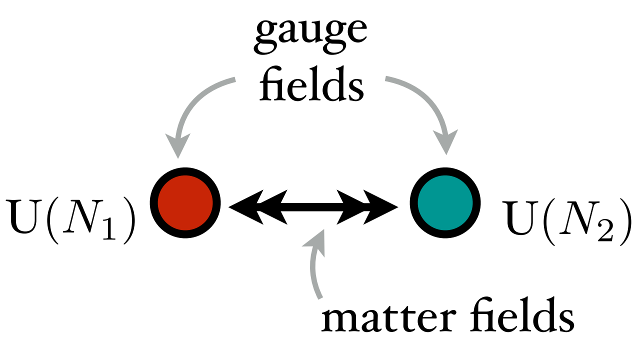
- for ex., conifold: ⇨



- 4d gauge theory is conformal for some choice of ranks:

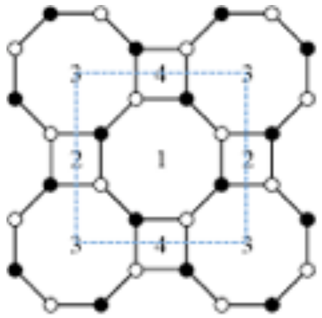
usually equal, but not always

[conifold: $N_1 = N_2$]

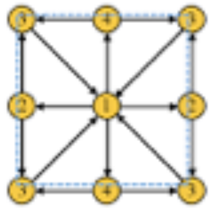
- How does one find the quiver associated to a singularity?
  - Orbifolds: algorithm with finite group rep. theory

    [Douglas, Moore '96]
  - Toric: algorithm involving dimers

    [Hanany, Kennaway '05; Franco, Hanany, Kennaway, Vegh, Wecht '05]



(a)



$$W = \epsilon_{ac}\epsilon_{bd} X_{31}^{ab} X_{12}^{c} X_{23}^{d} - \epsilon_{ac}\epsilon_{bd} X_{31}^{ab} X_{14}^{c} X_{43}^{d}$$

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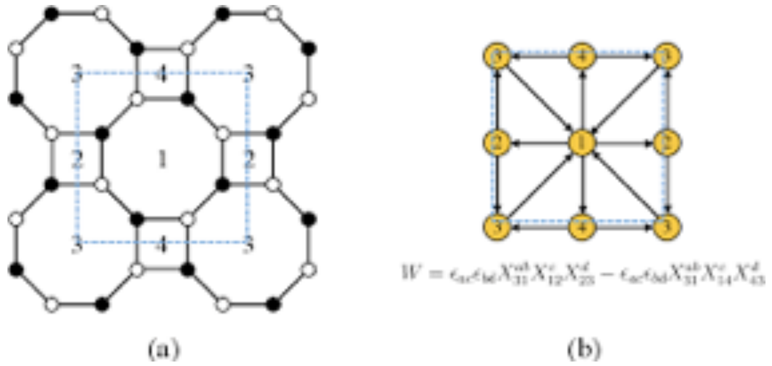

  - Reformulated as a mathematical problem: [Douglas '00; Berenstein, Leigh '01; ...]

    each type of D-brane is a module (=representation) of ring of functions on CY

    a 'basis' is made of co-kernels of maps Φ such that $\exists \Psi \mid \Psi\Phi = \Phi\Psi = f\,\mathrm{Id}$

    where $\{f = 0\}$ is CY; "matrix factorization" (MF) [Eisenbud '80; ...]

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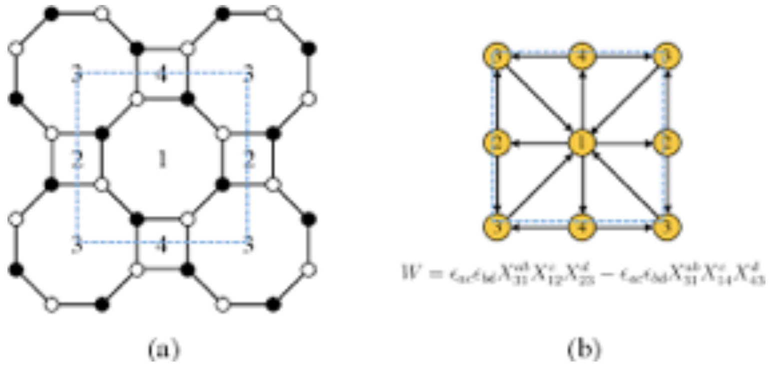

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  - example, conifold: $\Phi = \begin{pmatrix} x+iy & z+it \\ -z+it & x-iy \end{pmatrix}$ $\Psi = \begin{pmatrix} x-iy & -z-it \\ z-it & x+iy \end{pmatrix}$ $f = x^2 + y^2 + z^2 + t^2$

  - quiver is reduced to finding all MFs; this is called "non-commutative crepant resolution" (NCCR) [Van den Bergh '04]

III. Putting it together

Let’s see if these techniques can be applied at the same time.

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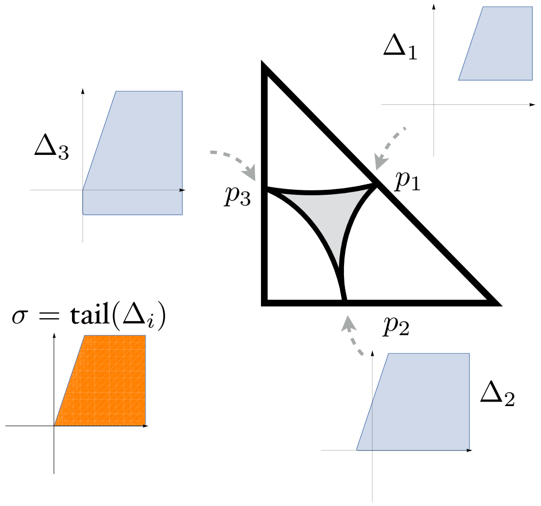
in practice, ‘easy’ to check for cases with isometry torus $\mathrm{U}(1)^2$

combinatorial methods exist
similar to toric case

[Altmann, Hausen ’03; …; Ilten, Süss ’17]



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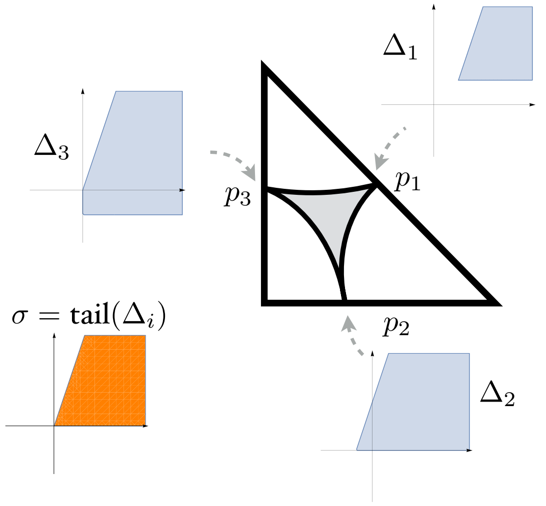

- where NCCR exists.

in principle algebraic, but lengthy; done for several classes [Iyama, Wemyss ’18]

- We first look among some spaces which are known to be K-stable [Collins, Szekelyhidi '15]
  - BP(p, q): $\{x^2 + y^2 + z^p + t^q = 0\}$

    [Collins, Szekelyhidi '15]

    K-stable if $\frac{1}{2} < \frac{p}{q} < 2$ ∞ many SE with topology S^5!

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  - Yau–Yu-II(p,q): $\{x^2+y^2+z^p+zy^q=0\}$ [Yau, Yu '03]

    K-stable if $\dfrac{p^2-1}{2p-1}<q<2(p-1)$

    [Collins, Szekelyhidi '15]

    NCCR exists if $p=q+1$ [Fazzi, AT '19]

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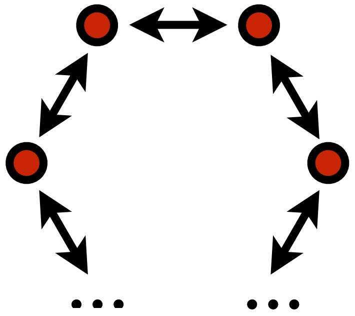
  - Yau–Yu-III(p,q): $\{x^2+y^2+z^pt+zt^q=0\}$

    ... similar results

- However, these cases are sort of boring: obtained from ‘generalized conifolds’ by int. out adjoints

[Gubser, Nekrasov, Shatashvili ’98]

quiver:



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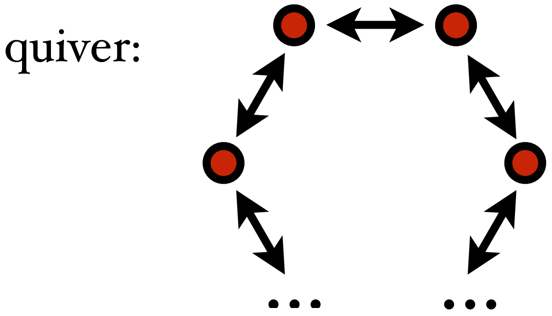
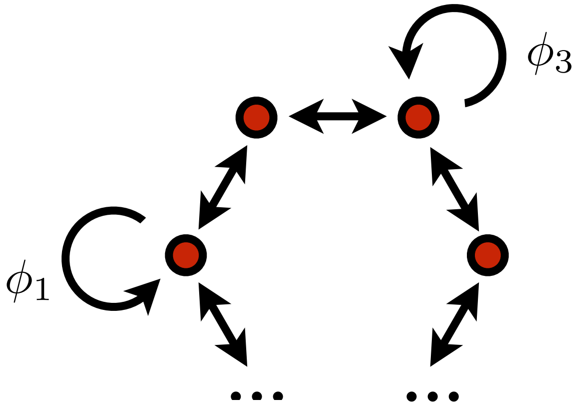


- NCCR exists for large class of similar (but not equal) quivers with some extra adjoints

superpotential $\mathrm{Tr}\phi_i^p$ for adjoints

K-stability restrictions are strong but not deadly



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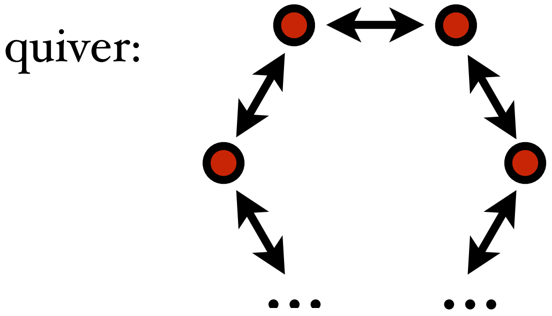

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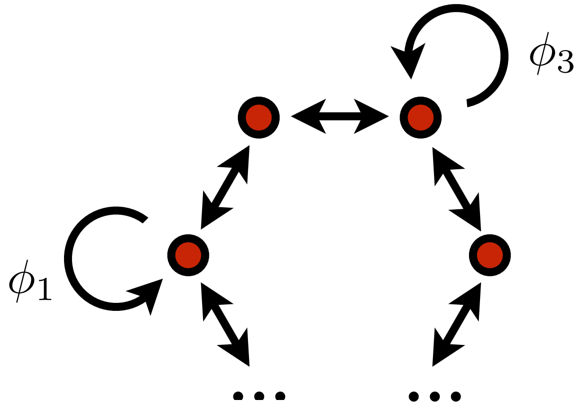


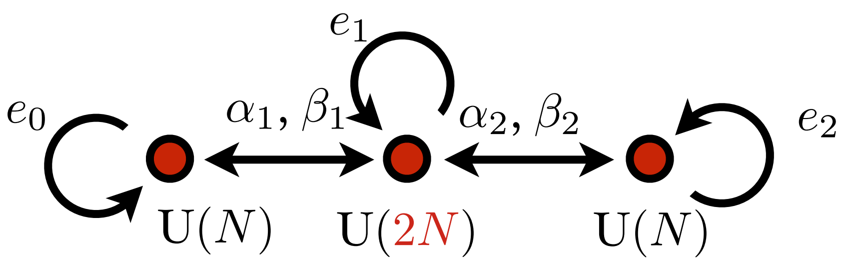
- One could look for more interesting examples by exploring systematically the Yau–Yu classification of hypersurfaces with at least one $\mathbb{C}^*$ action

[Yau, Yu ’03]

We found more examples starting from existing NCCRs

- a three-node quiver:

$$x^2 + ty^2 + t^2 z = x^2 + t(y^2 + tz) = 0$$



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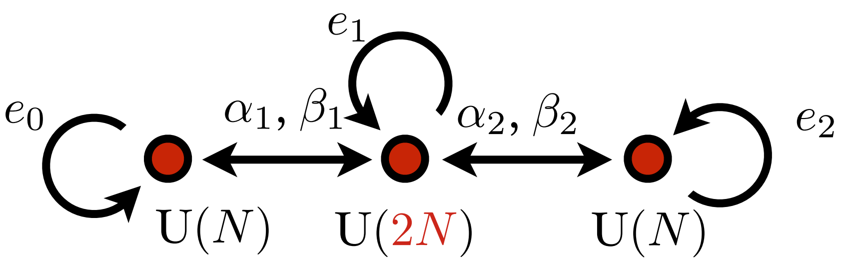

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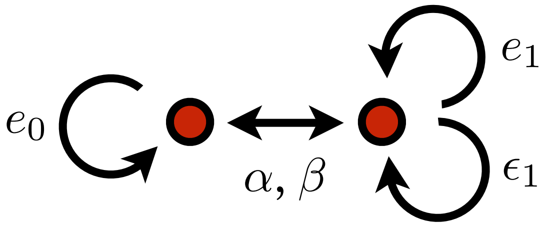

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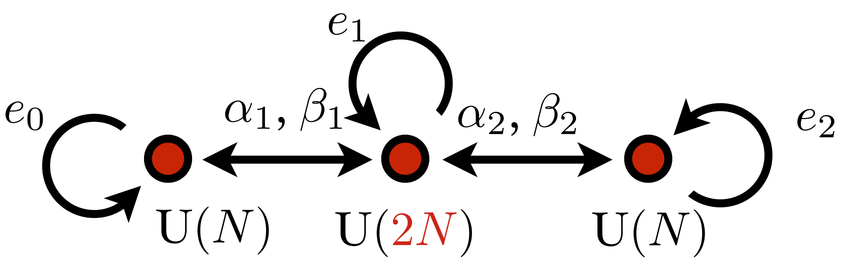

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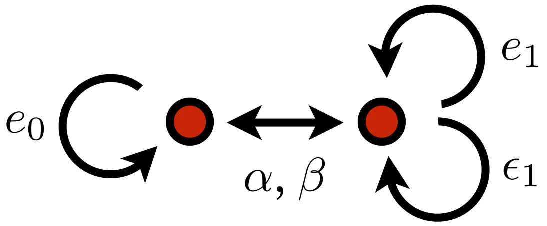
- an example related to the ‘Laufer’ singularity:

$$x^2 + y^3 + z^2 t = 0$$



$$W = \mathrm{Tr}\left(\beta e_0 \alpha + \alpha \epsilon_1^2 \beta + \epsilon_1 e_1^2\right)$$

- we also tried the original Laufer’s singularity

$$x^2 + y^3 + z^2 t + yt^3 = 0$$

but only one $\mathrm{U}(1)$ symmetry;

difficult to prove it’s really K-stable

tentative superpotential:
same as above +

$$\mathrm{Tr}(e_0^2 + e_1^4)$$

Conclusions

- Progress in finding conical CYs [K-stability]...

... and in how to find their duals [MF]

Proof of concept: several examples exist where both techniques apply [Fazzi, AT '19]

- K-stability: possible physical interpretation

Maybe lessons for CFTs more generally? [Collins, Xie, Yau '16; Benvenuti, Giacomelli '16]

- What about the CYs without an associated quiver?

we tried alternative mathematical procedures, but they failed to provide a CFT ['Maximal modification algebras']