non Abelian CS in $d = 2n + 1$, $n = 1, 2, 3$

$$\Omega_{\rm CS}^{(1)} = \varepsilon^{\mu\nu\lambda}\text{Tr}\, A_\lambda \left(F_{\mu\nu} - \frac{2}{3} F_\mu F_\nu \right) \tag{1}$$

$$\Omega_{\rm CS}^{(2)} = \varepsilon^{\mu\nu\rho\sigma\lambda}\, \text{Tr}\, A_\lambda \left(F_{\mu\nu} F_{\rho\sigma} - F_{\mu\nu} A_\rho A_\sigma + \frac{2}{5} A_\mu A_\nu A_\rho A_\sigma \right) \tag{2}$$

$$\begin{aligned} \Omega_{\rm CS}^{(3)} = \varepsilon_{\mu\nu\rho\sigma\tau\lambda\eta}\text{Tr}\, A_\eta \Big(& F_{\mu\nu} F_{\rho\sigma} F_{\tau\lambda} - \frac{4}{5} F_{\mu\nu} F_{\rho\sigma} A_\tau A_\lambda - \frac{2}{5} F_{\mu\nu} A_\rho F_{\sigma\tau} A_\lambda \\ & + \frac{4}{5} F_{\mu\nu} A_\rho A_\sigma A_\tau A_\lambda - \frac{8}{35} A_\mu A_\nu A_\rho A_\sigma A_\tau A_\lambda \Big) \,. \end{aligned} \tag{3}$$

Gauge variation of non Abelian CS

$$\Omega^{(2)}_{\rm CS} \to \bar{\Omega}^{(2)}_{\rm CS} = \Omega^{(2)}_{\rm CS} - \frac{2}{3}\varepsilon_{\lambda\mu\nu}\,{\rm Tr}\,\alpha_\lambda\alpha_\mu\alpha_\nu - 2\varepsilon_{\lambda\mu\nu}\,\partial_\lambda{\rm Tr}\,\alpha_\mu\,A_\nu$$

$$\begin{aligned}\Omega^{(3)}_{\rm CS} \to \bar{\Omega}^{(3)}_{\rm CS} &= \Omega^{(3)}_{\rm CS} - \frac{2}{5}\,\varepsilon_{\lambda\mu\nu\rho\sigma}{\rm Tr}\,\alpha_\lambda\alpha_\mu\alpha_\nu\alpha_\rho\alpha_\sigma \\ &+2\,\varepsilon_{\lambda\mu\nu\rho\sigma}\,\partial_\lambda{\rm Tr}\,\alpha_\mu\Big[A_\nu\left(F_{\rho\sigma}-\frac{1}{2}A_\rho A_\sigma\right)+\left(F_{\rho\sigma}-\frac{1}{2}A_\rho A_\sigma\right)A_\nu \\ &\qquad -\frac{1}{2}\,A_\nu\,\alpha_\rho\,A_\sigma - \alpha_\nu\,\alpha_\rho\,A_\sigma\Big]\end{aligned}$$

Higgs–Chern-Simons densities $d = 3, 5, 7$

$$\begin{aligned}
\Omega^{(3,6)}_{\rm HCS} &= \eta^2\tilde\Omega^{(1)}_{\rm CS} + \varepsilon^{\mu\nu\lambda}\mathrm{Tr}\,\gamma_5 D_\lambda\Phi\left(F_{\mu\nu}\,\Phi + F_{\mu\nu}\Phi\right)\,. \\
\Omega^{(5,8)}_{\rm HCS} &= \eta^2\tilde\Omega^{(2)}_{\rm CS} + \\
&+\varepsilon^{\mu\nu\rho\sigma\lambda}\,\mathrm{Tr}\,\gamma_7\Big[D_\lambda\Phi\left(\Phi F_{\mu\nu}F_{\rho\sigma} + F_{\mu\nu}\Phi F_{\rho\sigma} + F_{\mu\nu}F_{\rho\sigma}\Phi\right)\Big] \\
\Omega^{(7,10)}_{\rm HCS} &= \eta^2\tilde\Omega^{(3)}_{\rm CS} \\
&+\varepsilon^{\mu\nu\rho\sigma\tau\lambda\kappa}\,\mathrm{Tr}\,\gamma_9 D_\kappa\Phi(\Phi F_{\mu\nu}F_{\rho\sigma}F_{\tau\lambda} + F_{\mu\nu}\Phi F_{\rho\sigma}F_{\tau\lambda} \\
&+F_{\mu\nu}F_{\rho\sigma}\Phi F_{\tau\lambda} + F_{\mu\nu}F_{\rho\sigma}F_{\tau\lambda}\Phi)\,.
\end{aligned}$$

$$\begin{aligned}
\Omega^{(3,8)}_{\rm HCS} &= 6\eta^4\tilde\Omega^{(1)}_{\rm CS} - \varepsilon^{\mu\nu\lambda}\,\mathrm{Tr}\,\gamma_5\Big\{6\,\eta^2\left(\Phi\,D_\lambda\Phi - D_\lambda\Phi\,\Phi\right)\,F_{\mu\nu} \\
&-\Big[\left(\Phi^2\,D_\lambda\Phi\,\Phi - \Phi\,D_\lambda\Phi\,\Phi^2\right) - 2\left(\Phi^3\,D_\lambda\Phi - D_\lambda\Phi\,\Phi^3\right)\Big]F_{\mu\nu}\Big\}
\end{aligned}$$

Higgs–Chern-Simons densities $d = 4$

$$
\begin{aligned}
\Omega_{\mathrm{HCS}}^{(4,6)} &= \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\,F_{\mu\nu}\,F_{\rho\sigma}\,\Phi \\
\Omega_{\mathrm{HCS}}^{(4,8)} &= \varepsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}\Bigg[\Phi\left(\eta^2\,F_{\mu\nu}F_{\rho\sigma}+\frac{2}{9}\,\Phi^2\,F_{\mu\nu}F_{\rho\sigma}+\frac{1}{9}\,F_{\mu\nu}\Phi^2F_{\rho\sigma}\right) \\
&\quad -\frac{2}{9}\left(\Phi D_\mu\Phi D_\nu\Phi - D_\mu\Phi\Phi D_\nu\Phi + D_\mu\Phi D_\nu\Phi\Phi\right)F_{\rho\sigma}\Bigg]\,.
\end{aligned}
$$

The Higgs–Chern-Simons (HCS) densities

$$A_\mu = -\frac{1}{2}\,\omega_\mu^{ab}\,\gamma_{ab} \quad \Rightarrow \quad F_{\mu\nu} = -\frac{1}{2}R_{\mu\nu}^{ab}\,\gamma_{ab}$$

$$\Phi = \frac{1}{2}\phi^a\,\gamma_{a,d+1} \quad \Rightarrow \quad D_\mu\Phi = \frac{1}{2}D_\mu\phi^a\,.$$

$$D_\mu\phi^a = \partial_\mu\phi^a + \omega_\mu^{ab}\phi^b\,.$$

Contraction

$$\omega_\mu^{\alpha,d+1} = 0$$

Gravitational CS densities in 3 and 4

$$\Omega^{(3,6)}_{\rm GHCS} = \varepsilon^{\lambda\mu\nu}\left[\eta^2\,\omega_\lambda^{\alpha\beta}\left(R_{\mu\nu}^{\alpha\beta}-\frac{2}{3}\,(\omega_\mu\omega_\nu)^{\alpha\beta}\right)+4\phi^\alpha R_{\mu\nu}^{\alpha\beta}D_\lambda\phi^\beta\right]$$

$$\begin{aligned}\Omega^{(4,6)}_{\rm GHCS} &= \varepsilon^{\mu\nu\rho\sigma}\,R_{\mu\nu}^{\alpha\beta}\,R_{\rho\sigma}^{\gamma\delta}\,(\phi^\epsilon\,\mathrm{Tr}\,\Sigma_{\alpha\beta}\Sigma_{\gamma\delta}\Sigma_{\epsilon 6}+\phi\,\mathrm{Tr}\,\Sigma_{\alpha\beta}\Sigma_{\gamma\delta}\Sigma_{56})\\ &= \varepsilon^{\mu\nu\rho\sigma}\,\phi\left[\frac{1}{3}\varepsilon_{\alpha\beta\gamma\delta}\,R_{\mu\nu}^{\alpha\beta}\,R_{\rho\sigma}^{\gamma\delta}\,\mathrm{Tr}\,\Sigma_{56}^2-\frac{1}{2}\,R_{\mu\nu}^{\alpha\beta}\,R_{\rho\sigma}^{\alpha\beta}\,\mathrm{Tr}\,\Sigma_{56}\right]\end{aligned}$$

identity

$$\Sigma_{\alpha\beta}\Sigma_{\gamma\delta}=\frac{1}{3}\,\varepsilon_{\alpha\beta\gamma\delta}\Sigma_{56}+\frac{1}{2}\,[(\delta_{\alpha\gamma}\Sigma_{\beta\delta}-(\gamma,\delta))-(\alpha,\beta)]-\frac{1}{4}(\delta_{\alpha\gamma}\delta_{\beta\delta}-(\gamma,\delta))$$

$$\begin{aligned}
\Omega^{(4,6)}_{\rm GHCS} &= -\frac{1}{2}\,\varepsilon^{\mu\nu\rho\sigma}\,R^{\alpha\beta}_{\mu\nu}\,R^{\alpha\beta}_{\rho\sigma}\,\phi \\
\Omega^{(4,8)}_{\rm GHCS} &= \frac{1}{2}\left(\eta^2-\frac{1}{3}|\phi^a|^2\right)\Omega^{(4,6)}_{\rm GHCS} \\
&\quad -\frac{1}{6}\,\varepsilon^{\mu\nu\rho\sigma}\,R^{\alpha\beta}_{\mu\nu}\left(\phi^{\alpha\beta}_{\rho\sigma}\,\phi+4\phi^{\alpha}\phi^{\beta}_{\rho}\,\phi_{\sigma}\right)
\end{aligned}$$

where $|\phi^a|^2 = |\phi^\alpha|^2 + \phi^2$ and the abbreviated notations $\phi^\alpha_\mu = D_\mu\phi^\alpha$, $\phi_\mu = \partial_\mu\phi$ and $\phi^{\alpha\beta}_{\mu\nu} = D_{[\mu}\phi^\alpha D_{\nu]}\phi^\beta$ are used

nA gauge-Higgs to gravity: no contraction

$$A_\mu = -\frac{1}{2}\,\omega_\mu^{\alpha\beta}\,\gamma_{\alpha\beta} + \kappa\, e_\mu^\alpha\,\gamma_{\alpha,d+1} \Rightarrow F_{\mu\nu} = -\frac{1}{2}\left(R_{\mu\nu}^{\alpha\beta} - \kappa^2\, e_{[\mu}^\alpha\, e_{\nu]}^\beta\right)\gamma_{\alpha\beta}\,.$$

$$\frac{1}{2}\Phi = (\phi^\alpha\,\gamma_{\alpha,d+2} + \phi\,\gamma_{d+1,d+2}) \;\Rightarrow$$
$$\Rightarrow \frac{1}{2}D_\mu\Phi = (D_\mu\phi^\alpha - \kappa\, e_\mu^\alpha\,\phi)\gamma_{\alpha,d+2} + (\partial_\mu\phi + \kappa\, e_\mu^\alpha\,\phi^\alpha)\gamma_{d+1,d+2}$$

Chern-Simons Gravities in $d = 3, 5, 7$ (without Higgs)

$$
\begin{aligned}
\mathcal{L}^{(1)}_{\mathrm{CSG}} &= -\kappa\, \varepsilon^{\mu\nu\lambda} \varepsilon_{\alpha\beta\gamma} \left(R^{\alpha\beta}_{\mu\nu} - \frac{2}{3}\,\kappa^2\, e^{\alpha}_{\mu} e^{\beta}_{\nu} \right) e^{\gamma}_{\lambda} \\
\mathcal{L}^{(2)}_{\mathrm{CSG}} &= \kappa\, \varepsilon^{\mu\nu\rho\sigma\lambda} \varepsilon_{\alpha\beta\gamma\delta\epsilon} \left(\frac{3}{4}\, R^{\alpha\beta}_{\mu\nu}\, R^{\gamma\delta}_{\rho\sigma} - \kappa^2\, R^{\alpha\beta}_{\mu\nu}\, e^{\gamma}_{\rho} e^{\delta}_{\sigma} + \frac{3}{5}\,\kappa^4\, e^{a}_{\mu} e^{\beta}_{\nu} e^{\gamma}_{\rho} e^{\delta}_{\sigma} \right) \\
\mathcal{L}^{(3)}_{\mathrm{CSG}} &= -\kappa\, \varepsilon^{\mu\nu\rho\sigma\tau\kappa\lambda} \varepsilon_{\alpha\beta\gamma\delta\epsilon\theta\psi} \left(\frac{1}{8}\, R^{\alpha\beta}_{\mu\nu}\, R^{\gamma\delta}_{\rho\sigma}\, R^{\epsilon\theta}_{\tau\kappa} - \frac{1}{4}\,\kappa^2\, R^{\alpha\beta}_{\mu\nu}\, R^{\gamma\delta}_{\rho\sigma}\, e^{\epsilon}_{\tau} e^{\theta}_{\kappa} \right. \\
&\qquad \left. + \frac{3}{10}\,\kappa^4\, R^{\alpha\beta}_{\mu\nu}\, e^{\gamma}_{\rho} e^{\delta}_{\sigma} e^{\epsilon}_{\tau} e^{\theta}_{\kappa} - \frac{1}{7}\,\kappa^6\, e^{\alpha}_{\mu} e^{\beta}_{\nu} e^{\gamma}_{\rho} e^{\delta}_{\sigma} e^{\epsilon}_{\tau} e^{\theta}_{\kappa} \right) e^{\psi}_{\lambda}
\end{aligned}
$$

Chern-Simons Gravities in $d = 3, 5, 7$ (with Higgs)

$$
\begin{aligned}
\bar{R}_{\mu\nu}^{\alpha\beta} &= R_{\mu\nu}^{\alpha\beta} - \kappa^2\, e_{[\mu}^{\alpha} e_{\nu]}^{\beta} \\
\phi_{\mu}^{\alpha} &= D_{\mu}\phi^{\alpha} - \kappa\, e_{\mu}^{\alpha}\phi \\
\phi_{\mu} &= \partial_{\mu}\phi + \kappa\, e_{\mu}^{\alpha}\phi^{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\mathrm{HCSG}}^{(3,6)} &= \eta^2\kappa\,\mathcal{L}_{\mathrm{CSG}}^{(1)} + \varepsilon^{\mu\nu\lambda}\varepsilon_{\alpha\beta\gamma}\bar{R}_{\mu\nu}^{\alpha\beta}(\phi\,\phi_{\lambda}^{\gamma} - \phi^{\gamma}\phi_{\lambda}) \\
\mathcal{L}_{\mathrm{HCSG}}^{(5,8)} &= \eta^2\kappa\,\mathcal{L}_{\mathrm{CSG}}^{(2)} - \frac{3}{4}\varepsilon^{\mu\nu\rho\sigma\lambda}\varepsilon_{\alpha\beta\gamma\delta\epsilon}\bar{R}_{\mu\nu}^{\alpha\beta}\bar{R}_{\rho\sigma}^{\gamma\delta}(\phi\,\phi_{\lambda}^{\epsilon} - \phi^{\epsilon}\phi_{\lambda}) \\
\mathcal{L}_{\mathrm{HCSG}}^{(7,10)} &= \eta^2\kappa\,\mathcal{L}_{\mathrm{CSG}}^{(3)} + 2\varepsilon^{\mu\nu\rho\sigma\tau\lambda\kappa}\varepsilon_{\alpha\beta\gamma\delta\epsilon\theta\psi}\bar{R}_{\mu\nu}^{\alpha\beta}\bar{R}_{\rho\sigma}^{\gamma\delta}\bar{R}_{\tau\lambda}^{\epsilon\theta}(\phi\,\phi_{\lambda}^{\psi} - \phi^{\psi}\phi_{\lambda})
\end{aligned}
$$

Chern-Simons Gravities in $d = 4$ (with Higgs)

$$
\begin{aligned}
\mathcal{L}^{(4,6)}_{\mathrm{HCSG}} &= -\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}\,\phi\,\bar{R}^{\alpha\beta}_{\mu\nu}\,\bar{R}^{\gamma\delta}_{\rho\sigma} \\
\mathcal{L}^{(4,8)}_{\mathrm{HCSG}} &= \left(\eta^2 - \frac{1}{3}|\phi^a|^2\right)\mathcal{L}^{(4,6)}_{\mathrm{HGCS}} - \\
&\quad -\frac{2}{3}\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}\bar{R}^{\alpha\beta}_{\mu\nu}\,D_\rho\phi^\gamma(\phi D_\sigma\phi^\delta - 2\phi^\delta\partial_\sigma\phi)
\end{aligned}
$$

Black hole solutions: $d = 3$

$$ds^2 = A^2(r)dr^2 + r^2 d\varphi^2 - B^2(r)dt^2,$$

Dreibeine $e^\alpha_\mu = (e^A_\mu, e^3_\mu)$, with $\alpha = A, 3$; $A = 1, 2$, with $\mu = (r, \varphi, t)$

$$e^A_r = A\, n^A_{(1)} \,, \qquad e^A_\varphi = r\, n^A_{(2)} \,, \qquad e^3_t = B,$$

$n^A_{(1)}$ and $n^A_{(2)}$ complete orthonormal set of vectors of unit length.
Ansatz for the frame vector field $\phi^\alpha = (\phi^A, \phi^3)$ and scalar field ϕ

$$\begin{aligned} \phi^A &= f(r)\, n^A_{(1)} \,, \quad \phi^3 = g(r), \\ \phi &= h(r)\,. \end{aligned} \tag{4}$$

The closed-form solution

$$\begin{aligned} B^2(r) &= \frac{1}{A^2(r)} = \kappa^2 r^2 + c_t, \\ h(r) &= c_0 r \,, \quad g(r) = 0 \,, \quad f(r) = \frac{c_0}{\kappa} B(r), \end{aligned} \tag{5}$$

c_t, c_0 are free parameters

Black hole solutions: odd d

$$
\begin{aligned}
\left(\frac{N'}{2N} + 2\kappa^2\, r\, N\right) &= 0 \qquad (6)\\
\left[2\kappa^2\, r^2 - \left(1 - \frac{1}{N}\right)\right]\left(\frac{N'}{2N} + 2\kappa^2\, r\, N\right) &= 0\\
\left[-4\kappa^4\, r^4 + 4\kappa^2\, r^2 \left(1 - \frac{1}{N}\right) - \left(1 - \frac{1}{N}\right)^2\right]\left(\frac{N'}{2N} + 2\kappa^2\, r\, N\right) &= 0
\end{aligned}
$$

which all yield the one solution

$$
N = \frac{1}{2(\kappa^2 r^2 + c^2)} \tag{7}
$$

Skyrme–Chern-Pontryagin (SCP) densities: $\bar{D} = 2, 3, 4$

Remarks on Skyrme-CP and Skyrme CS densities

The Skyrme–Chern-Simons (SCS) densities