Braneworld gravity in a hyperbolic transverse space with a mass gap

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Trying to understand what lies beyond our local world

In Armenia, attempts to understand the structure of the universe beyond our immediate local world date back to the Bronze Age.

Braneworld localized gravity

The idea of formulating the cosmology of our universe on a brane embedded in a higher-dimensional spacetime dates was initially considered, among others, by Rubakov and Shaposhnikov.

Phys. Lett. B125 (1983), 136

An approach to gravity localization: Salam-Sezgin theory and its embedding

Abdus Salam and Ergin Sezgin constructed in 1984 a version of 6D minimal (chiral, *i.e.* (1,0)) supergravity coupled to a 6D 2-form tensor multiplet and a 6D super-Maxwell multiplet which gauges the U(1) R-symmetry of the theory. Phys.Lett. B147 (1984) 47 This Einstein-tensor-Maxwell system has the bosonic Lagrangian

$$
\begin{aligned}
\mathcal{L}_{\mathrm{SS}} &= \tfrac{1}{2}R - \tfrac{1}{4g^2}e^{\sigma}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{6}e^{-2\sigma}G_{\mu\nu\rho}G^{\mu\nu\rho} - \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - g^2e^{-\sigma} \\
G_{\mu\nu\rho} &= 3\partial_{[\mu}B_{\nu\rho]} + 3F_{[\mu\nu}A_{\rho]}
\end{aligned}
$$

Note the *positive* potential term for the scalar field σ. This is a key feature of all R-symmetry gauged models generalizing the Salam-Sezgin model, leading to models with noncompact symmetries. For example, upon coupling to yet more vector multiplets, the sigma-model target space can have a structure $SO(p,q)/(SO(p)\times SO(q))$.

The Salam-Sezgin theory does not admit a maximally symmetric 6D solution, but it does admit a $(\mathrm{Minkowski})_4 \times S^2$ solution with the flux for a $U(1)$ monopole turned on in the S^2 directions

$$
\begin{aligned}
ds^2 &= \eta_{\mu\nu} dx^\mu dx^\nu + a^2 (d\theta^2 + \sin^2\theta d\phi^2) \\
A_m dy^m &= (n/2g)(\cos\theta \mp 1) d\phi \\
\sigma &= \sigma_0 = \text{const}\ , \qquad B_{\mu\nu} = 0 \\
g^2 &= \frac{e^{\sigma_0}}{2a^2}\ , \qquad n = \pm 1
\end{aligned}
$$

$\mathcal{H}^{(2,2)}$ embedding of the Salam-Sezgin theory

A way to obtain the Salam-Sezgin theory from M theory was given by Cvetič, Gibbons & Pope. Nucl. Phys. B677 (2004) 164 This employed a reduction from 10D type IIA supergravity on the space $\mathcal{H}^{(2,2)}$, or, equivalently, from 11D supergravity on $S^1 \times \mathcal{H}^{(2,2)}$. The $\mathcal{H}^{(2,2)}$ space is a cohomogeneity-one 3D hyperbolic space which can be obtained by embedding into R^4 via the condition $\mu_1^2 + \mu_2^2 - \mu_3^2 - \mu_4^2 = 1$. This embedding condition is SO(2, 2) invariant, but the embedding $\mathbb{R}^4$ space has SO(4) symmetry, so the linearly realized isometries of this space are just $\mathrm{SO}(2,2) \cap \mathrm{SO}(4) = \mathrm{SO}(2) \times \mathrm{SO}(2)$. The cohomogeneity-one $\mathcal{H}^{(2,2)}$ metric can be written $ds_3^2 = \cosh 2\rho d\rho^2 + \cosh^2\rho d\alpha^2 + \sinh^2\rho d\beta^2$.

Since $\mathcal{H}^{(2,2)}$ admits a natural $SO(2,2)$ group action, the resulting 7D supergravity theory has maximal (32 supercharge) supersymmetry and a gauged $SO(2,2)$ symmetry, linearly realized on $SO(2) \times SO(2)$. Note how this fits neatly into the general scheme of extended Salam-Sezgin gauged models.

The Kaluza-Klein spectrum

Reduction on the non-compact $\mathcal{H}^{(2,2)}$ space from ten to seven dimensions, despite its mathematical consistency, does not provide a full physical basis for compactification to 4D. The chief problem is that the massive Kaluza-Klein modes form a *continuum* instead of a discrete set with mass gaps.

Moreover, the wavefunction of "reduced" 4D states when viewed from 10D or 11D includes a non-normalizable factor owing to the infinite $\mathcal{H}^{(2,2)}$ directions. Accordingly, the higher-dimensional supergravity theory does not naturally localize gravity in the lower-dimensional subspace when treated by ordinary Kaluza-Klein methods.

Expansion about the Salam-Sezgin background

The $D = 10$ lift of the Salam-Sezgin "vacuum" solution yields the metric

$$
\begin{aligned}
ds_{10}^2 &= (\cosh 2\rho)^{1/4}\Big[e^{-\frac{1}{4}\bar{\phi}}\, d\bar{s}_6^2 + e^{\frac{1}{4}\bar{\phi}}\, dy^2 + \tfrac{1}{2}\bar{g}^{-2}\, e^{\frac{1}{4}\bar{\phi}}\Big(d\rho^2 \\
&\quad + \tfrac{1}{4}[d\psi + \mathrm{sech} 2\rho(d\chi - 2\bar{g}\bar{A})]^2 + \tfrac{1}{4}(\tanh 2\rho)^2\,(d\chi - 2\bar{g}\bar{A})^2\Big)\Big] \\
\bar{A}_{(1)} &= -\frac{1}{2\bar{g}}\,\cos\theta\, d\varphi
\end{aligned}
$$

in which the $d\bar{s}_6^2$ metric has Minkowski$_4 \times S^2$ structure

$$d\bar{s}_6^2 = dx^\mu dx^\nu \eta_{\mu\nu} + \frac{1}{8\bar{g}^2}\,(d\theta^2 + \sin^2\theta d\varphi^2)$$

The inclusion of gravitational fluctuations about this background is then accomplished by replacing

$$\eta_{\mu\nu} \longrightarrow \eta_{\mu\nu} + h_{\mu\nu}(x, z)$$

where z^p are the coordinates transverse to the 4D coordinates x^μ.

Bound states and mass gaps Crampton, Pope & K.S.S.

An approach to obtaining the localization of gravity on the 4D subspace is then to look for a *normalizable* transverse-space wavefunction $\xi(z)$ for $h_{\mu\nu}(x,z) = h_{\mu\nu}(x)\xi(z)$ with a *mass gap* before the onset of the continuous massive Kaluza-Klein spectrum. This could be viewed as analogous to an effective field theory for electrons confined to a metal by a nonzero work function.

General study of the fluctuation spectra about brane solutions shows that the mass spectrum of the spin-two fluctuations about a brane background is given by the spectrum of the scalar Laplacian in the transverse embedding space of the brane (with $g = \sqrt{2}\bar{g}$ now) Csáki, Erlich, Hollowood & Shirman, Nucl.Phys. B581 (2000) 309; Bachas & Estes, JHEP 1106 (2011) 005

$$\begin{aligned} \Box_{(10)} F &= \frac{1}{\sqrt{-\det g_{(10)}}}\partial_M\left(\sqrt{-\det g_{(10)}}\, g^{MN}_{(10)}\partial_N F\right) \\ &= H_{SS}^{\frac{1}{4}}(\Box_{(4)} + g^2\triangle_{\theta,\phi,y,\psi,\chi} + g^2\triangle_{KK}) \\ H_{SS} &= (\cosh 2\rho)^{-1} \text{ warp factor;} \quad \triangle_{KK} = \frac{\partial^2}{\partial\rho^2} + \frac{2}{\tanh(2\rho)}\frac{\partial}{\partial\rho} \end{aligned}$$

The z^p directions θ, ϕ, y, ψ & χ are all compact, and one can employ ordinary Kaluza-Klein methods for reduction on them, truncating to the invariant sector for these coordinates, but still allowing dependence on the noncompact coordinate ρ.

To handle the noncompact direction ρ, one needs to expand all fields in eigenmodes of $\triangle_{\mathsf{KK}}$:

$$\phi(x^\mu, \rho) = \sum_i \phi_{\lambda_i}(x^\mu)\xi_{\lambda_i}(\rho) + \int_\Lambda^\infty d\lambda \phi_\lambda(x^\mu)\xi_\lambda(\rho)$$

where the ϕ_{λ_i} are discrete eigenmodes and the ϕ_λ are the continuous Kaluza-Klein eigenmodes. Their eigenvalues give the Kaluza-Klein masses in 4D from the wave equation $\Box_{(10)}\phi_\lambda = 0$ using $\triangle_{\theta,\phi,y,\psi,\chi}\phi_\lambda = 0$

$$\begin{aligned} \triangle_{\mathsf{KK}}\xi_\lambda &= -\lambda\xi_\lambda \\ \Box_{(4)}\phi_\lambda &= (g^2\lambda)\phi_\lambda \end{aligned}$$

The Schrödinger equation for $\mathcal{H}^{(2,2)}$ eigenfunctions

One can rewrite the $\triangle_{\mathsf{KK}}$ eigenvalue problem as a Schrödinger equation by making the substitution

$$\Psi_\lambda = \sqrt{\sinh(2\rho)}\xi_\lambda$$

after which the eigenfunction equation takes the Schrödinger equation form

$$-\frac{d^2\Psi_\lambda}{d\rho^2} + V(\rho)\Psi_\lambda = \lambda\Psi_\lambda$$

where the potential is

$$V(\rho) = 2 - \frac{1}{\tanh^2(2\rho)}$$

The SS Schrödinger equation potential $V(\rho)$ asymptotes to the value 1 for large ρ. In this limit, the Schrödinger equation becomes

$$\frac{d^2\Psi_\lambda}{d\rho^2} + (\lambda - 1)\Psi_\lambda = 0$$

giving scattering-state solutions for $\lambda > 1$:

$$\Psi_\lambda(\rho) \sim \left(A_\lambda e^{i\sqrt{\lambda-1}\rho} + B_\lambda e^{-i\sqrt{\lambda-1}\rho}\right) \quad \text{for large } \rho$$

while for $\lambda < 1$, one can have L^2 normalizable bound states. Recalling the ρ dependence of the measure $\sqrt{-g_{(10)}} \sim (\cosh(2\rho))^{\frac{1}{4}} \sinh(2\rho)$, one finds for large ρ

$$\int_{\rho_1 \gg 1}^{\infty} |\Psi_\lambda(\rho)|^2 d\rho < \infty \Rightarrow \Psi_\lambda \sim B_\lambda e^{-\sqrt{1-\lambda}\rho} \text{ for } \lambda < 1$$

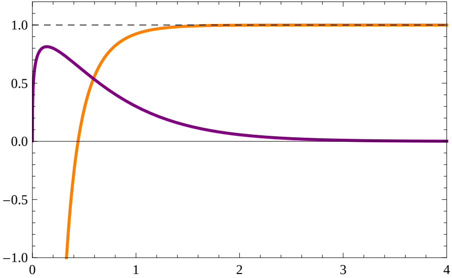
So for $\lambda < 1$ we can have candidate bound states, with a *mass gap* up to the edge of the scattering states' continuum spectrum.

The zero-mode bound state

The 1-D quantum mechanical system with a $V(\rho) = 2 - \coth^2(2\rho)$ potential belongs to a special class of Pöschl-Teller integrable systems. Neither normalizability nor self-adjointness are by themselves sufficient to completely determine the transverse wavefunction for the reduced effective theory. A key feature of such systems, however is 1-D *supersymmetry* and requiring that this be unbroken by the transverse wavefunction Ψ_λ selects the value $\lambda = 0$.

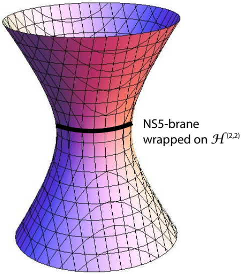
Happily, for $\lambda = 0$ the Schrödinger equation can be solved exactly. The normalized result is

$$\Psi_0(\rho) = \sqrt{\sinh(2\rho)}\xi_0(\rho) = \frac{2\sqrt{3}}{\pi}\sqrt{\sinh(2\rho)}\log(\tanh\rho)$$



$\mathcal{H}^{(2,2)}$ Schrödinger equation potential (orange) and zero-mode ξ_0 (purple)

The asymptotic structure of the Salam-Sezgin background as $\rho \to 0$ limits to the horizon structure of a NS-5 brane. This also allows for the inclusion of an additional NS-5 brane source as $\rho \to 0$. After such an inclusion, the zero-mode transverse wavefunction ξ_0 remains *unchanged*. Moreover, inclusion of such an additional NS-5 brane does not alter the 8 unbroken space-time supersymmetries possessed by the Salam-Sezgin background. The NS-5 modified 10-D supergravity solution can still be given explicitly.



$\mathcal{H}^{(2,2)}$ space with an NS-5 brane source wrapped around its 'waist' and smeared on a transverse S^2

Braneworld effective gravity

The effective action for 4D gravity reduced on the background SS solution is obtained by letting the higher dimensional metric take the form $d\hat{s}^2 = e^{2A(z)}(\eta_{\mu\nu} + h_{\mu\nu}(x)\xi_0(\rho))dx^\mu dx^\nu + \hat{g}_{ab}(z)dz^a dz^b$, where the warp factor $A(z)$ and the transverse metric $\hat{g}_{ab}(z)$ are given by the SS background.

Starting from the 10D Einstein gravitational action

$$I_{10} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{\hat{g}} \hat{R}(\hat{g})$$

and making the reduction to 4D, one obtains, at quadratic order in $h_{\mu\nu}$, the linearized 4D Einstein (*i.e.* massless Fierz-Pauli) action with a prefactor υ_0^{-2}

$$I_{\mathrm{lin}\,4} = \frac{1}{\upsilon_0^2} \int d^4x \left(-\tfrac{1}{2}\partial_\sigma h_{\mu\nu}\partial^\sigma h^{\mu\nu} + \tfrac{1}{2}\partial_\mu h^\sigma{}_\sigma \partial^\mu h^\tau{}_\tau + \partial^\nu h_{\mu\nu}\partial^\sigma h^\mu{}_\sigma + h^\sigma{}_\sigma \partial^\mu \partial^\nu h_{\mu\nu}\right)$$

The normalizing factor $\upsilon_0 = \left(\frac{16\pi G_{10} g^5}{\pi^2 \ell_y I_2}\right)^{\frac{1}{2}}$ involves the first of a series of integrals involving products of the transverse wavefunction ξ_0. For υ_0 one needs

$$I_2 = \int_0^\infty d\rho \sinh 2\rho \, \xi_0^2 = \frac{\pi^2}{12}$$

The ability to evaluate explicitly such integrals of products of transverse wave functions is directly related to the integrable-model Pöschl-Teller structure of the transverse wavefunction Schrödinger equation with $V(\rho) = 2 - \coth^2(2\rho)$.

In order to obtain the effective 4D Newton's constant, one needs to rescale $h_{\mu\nu} = \upsilon_0 \tilde{h}_{\mu\nu}$ in order to obtain a canonically-normalised kinetic term for $\tilde{h}_{\mu\nu}$. Then the leading effective 4D coupling $\kappa_4 = \sqrt{32\pi G_4}$ for gravitational self-interactions is obtained from the coefficient in front of the trilinear terms in $\tilde{h}_{\mu\nu}$ in the 4D effective action.

These involve the integral

$$I_3 = \int_0^\infty d\rho \sinh 2\rho\, \xi_0^3 = -\frac{3\zeta(3)}{4} \; ;$$

accordingly, the 4D Newton constant is given by

$$G_4 = \frac{486\, \zeta(3)^2\, G_{10} g^5}{\pi^8 \ell_y}$$

with corresponding 4D κ_4 expansion coupling

$$\kappa_4 = 72\sqrt{3}\zeta(3) \left(\frac{G_{10} g^5}{\pi^7 \ell_y} \right)^{\frac{1}{2}} \; .$$

The Pöschl-Teller integrable structure of the transverse Schrödinger problem enables higher order terms in the effective action to be evaluated explicitly as well. For the ξ_0^n integrals needed in evaluating the effective theory interactions, one finds

$$I_n \equiv \int_0^\infty d\rho \, \sinh 2\rho \, \xi_0^n(\rho) = (-1)^n \, n! \, 2^{-n} \, \zeta(n)$$

Moreover, integrating out the continuum of massive modes also requires performing integrals like

$$\int_0^\infty d\rho \, \sinh 2\rho \, \xi_0^n(\rho) \xi_\lambda(\rho)$$

which can also be evaluated and the results given in terms of Legendre functions. Integrating out the ξ_λ contributions then produces a series of corrections to the leading-order effective theory.

Life in an inconsistent truncation

The "inconsistency" of the mass-gapped reduction to $D = 4$ is revealed in the types of corrections to the lower-dimensional effective theory that can arise from integrating out the transverse coordinate dependence.

There are some similarities here to compactification on Calabi-Yau spaces. M.J. Duff, S. Ferrara, C.N. Pope & K.S.S., Nucl.Phys. B333 (1990) 783 However, in such CY compactifications, if one focuses on parts of the leading order effective theory without scalar potentials, the result of integrating out the massive KK modes is purely to generate higher-derivative corrections to the leading-order effective theory.

In the present case, however, one has to be alert to corrections in the leading-order two-derivative part of the effective theory. One can see this thanks to the special integrability features of the Pöschl-Teller transverse wavefunctions, which allow for transverse integrals actually to be done explicitly.

Note, for example that quartic terms in $h_{\mu\nu}(x)$ involve the integral $I_4 = 4!\, 2^{-4}\, \zeta(4)$. This, however, does not yet yield the expected quartic term with a coefficient $(\kappa_4)^2$: I_4 involves $\zeta(4)$, while $(\kappa_4)^2$ involves $(\zeta(3))^2$.

This poses a key question: Is general covariance somehow spontaneously broken in such systems?

A Toy model with similar features

The puzzling features of the way in which higher-dimensional diffeomorphism invariance projects into the lower dimensional braneworld can be replicated in a simpler 5-dimensional Kaluza-Klein model with unusual boundary conditions.

Consider reduction from $D = 5$ to $D = 4$ on a manifold $\mathfrak{M}_5 = \mathbb{R}^{1,3} \times \mathbb{I}$ where $\mathbb{I} \; = \; \left\{\rho \in \mathbb{R} \;\middle|\; 0 < \rho < 1\right\}$. Here, one simply expands around flat $D = 5$ spacetime $g_{MN}(x,\rho) \; = \; \eta_{MN}$ in a fashion similar to the main construction above: $g_{MN} \; = \; \eta_{MN} \; + \; H_{MN}(x,\rho)$.

One anticipates expansion in transverse eigenmodes $H(x,\rho) \; = \; \sum_i h^i(x)\, \xi_i(\rho)$ where now the relevant transvere Schrödinger problem is simply $\Delta\, \xi_i(\rho) \; = \; \frac{d^2}{d\rho^2}\, \xi_i(\rho) \; = \; -\lambda_i\, \xi_i(\rho)$.

Instead of the usual KK periodic boundary conditions in the $5^{\rm th}$ dimension, however, one now requires the boundary, normalization and orthogonality conditions

$$\xi_i(0) \ = \ 0 \, , \qquad \int_{\mathbb{I}} (\xi_i)^2 \, d\rho = 1 \, , \qquad \int_{\mathbb{I}} \xi_i \xi_j d\rho|_{i \neq j} = 0 \ .$$

Instead of the usual constant zero-mode, one now has $\xi_0(\rho) \ = \sqrt{3}\rho$; moreover, the orthogonality condition can be recast using $\int_{\mathbb{I}} \xi_0 \xi_i d\rho \ = \ \frac{1}{\lambda_i} \left(\xi_0 \xi_i' - \xi_0' \xi_i \right) \Big|_{\partial \mathbb{I}}$ so requiring $\int_{\mathbb{I}} \xi_0 \xi_j d\rho|_{j \neq 0} = 0$ yields the eigenvalue condition $\tan\left(\sqrt{\lambda_i}\right) = \sqrt{\lambda_i}$.

The admissible eigenfunctions of the transverse wavefunction problem are consequently $\xi_0 = \sqrt{3}\rho$, $\xi_i(\rho) = n_i \sin\left(\sqrt{\lambda_i}\rho\right)$ with the constants

$$n_i = \left(\frac{4\sqrt{\lambda_i}}{2\sqrt{\lambda_i} - \sin(2\sqrt{\lambda_i})}\right)^{\frac{1}{2}} , \; \sqrt{\lambda_i} = \tan(\sqrt{\lambda_i}) , \; \lambda_i > 0 .$$

Note that the boundary condition at $\rho = 1$ can be recast as $d\xi_i/d\rho - \xi_i = 0$, which is known as a Robin boundary condition. Accordingly this modified KK system uses a kind of mixed Dirichlet-Robin set of boundary conditions at the two ends.

Analysis of the effective theory expansion of this Toy Model shows that it shares with the $\mathcal{H}^{(2,2)}$ braneworld the same difficulties with expansion coefficients in the realization of diffeomorphism invariance, starting at the quadrilinear order in $h_{\mu\nu}(x)$.

The would-be Kaluza-Klein vector

A peculiar feature of both the $\mathcal{H}^{(2,2)}$ braneworld and the Toy Model is the behavior of $K_\mu = h_{\mu\rho}$. In a standard Kaluza-Klein reduction on a space where this braneworld vector corresponds to a Killing vector, this field simply becomes a massless vector on the lower-dimensional braneworld spacetime.

In both of the present cases, however, there is no Killing vector symmetry in the ρ direction. Moreover, the non-trivial dependence of ξ_0 on the radial transverse coordinate ρ gives rise to a Stückelberg transformation of K_μ at lowest order in the diffeomorphism parameter ζ_μ, *i.e.* $\delta K_\mu = \zeta_\mu + \ldots$.

Starting from the $D = 10$ diffeomorphism symmetry of the type IIA supergravity theory, there inevitably is an imprint of this symmetry in the lower-dimensional braneworld.

Analysis of the effective theory expansion shows that, once the higher-order massive spin-two modes are integrated out, the would-be Kaluza-Klein vector doesn't appear in the expansion before the quadrilinear level in zero-mode fields. So the structure of the effective theory at bilinear and trilinear level is exactly what one expects for a massless effective theory of braneworld gravity. What happens from then on is not yet clear. There would seem to be two possibilities:

- The K_μ field does appear in higher order and causes a spontaneous breakdown of diffeomorphism invariance at the quadrilinear level in interactions and higher.
- There may be some field redefinition which absorbs the K_μ field and allows for a diffeomorphism invariant braneworld effective gravity after all.

Conclusions

- Braneworld gravity on a subsurface of the Salam-Sezgin hyperbolic vacuum spacetime can successfully be localized within an infinite transverse space. This is in contrast to the situation with asymptotically maximally symmetric spacetimes where localization has failed.
- There is a mass gap between the zero mode and the edge of the continuous massive spectrum: gravity is localized on the 6D brane worldvolume. Further standard Kaluza-Klein compactification to 4D then gives a localized 4D braneworld gravity.
- Such reductions involve inconsistent Kaluza-Klein truncations, and the consequences of this, after integrating out the massive fields, display subtleties in the realization of diffeomorphism invariance (and evidently, of local supersymmetry as well).