Integrable non-linear differential equations with matrix variables

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Let us consider integrable systems of ODEs of the form

$$\frac{du_i}{dt} = \sum_{j,k} C^i_{jk}\, u_j\, u_k, \qquad i, j, k = 1, \ldots, n, \tag{1}$$

where $C^i_{jk} = C^i_{kj}$ are constants.

The main feature of integrable systems of the form (1) is the existence of infinitesimal polynomial symmetries and first integrals.

Another evidence of the integrability is the absent of moveable singularities in solutions for complex t. The so-called Painlevé approach is based on this assumption. We use the Kowalevski-Lyapunov test, which is one of incarnations of the Painlevé approach.

Infinitesimal symmetries and first integrals

Suppose we have a dynamical system

$$\frac{d\,u^i}{dt} = F_i(u^1, \dots, u^n), \qquad i = 1, \dots, n\,. \tag{2}$$

Definition. The dynamical system

$$\frac{d\,u^i}{d\tau} = G_i(u^1, \dots, u^n), \qquad i = 1, \dots, n \tag{3}$$

is called (infinitesimal) symmetry for (2) iff (2) and (3) are compatible.

The compatibility means that

$$XY - YX = 0,$$

where $X = \sum F_i \frac{\partial}{\partial u^i}, \quad Y = \sum G_i \frac{\partial}{\partial u^i}.$

Definition. The function $I(u_1, \ldots, u_n)$ is called first integral for the system (2) iff

$$X(I) = 0.$$

Kowalevski-Lyapunov integrability test

Systems of the form (1) possess special Kowalevski solutions of the form

$$u_i(t) = \frac{z_i}{t},$$

where

$$-z_i = \sum_{j,k} C^i_{jk}\, z_j\, z_k.$$

For any Kowalevski solution consider a formal solution of the form

$$u_i = \frac{z_i}{t} + \varepsilon\, p_i\, t^s + \cdots$$

The numbers p_i are defined from the system of linear equations

$$s\, p_i = \sum_{j,k} C^i_{jk}\, (p_j z_k + \, p_k z_j).$$

We see that the Kowalevski exponents s are eigen-values and $(p_1, \ldots p_n)$ are corresponding eigen-vectors of a matrix defined by the Kowalevski solution.

Definition. System (1) satisfies the Kowalevski-Lyapunov integrability test iff for any Kowalevski solution all exponents are integers.

Two-component case

Consider systems of the form

$$\left\{\begin{array}{lll} u_t & = & \alpha_1 u^2 + \alpha_2 uv + \alpha_3 v^2, \\ v_t & = & \beta_1 v^2 + \beta_2 uv + \beta_3 u^2. \end{array}\right. \tag{4}$$

Two systems related to each other by a linear transformation of the form

$$\hat{u} = r_1 u + r_2 v\,, \qquad \hat{v} = r_3 u + r_4 v\,, \qquad r_1 r_4 - r_2 r_3 \neq 0 \tag{5}$$

are called equivalent.

Consider polynomial first integrals of systems (4).

Lemma 1. Let I be a first integral of a system (4). Then for any N

$$\left\{\begin{array}{lll} u_\tau & = & I^N\,(\alpha_1 u^2 + \alpha_2 uv + \alpha_3 v^2), \\ v_\tau & = & I^N\,(\beta_1 v^2 + \beta_2 uv + \beta_3 u^2) \end{array}\right.$$

is a symmetry of (4).

Any polynomial integral can be written in the form

$$I = \prod_{i=1}^{m} (u - \mu_i v)^{k_i}, \qquad k_i \in \mathbb{N}.$$

Proposition. $m \leqslant 3$.

Consider the case $m = 3$. Using transformations (5), we send μ_i to $0, 1$ and ∞. For such a normalization we have

$$I = u^{k_1} (u - v)^{k_2} v^{k_3}, \tag{6}$$

where k_i are natural numbers which are defined up to the permutations. Without loss of generality we may assume that k_1, k_2, k_3 have no a non-trivial common devisor.

Proposition. Suppose that a system (4) has an integral (6). Then it is given (up to a scaling $u \to \mu u$, $v \to \mu v$) by

$$\begin{cases} u_t &= -k_3\, u^2 + (k_2 + k_3)\, uv \\ v_t &= -k_1\, v^2 + (k_1 + k_2)\, uv. \end{cases} \tag{7}$$

Theorem. The Kowalevski exponents for (7) have the form

$$s_1 = \frac{k_2 + k_3}{k_1}, \qquad s_2 = \frac{k_1 + k_3}{k_2}, \qquad s_3 = \frac{k_1 + k_2}{k_3}. \tag{8}$$

The system passes through the Kowalevski-Lyapunov test iff these numbers are natural. The natural numbers s_i satisfy the identity

$$\sum_i \frac{1}{s_i + 1} = 1$$

well known in elementary geometry.

Lemma 2. There exist only three admissible sets k_1, k_2, k_3. They are:

- Case 1. $k_1 = k_2 = k_3 = 1$;
- Case 2. $k_1 = k_3 = 1,\ k_2 = 2$;
- Case 3. $k_1 = 1,\ k_2 = 2,\ k_3 = 3$.

Remark. In the case 1 the symmetry orders are $2 + 3m$, for the case 2 they are $2 + 4m$ and in the case 3 we get $2 + 6m$.

Non-abelian ODEs

c It is easy to see that all components of the matrix $uv - vu$ are first integrals for the non-abelian system.

In the case $m = 1$ we have a system of two ODEs which can be written in the Hamiltonian form

$$u_t = -\frac{\partial H}{\partial v}, \qquad v_t = \frac{\partial H}{\partial u}$$

with the Hamiltonian

$$H = \frac{1}{3}u^3 - \frac{1}{3}v^3 - cuv + bu - av.$$

For generic a, b, c the relation $H = const$ is an elliptic curve and the dynamical system describes the motion of its point.

In the case of arbitrary m the system remains to be Hamiltonian with the Hamiltonian

$$H = \operatorname{tr}\left(\frac{1}{3}u^3 - \frac{1}{3}v^3 - cuv + bu - av\right)$$

and non-abelian constant Poisson bracket.

The homogeneous non-abelian system possesses the following Lax (L, A)-pair

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix} \lambda + \begin{pmatrix} 0 & 3\varepsilon u & 3v \\ v & 0 & (\varepsilon - 1)u \\ u & (2\varepsilon + 1)v & 0 \end{pmatrix},$$

$$A = -\frac{1}{3}\begin{pmatrix} \varepsilon^2 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \lambda + \frac{1}{3}\begin{pmatrix} 0 & 3\varepsilon^2 u & 3v \\ \varepsilon v & 0 & (\varepsilon + 2)u \\ u & (1 - \varepsilon)v & 0 \end{pmatrix}$$

where

$$\varepsilon^2 + \varepsilon + 1 = 0.$$

Proposition 1. The Lax equation

$$\bar{L}_t = [A,\, \bar{L}],$$

where $\bar{L} = \lambda\, L + \lambda\, c\, P + a\, Q + b\, R$,

$$P = \begin{pmatrix} \varepsilon + 2 & 0 & 0 \\ 0 & -2\varepsilon - 1 & 0 \\ 0 & 0 & \varepsilon - 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 3(\varepsilon + 2) & 0 \\ 0 & 0 & -3 \\ \varepsilon - 1 & 0 & 0 \end{pmatrix},$$

$$R = \begin{pmatrix} 0 & 0 & 3(1 - \varepsilon) \\ 2\varepsilon + 1 & 0 & 0 \\ 0 & -3\varepsilon & 0 \end{pmatrix}$$

is equivalent to the non-abelian system (**??**).

Let us consider “ODE systems” of the form

$$\frac{dx_\alpha}{dt} = F_\alpha(\mathbf{x}), \qquad \mathbf{x} = (x_1, ..., x_N), \tag{9}$$

where $x_i(t)$ are $m \times m$ matrices, F_α are (non-commutative) polynomials with constant scalar coefficients. As usual, a symmetry is defined as an equation

$$\frac{dx_\alpha}{d\tau} = G_\alpha(\mathbf{x}), \tag{10}$$

compatible with (9).

We call system (9) integrable if it possesses infinitely many linearly independent symmetries.

The simplest class of such non-abelian systems are quadratic systems of the form

$$\begin{cases} u_t = \alpha_1 u\,u + \alpha_2 u\,v + \alpha_3 v\,u + \alpha_4 v\,v, \\ v_t = \beta_1 v\,v + \beta_2 v\,u + \beta_3 u\,v + \beta_4 u\,u. \end{cases} \tag{11}$$

Definition. Two systems of the form (11) are called equivalent if they are related by a transformation (5) or by an involution $\star$ (the "transposition") defined by the formulas

$$u^\star = u\,, \qquad v^\star = v\,, \qquad (a\,b)^\star = b^\star\,a^\star. \tag{12}$$

Some experiments with non-triangular systems (11) having symmetries of a small degree have been made in paper [1]. One of the results is:

Theorem. Any non-triangular equation (11) possessing a symmetry of the form

$$\begin{cases} u_\tau = \gamma_1 u\,u\,u + \gamma_2 u\,u\,v + \gamma_3 u\,v\,u + \gamma_4 v\,u\,u + \\ \qquad \gamma_5 u\,v\,v + \gamma_6 v\,u\,v + \gamma_7 v\,v\,u + \gamma_8 v\,v\,v, \\ v_\tau = \delta_1 u\,u\,u + \delta_2 u\,u\,v + \delta_3 u\,v\,u + \delta_4 v\,u\,u + \\ \qquad \delta_5 u\,v\,v + \delta_6 v\,u\,v + \delta_7 v\,v\,u + \delta_8 v\,v\,v \end{cases}$$

is equivalent to one of the following:

$$a): \begin{cases} u_t = u\,u - u\,v, \\ v_t = v\,v - u\,v + v\,u, \end{cases} \qquad b): \begin{cases} u_t = u\,v, \\ v_t = v\,u, \end{cases}$$

$$c): \begin{cases} u_t = u\,u - u\,v, \\ v_t = v\,v - u\,v, \end{cases} \qquad d): \begin{cases} u_t = -u\,v, \\ v_t = v\,v + u\,v - v\,u, \end{cases}$$

$$e): \begin{cases} u_t = u\,v - v\,u, \\ v_t = u\,u + u\,v - v\,u, \end{cases} \qquad f): \begin{cases} u_t = v\,v, \\ v_t = u\,u. \end{cases}$$

It is a remarkable fact that a requirement of the existence of just one cubic symmetry selects a finite list of equations with no free parameters (or more precisely, all possible parameters can be removed by linear transformations (5)).

The five non–equivalent equations with quartic symmetries are given by

$$\begin{cases} u_t = -u\,v, \\ v_t = v\,v + u\,v, \end{cases} \qquad \begin{cases} u_t = -v\,u, \\ v_t = v\,v + u\,v, \end{cases} \qquad \begin{cases} u_t = u\,u - 2v\,u, \\ v_t = v\,v - 2v\,u, \end{cases}$$

$$\begin{cases} u_t = u\,u - u\,v - 2v\,u, \\ v_t = v\,v - v\,u - 2u\,v, \end{cases} \qquad g): \begin{cases} u_t = u\,u - 2u\,v, \\ v_t = v\,v + 4v\,u. \end{cases}$$

Using computer algebra system CRACK, T.Wolf verified that it is a complete list of non–triangular systems that have quartic (but have no cubic) symmetries.

Possibly equatiosn a) and g) have only one symmetry while the other 9 have infinitely many.

Attempts to describe systems (11) with fifth order symmetries by a straightforward computation looks rather hopeless. One of the reasons is that the coefficients of (11) turn out to be related by algebraic relations. Even if they can be resolved, the coefficients very often becomes algebraic numbers.

Example. The non-abelian system

$$\begin{cases} u_t = 11\sqrt{7}\,u\,u - 7\sqrt{7}\,v\,v, \\ v_t = -4\sqrt{7}\,v\,u - 4\sqrt{7}\,u\,v + 30\,u\,u \end{cases}$$

has a symmetry of fifth order.

We say that the system (11) is a non-abelization of a system (7) if in the abelian limit (11) coincides with (7).

New state of the classification problem.

Let us assume the the abelian limit of the system (11):

- **1** has a polynimial integral I (and therefore possesses an ibfinite sequance of polynomial symmetries);
- **2** satisfies the Kowalevski-Lyapunov test.

We also assume that (11) has symmetries such that their abelean limits coincides with symmetries described in Item **1**.

Consider the abelian systems

$$\begin{cases} u_t &= -k_3\, u^2 + (k_2 + k_3)\, uv \\ v_t &= -k_1\, v^2 + (k_1 + k_2)\, uv, \end{cases}$$

These systems have first integral

$$I = u^{k_1}(u - v)^{k_2} v^{k_3}.$$

The following three possibilities follows from the Kowalevski-Lyapunov test:

- Case 1. $k_1 = k_2 = k_3 = 1;$
- Case 2. $k_1 = k_3 = 1,\ k_2 = 2;$
- Case 3. $k_1 = 1,\ k_2 = 2,\ k_3 = 3.$

It is clear that any non-abelization this system has the form

$$\begin{cases} u_t &= -k_3\, u^2 + (k_2 + k_3)\, uv + \alpha(uv - vu) \\ v_t &= -k_1\, v^2 + (k_1 + k_2)\, vu + \beta(vu - uv), \end{cases} \tag{13}$$

where α and β are some constants.

Involution (12) transforms α and β in (13) as follows:

$$\alpha \to -\alpha - k_2 - k_3, \qquad \beta \to -\beta - k_2 - k_1.$$

A non-abelization is called integrable if it has symmetries whose abelian limit coincides with the symmetries of (7).

In the case $k_1 = k_2 = k_3 = 1$ the abelian system (7) has a fifth order symmetry.

Theorem 1. In the case $k_1 = k_2 = k_3 = 1$ there exist the following non-equivalent integrable non-abelizations:

- **1.** $\alpha = -1, \quad \beta = -1;$
- **2.** $\alpha = 0, \quad \beta = -1;$
- **3.** $\alpha = 0, \quad \beta = -2;$
- **4.** $\alpha = 0, \quad \beta = 0;$
- **5.** $\alpha = 0, \quad \beta = -3.$

Remark. The system **1** is equivalent to the system from Example 1. The systems **4** and **5** are new.

In the case $k_1 = k_3 = 1$, $k_2 = 2$ the abelian system (7) has a symmetry of order 6.

Theorem 2. In the case $k_1 = k_3 = 1$, $k_2 = 2$ there exist the following non-equivalent integrable non-abelizations:

- **1.** $\alpha = -1, \quad \beta = -1;$
- **2.** $\alpha = 0, \quad \beta = -2;$
- **3.** $\alpha = 0, \quad \beta = 0;$
- **4.** $\alpha = 0, \quad \beta = -4.$

Remark. The systems **2, 3** and **4** are new.

In the case $k_1 = 1$, $k_2 = 2$, $k_3 = 3$ the abelian system (7) has a symmetry of order 8.

Theorem 3. In the case $k_1 = 1$, $k_2 = 2$, $k_3 = 3$ there exist the following non-equivalent integrable non-abelizations:

- **1.** $\alpha = -2, \quad \beta = 0;$
- **2.** $\alpha = -4, \quad \beta = 0;$
- **3.** $\alpha = -6, \quad \beta = 0;$
- **4.** $\alpha = 0, \quad \beta = -6.$
- **5.** $\alpha = 0, \quad \beta = 0.$

Remark. All these systems are new.

Consider the case of abelian systems with a two-root first integral. By a linear transformation it can be reduced to

$$I = u^{k_1} v^{k_2}.$$

The Kowalevskaya-Lyapunov test leads to $k_1 = k_2 = 1$. The orders of symmetries are $2 + 2n$, i.e. the simplest symmetry has order 4. The possible non-abelizations are given by

$$\left\{ \begin{array}{rcl} u_t & = & -b_2\, u^2 + a_2\, uv + \alpha(uv - vu) \\ v_t & = & -a_2\, v^2 + b_2\, vu + \beta(vu - uv). \end{array} \right. \tag{14}$$

Theorem 4. There exist the following non-triangular integrable non-equaivalent systems of the form (14):

- **1.** $a_2 = b_2 = 1, \quad \alpha = 0, \quad \beta = 0;$
- **2.** $a_2 = b_2 = 1, \quad \alpha = 0, \quad \beta = -2;$
- **3.** $a_2 = b_2 = 1, \quad \alpha = 0, \quad \beta = -1;$
- **4.** $a_2 = 1,\ b_2 = 0, \quad \alpha = 0, \quad \beta = 1.$

In the case of abelian systems with a one-root first integral we have

$$I = u.$$

Their non-abelizataion is given by

$$\begin{cases} u_t &= \alpha(uv - vu) \\ v_t &= b_1\, v^2 + b_2\, vu + b_3 u^2 + \beta(vu - uv). \end{cases} \tag{15}$$

The simplest symmetry has order 3.

Theorem 5. There exist the following integrable non-triangular non-equivalent systems (15):

- **1.** $b_1 = b_2 = 1, \quad b_3 = 0, \qquad \alpha = 1, \qquad \beta = 0;$
- **2.** $b_1 = b_2 = 0, \quad b_3 = 1, \qquad \alpha = 1, \qquad \beta = 0.$

Conjecture. Theorems 1-5 give a complete list of integrable systems (11).

Triangular Laurent transformations.

Some of systems (11) admit invertible Laurent transformations. Consider systems of the form

$$\left\{\begin{array}{lcl} u_t & = & -p\,u^2 + q\,uv \\ v_t & = & -a\,v^2 + b\,vu + c\,uv. \end{array}\right. \tag{16}$$

It can be easily verified that the composition of the transformation

$$u = \bar{u}, \qquad v = \bar{u}^{-1}\bar{v}\bar{u}$$

and the involution (12) maps (16) to

$$\left\{\begin{array}{lcl} u_t & = & -p\,u^2 + q\,uv \\ v_t & = & -a\,v^2 + (c-p)\,vu + (b+p)\,uv. \end{array}\right. \tag{17}$$

Thus we have an involution $\tau : (16) \to (17)$ on the set of systems of the form (16).

In the case of Theorem 1 $\tau : \beta \mapsto -\beta - 3$ and therefore the systems 2,3 and 4,5 are dual with respect to τ.

In the case of Theorem 2 the involution τ corresponds to $\beta \mapsto -\beta - 4$. The cases 3,4 are dual and the case 2 is the self-dual.

In the case of Theorem 3 the systems 1,2 and 4,5 are dual and 3 is self-dual.

In the case of Theorem 4 the systems 1,2 and 4,5 are dual and 3 is self-dual.

The first system of Theorem 5 can be reduced to a triangular system

$$\left\{ \begin{array}{lcl} u_t & = & 0 \\ v_t & = & v^2 + vu \end{array} \right.$$

by a Laurent triangular transformation The latter system can be easily integrated in quadratures.

The second system of Theorem 5 implies

$$v_{tt} = [v_t,\, v]. \tag{18}$$

In the matrix case this equation can be reduce to a linear equation by the following way. If Y is a matrix solution of the equation

$$Y_t = Y(c_1 t + c_2),$$

where c_i are arbitrary constant matrices, then $v = -Y_t Y^{-1}$ is a general solution of (18).