On properties of parafermionic hyperbolic gamma functions.

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August 27, 2019

Based on works

G. Sarkissian and V. P. Spiridonov, "From rarefied elliptic beta integral to parafermionic star-triangle relation," JHEP **1810** (2018) 097

H. Poghosyan and G. Sarkissian, "Comments on fusion matrix in N=1 super Liouville field theory," Nucl. Phys. B **909** (2016) 458

Hyperbolic gamma function

Here we review double Gamma $\Gamma_b(x)$ and double Sine $S_b(x)$ functions .
$\Gamma_b(x)$ can be defined by means of the integral representation

$$\log\Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-xt}-e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q-2x)^2}{8e^t} - \frac{Q-2x}{t}\right] \quad (1.1)$$

It has the property:

$$\Gamma_b(x+b) = \sqrt{2\pi}\, b^{bx-\frac{1}{2}}\Gamma^{-1}(bx)\Gamma_b(x)\ . \qquad (1.2)$$

Hyperbolic gamma function

The double Sine function $S_b(x)$ may be defined in term of $\Gamma_b(x)$ as

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q-x)}\,. \tag{1.3}$$

It has an integral representation:

$$\log S_b(x) = \int_0^\infty \frac{dt}{t}\left(\frac{\sinh t(Q-2x)}{2\sinh bt \sinh b^{-1}t} - \frac{Q-2x}{2t}\right) \tag{1.4}$$

and the properties:

$$S_b(x+b) = 2\sin(\pi bx) S_b(x)\,, \tag{1.5}$$

$$S_b(x+1/b) = 2\sin(\pi x/b) S_b(x)\,. \tag{1.6}$$

Star-triangle relation for Hyperbolic gamma function

$$\int \frac{dx}{i} \prod_{i=1}^{3} S_b(x+a_i) S_b(-x+b_i) = \prod_{i,j=1} S_b(a_i+b_j) \tag{1.7}$$

$$\sum_i (a_i+b_i) = Q \tag{1.8}$$

$$\int_{-i\infty}^{i\infty} \exp\left(i\pi\left[(y(f_1+f_2+g_1)+f_1 f_2\right]\right)$$
$$S_b(y+f_1)S_b(y+f_2)S_b(-y+g_1)S_b(-y)\frac{dy}{i} =$$
$$S_b(Q-f_1-f_2-g_1)S_b(f_1+g_1)S_b(f_1)S_b(f_2+g_1)S_b(f_2).$$

$$\int_{-i\infty}^{i\infty} \exp i\pi(y(2\alpha + 2f - Q))S_b(y+f)S_b(-y+f)\frac{dy}{i}$$
$$= S_b(Q-\alpha-2f)S_b(\alpha)S_b(2f).$$

$$\int_{-i\infty}^{i\infty} e^{-\frac{i\pi}{24}(Q^2-2)} e^{\frac{i\pi}{2}y^2} e^{-2i\pi y(\frac{Q}{4}-\alpha)} S_b(y)\frac{dy}{i} = e^{-\frac{i\pi}{2}(\frac{Q}{2}-\alpha)^2} S_b(\alpha). \tag{1.9}$$

An integral representation convergent in the strip $0 < \mathrm{Re}(x) < Q$ is

$$\log \Upsilon_b(x) = \int_0^\infty \frac{dt}{t}\left[\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2(\frac{Q}{2}-x)\frac{t}{2}}{\sinh\frac{bt}{2}\sinh\frac{t}{2b}}\right]. \quad (1.10)$$

Important properties of $\Upsilon_b(x)$ are

1. Functional equation: $\Upsilon_b(x+b) = b^{1-2bx}\frac{\Gamma(bx)}{\Gamma(1-bx)}\Upsilon_b(x)$.
2. Analyticity: $\Upsilon_b(x)$ is entire analytic, zeros:
   $x = -nb - mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$,
   $x = Q + nb + mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}$.
3. Self-duality: $\Upsilon_b(x) = \Upsilon_{1/b}(x)$.
4. $\Upsilon_b(x) = \Upsilon_b(Q-x)$
5. $\Upsilon_b(x) = \frac{1}{\Gamma_b(x)\Gamma_b(Q-x)}$

Bootstrap equation

Bulk OPE has the form

$$\Phi_{(i\bar{i})}(z_1,\bar{z}_1)\Phi_{(j\bar{j})}(z_2,\bar{z}_2) = \sum_{k,\bar{k}} \frac{C^{(k\bar{k})}_{(i\bar{i})(j\bar{j})}}{(z_1-z_2)^{\Delta_i+\Delta_j-\Delta_k}(\bar{z}_1-\bar{z}_2)^{\Delta_{\bar{i}}+\Delta_{\bar{j}}-\Delta_{\bar{k}}}}\Phi_{(k\bar{k})}(z_2,\bar{z}_2) + \ldots \tag{2.1}$$

By the usual BPZ arguments we have for 4-point correlation function $\langle\Phi_i\Phi_k\Phi_j\Phi_l\rangle$ in s channel

$$\sum_{p\bar{p}} C^{p\bar{p}}_{j\bar{j}l\bar{l}} C^{i\bar{i}}_{k\bar{k}p\bar{p}} \mathcal{F}^s_p \begin{bmatrix} k & j \\ i & l \end{bmatrix} \mathcal{F}^s_{\bar{p}} \begin{bmatrix} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{bmatrix} \tag{2.2}$$

Bootstrap equation

and t channel

$$\sum_{q\bar{q}} C^{q\bar{q}}_{k\bar{k}j\bar{j}} C^{i\bar{i}}_{q\bar{q}l\bar{l}} \mathcal{F}^t_q \begin{bmatrix} k & j \\ i & l \end{bmatrix} \mathcal{F}^t_{\bar{q}} \begin{bmatrix} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{bmatrix} \qquad (2.3)$$

where $\mathcal{F}^s_p \begin{bmatrix} k & j \\ i & l \end{bmatrix}$ and $\mathcal{F}^t_q \begin{bmatrix} k & j \\ i & l \end{bmatrix}$ s and t channels conformal blocks correspondingly.
Conformal blocks in s and t channels are related by the fusing matrix

$$\mathcal{F}^s_p \begin{bmatrix} k & j \\ i & l \end{bmatrix} = \sum_q F_{p,q} \begin{bmatrix} k & j \\ i & l \end{bmatrix} \mathcal{F}^t_q \begin{bmatrix} k & j \\ i & l \end{bmatrix} \qquad (2.4)$$

$$\sum_{p\bar{p}} C^{p\bar{p}}_{j\bar{j}l\bar{l}} C^{i\bar{i}}_{k\bar{k}p\bar{p}} F_{p,q}\left[\begin{array}{cc} k & j \\ i & l \end{array}\right] F_{\bar{p},\bar{q}}\left[\begin{array}{cc} \bar{k} & \bar{j} \\ \bar{i} & \bar{l} \end{array}\right] = C^{q\bar{q}}_{k\bar{k}j\bar{j}} C^{i\bar{i}}_{q\bar{q}l\bar{l}} \tag{2.5}$$

Bootstrap equation implies

$$C^{p}_{ij} = \frac{\eta_i\eta_j}{\eta_0\eta_p} F_{0,p}\left[\begin{array}{cc} j & i \\ j & i \end{array}\right] \tag{2.6}$$

and

$$C^{p}_{ij} = \frac{\xi_i\xi_j}{\xi_0\xi_p} \frac{1}{F_{p,0}\left[\begin{array}{cc} j & j \\ i & i \end{array}\right]} \tag{2.7}$$

where

$$\eta_i = \sqrt{C_{ii}/F_i} \quad \xi_i = \sqrt{C_{ii}F_i} \quad F_i \equiv F_{0,0}\left[\begin{array}{cc} i & i \\ i & i \end{array}\right] \tag{2.8}$$

Liouville field theory

Liouville field theory is defined on a two-dimensional surface with metric g_{ab} by the local Lagrangian density

$$\mathcal{L} = \frac{1}{4\pi} g_{ab}\partial_a\varphi\partial_b\varphi + \mu e^{2b\varphi} + \frac{Q}{4\pi} R\varphi \tag{3.1}$$

where R is associated curvature and $Q = b + b^{-1}$. Primary fields V_α in this theory, which are associated with exponential fields $e^{2\alpha\varphi}$, have conformal dimensions

$$\Delta_\alpha = \alpha(Q - \alpha) \tag{3.2}$$

DOZZ structure constants

Structure constants are given by the the famous DOZZ formula

$$C(\alpha_1,\alpha_2,\alpha_3)=\lambda^{(Q-\sum_{i=1}^{3}\alpha_i)/b}\times \frac{\Upsilon_b(b)\Upsilon_b(2\alpha_1)}{\Upsilon_b(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_b(\alpha_1+\alpha_2-\alpha_3)}\times \frac{\Upsilon_b(2\alpha_2)\Upsilon_b(2\alpha_3)}{\Upsilon_b(\alpha_2+\alpha_3-\alpha_1)\Upsilon_b(\alpha_3+\alpha_1-\alpha_2)}, \tag{3.3}$$

where

$$\lambda=\pi\mu\gamma(b^2)b^{2-2b^2} \tag{3.4}$$

Ponsot-Teschner formula

$$F_{\alpha_s,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix} = \tag{3.5}$$

$$\frac{\Gamma_b(2Q-\alpha_t-\alpha_2-\alpha_3)\Gamma_b(Q-\alpha_t+\alpha_3-\alpha_2)}{\Gamma_b(2Q-\alpha_1-\alpha_s-\alpha_2)\Gamma_b(Q-\alpha_s-\alpha_2+\alpha_1)}$$

$$\times \quad \frac{\Gamma_b(Q+\alpha_t-\alpha_2-\alpha_3)\Gamma_b(\alpha_3+\alpha_t-\alpha_2)}{\Gamma_b(Q-\alpha_1-\alpha_2+\alpha_s)\Gamma_b(\alpha_s+\alpha_1-\alpha_2)}$$

$$\times \quad \frac{\Gamma_b(Q-\alpha_t-\alpha_1+\alpha_4)\Gamma_b(\alpha_1+\alpha_4-\alpha_t)}{\Gamma_b(Q-\alpha_s-\alpha_3+\alpha_4)\Gamma_b(\alpha_3+\alpha_4-\alpha_s)}$$

$$\frac{\Gamma_b(\alpha_t+\alpha_4-\alpha_1)\Gamma_b(\alpha_t+\alpha_1+\alpha_4-Q)}{\Gamma_b(\alpha_s+\alpha_4-\alpha_3)\Gamma_b(\alpha_s+\alpha_3+\alpha_4-Q)}$$

$$\times \frac{\Gamma_b(2Q-2\alpha_s)\Gamma_b(2\alpha_s)}{\Gamma_b(Q-2\alpha_t)\Gamma_b(2\alpha_t-Q)}\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau J_{\alpha_s,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}$$

$$J_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right] = \tag{3.6}$$

$$\frac{S_b(Q+\tau-\alpha_1)S_b(\tau+\alpha_4+\alpha_2-\alpha_3)}{S_b(Q+\tau+\alpha_4-\alpha_t)S_b(\tau+\alpha_4+\alpha_t)}$$

$$\times \quad \frac{S_b(\tau+\alpha_1)S_b(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_b(Q+\tau+\alpha_2-\alpha_s)S_b(\tau+\alpha_2+\alpha_s)}$$

with the help of (3.16) one can show

$$F_{0,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_1\\ \alpha_3 & \alpha_1\end{array}\right] = C(\alpha_t,\alpha_1,\alpha_3)\frac{W(Q)W(\alpha_t)}{4\pi W(Q-\alpha_1)W(Q-\alpha_3)} \tag{3.7}$$

In the limit

$$\alpha_t = \epsilon \to 0, \qquad \alpha_3 = \alpha_2, \qquad \alpha_4 = \alpha_1 \tag{3.8}$$

$$\lim_{\epsilon\to 0} F_{\alpha_s,\epsilon}\left[\begin{array}{cc}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{array}\right] = \tag{3.9}$$

$$\frac{1}{\pi\epsilon^2 C(\alpha_s,\alpha_2,\alpha_1)}\frac{W(0)W(Q-\alpha_s)}{W(\alpha_1)W(\alpha_2)}$$

Super Barnes and Upsilon functions

In the super Liouville theory are important the functions

$$\Gamma_1(x) \equiv \Gamma_{\rm NS}(x) = \Gamma_b\left(\frac{x}{2}\right)\Gamma_b\left(\frac{x+Q}{2}\right), \tag{3.10}$$

$$\Gamma_0(x) \equiv \Gamma_{\rm R}(x) = \Gamma_b\left(\frac{x+b}{2}\right)\Gamma_b\left(\frac{x+b^{-1}}{2}\right). \tag{3.11}$$

The structure constants in the super Liouville theory are defined in terms of the functions:

$$\Upsilon_1(x) \equiv \Upsilon_{\rm NS}(x) = \Upsilon_b\left(\frac{x}{2}\right)\Upsilon_b\left(\frac{x+Q}{2}\right) = \frac{1}{\Gamma_{\rm NS}(x)\Gamma_{\rm NS}(Q-x)}, \tag{3.12}$$

$$\Upsilon_0(x) \equiv \Upsilon_{\rm R}(x) = \Upsilon_b\left(\frac{x+b}{2}\right)\Upsilon_b\left(\frac{x+b^{-1}}{2}\right) = \frac{1}{\Gamma_{\rm R}(x)\Gamma_{\rm R}(Q-x)}. \tag{3.13}$$

Super hyperbolic gamma functions

To write fusion matrix we need also the functions:

$$S_1(x) \equiv S_{\rm NS}(x) = S_b\left(\frac{x}{2}\right) S_b\left(\frac{x}{2}+\frac{Q}{2}\right) = \frac{\Gamma_{\rm NS}(x)}{\Gamma_{\rm NS}(Q-x)}\,, \quad (3.14)$$

$$S_0(x) \equiv S_{\rm R}(x) = S_b\left(\frac{x}{2}+\frac{b}{2}\right) S_b\left(\frac{x}{2}+\frac{b^{-1}}{2}\right) = \frac{\Gamma_{\rm R}(x)}{\Gamma_{\rm R}(Q-x)}\,. \quad (3.15)$$

Star-triangle relation for Super Barnes functions

$$\sum_{m=0,1}(-)^{m(1+\sum_a(\nu_a+\mu_a))/2}\int\frac{dx}{i}\prod_{a=1}^{3}S_{m+\nu_a}(x+f_a)S_{1+m+\mu_a}(-x+g_a)$$

$$=2(-1)^{(\sum\mu_a)(1+\sum_a(\nu_a+\mu_a))/2}\prod_{a,b=1}^{3}S_{\nu_a+\mu_b}(f_a+g_b)\,, \quad (3.16)$$

$$\sum_a(\nu_a+\mu_a)=1 \bmod 2\,, \quad (3.17)$$

and

$$\sum_a(f_a+g_a)=Q\,. \quad (3.18)$$

Primaries in super Liouville fielf theory

NS-NS primary fields $N_\alpha(z,\bar z)$ in this theory, $N_\alpha(z,\bar z)=e^{\alpha\varphi(z,\bar z)}$, have conformal dimensions

$$\Delta^{NS}_\alpha=\frac{1}{2}\alpha(Q-\alpha)\,. \tag{4.1}$$

Introduce also the field

$$\tilde N_\alpha(z,\bar z)=G_{-1/2}\bar G_{-1/2}N_\alpha(z,\bar z)\,. \tag{4.2}$$

The R-R is defined as

$$R_\alpha(z,\bar z)=\sigma(z,\bar z)e^{\alpha\varphi(z,\bar z)}\,, \tag{4.3}$$

where σ is the spin field.

The dimension of the R-R operator is

$$\Delta^{R}_\alpha=\frac{1}{16}+\frac{1}{2}\alpha(Q-\alpha)\,. \tag{4.4}$$

Structure constants in SL

The structure constants in $N=1$ super Liouville field theory are computed in [R. H. Poghossian, hep-th/9607120] and [R. C. Rashkov and M. Stanishkov, hep-th/9602148]

$$\langle N_{\alpha_1}(z_1,\bar{z}_1)N_{\alpha_2}(z_2,\bar{z}_2)N_{\alpha_3}(z_3,\bar{z}_3)\rangle = \frac{C_{NS}(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_2}-\Delta^N_{\alpha_3})}|z_{23}|^{2(\Delta^N_{\alpha_2}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_1})}|z_{13}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_2})}}, \tag{4.5}$$

$$\langle \tilde{N}_{\alpha_1}(z_1,\bar{z}_1)N_{\alpha_2}(z_2,\bar{z}_2)N_{\alpha_3}(z_3,\bar{z}_3)\rangle = \frac{\tilde{C}_{NS}(\alpha_1,\alpha_2,\alpha_3)}{|z_{12}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_2}-\Delta^N_{\alpha_3}+1/2)}|z_{23}|^{2(\Delta^N_{\alpha_2}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_1}-1/2)}} \times \frac{1}{|z_{13}|^{2(\Delta^N_{\alpha_1}+\Delta^N_{\alpha_3}-\Delta^N_{\alpha_2}+1/2)}}, \tag{4.6}$$

$$C_{NS}(\alpha_1,\alpha_2,\alpha_3) = \lambda^{(Q-\sum_{i=1}^{3}\alpha_i)/b} \times \frac{\Upsilon'_{NS}(0)\Upsilon_{NS}(2\alpha_1)\Upsilon_{NS}(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_{NS}(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_{NS}(\alpha_1+\alpha_2-\alpha_3)} \times \frac{1}{\Upsilon_{NS}(\alpha_2+\alpha_3-\alpha_1)\Upsilon_{NS}(\alpha_3+\alpha_1-\alpha_2)}, \tag{4.7}$$

$$\tilde{C}_{NS}(\alpha_1,\alpha_2,\alpha_3) = \lambda^{(Q-\sum_{i=1}^{3}\alpha_i)/b} \times \frac{\Upsilon'_{NS}(0)\Upsilon_{NS}(2\alpha_1)\Upsilon_{NS}(2\alpha_2)\Upsilon_{NS}(2\alpha_3)}{\Upsilon_{R}(\alpha_1+\alpha_2+\alpha_3-Q)\Upsilon_{R}(\alpha_1+\alpha_2-\alpha_3)} \times \frac{1}{\Upsilon_{R}(\alpha_2+\alpha_3-\alpha_1)\Upsilon_{R}(\alpha_3+\alpha_1-\alpha_2)}, \tag{4.8}$$

Fusion matrix in the NS sector is computed in [L. Hadasz arXiv:0707.3384], [D. Chorazkiewicz and L. Hadasz, arXiv:0811.1226] Let us denote

Fusion matrix in SL

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^1_1 \equiv F_{N_{\alpha_s},N_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix} \tag{4.9}$$

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^2_1 \equiv F_{N_{\alpha_s},\tilde{N}_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix}, \tag{4.10}$$

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^1_2 \equiv F_{\tilde{N}_{\alpha_s},N_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix} \tag{4.11}$$

$$F_{\alpha_s,\alpha_t}\begin{bmatrix}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{bmatrix}^2_2 \equiv F_{\tilde{N}_{\alpha_s},\tilde{N}_{\alpha_t}}\begin{bmatrix}N_{\alpha_3} & N_{\alpha_2}\\ N_{\alpha_4} & N_{\alpha_1}\end{bmatrix}. \tag{4.12}$$

To write the fusion matrix we use the following convention. The functions $\Upsilon_i, \Gamma_i, S_i$ will be understood $\Upsilon_{NS}, \Gamma_{NS}, S_{NS}$ for $i = 1 \bmod 2$, and $\Upsilon_R, \Gamma_R, S_R$ for $i = 0 \bmod 2$. Now we can write the fusion matrix:

$$F_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]_j^i = \tag{4.13}$$

$$\frac{\Gamma_i(2Q-\alpha_t-\alpha_2-\alpha_3)\Gamma_i(Q-\alpha_t+\alpha_3-\alpha_2)}{\Gamma_j(2Q-\alpha_1-\alpha_s-\alpha_2)\Gamma_j(Q-\alpha_s-\alpha_2+\alpha_1)}$$

$$\times\ \frac{\Gamma_i(Q+\alpha_t-\alpha_2-\alpha_3)\Gamma_i(\alpha_3+\alpha_t-\alpha_2)}{\Gamma_j(Q-\alpha_1-\alpha_2+\alpha_s)\Gamma_j(\alpha_s+\alpha_1-\alpha_2)}\frac{\Gamma_i(Q-\alpha_t-\alpha_1+\alpha_4)\Gamma_i(\alpha_1+\alpha_4-\alpha_t)}{\Gamma_j(Q-\alpha_s-\alpha_3+\alpha_4)\Gamma_j(\alpha_3+\alpha_4-\alpha_s)}\frac{\Gamma_i(\alpha_t+\alpha_4-\alpha_1)\Gamma_i(\alpha_t+\alpha_1+\alpha_4-Q)}{\Gamma_j(\alpha_s+\alpha_4-\alpha_3)\Gamma_j(\alpha_s+\alpha_3+\alpha_4-Q)}$$

$$\times\frac{\Gamma_{\rm NS}(2Q-2\alpha_s)\Gamma_{\rm NS}(2\alpha_s)}{\Gamma_{\rm NS}(Q-2\alpha_t)\Gamma_{\rm NS}(2\alpha_t-Q)}\frac{1}{i}\int_{-i\infty}^{i\infty} d\tau J_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]_j^i,$$

$$
J_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]_1^1 = \tag{4.14}
$$

$$
\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)}\times
\frac{S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)}
$$

$$
+\quad\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)}\times
\frac{S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)},
$$

Fusion matrix in NS sector

$$
J_{\alpha_s,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_2 \\ \alpha_4 & \alpha_1 \end{bmatrix}_2^1 = \tag{4.15}
$$

$$
\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)}\times \frac{S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)}
$$

$$
- \quad \frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)}\times \frac{S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)},
$$

$$J_{\alpha_s,\alpha_t}\left[\begin{array}{cc}\alpha_3 & \alpha_2\\ \alpha_4 & \alpha_1\end{array}\right]^2_1 = \tag{4.16}$$

$$\frac{S_{NS}(Q+\tau-\alpha_1)S_{\mathrm{NS}}(\tau+\alpha_4+\alpha_2-\alpha_3)}{S_R(Q+\tau+\alpha_4-\alpha_t)S_R(\tau+\alpha_4+\alpha_t)}\times$$
$$\frac{S_{NS}(\tau+\alpha_1)S_{NS}(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_{NS}(Q+\tau+\alpha_2-\alpha_s)S_{NS}(\tau+\alpha_2+\alpha_s)}$$
$$-\quad\frac{S_R(Q+\tau-\alpha_1)S_R(\tau+\alpha_4+\alpha_2-\alpha_3)}{S_{NS}(Q+\tau+\alpha_4-\alpha_t)S_{NS}(\tau+\alpha_4+\alpha_t)}\times$$
$$\frac{S_R(\tau+\alpha_1)S_R(\tau+\alpha_4+\alpha_2+\alpha_3-Q)}{S_R(Q+\tau+\alpha_2-\alpha_s)S_R(\tau+\alpha_2+\alpha_s)},$$

Fusion matrix in NS sector

$$F_{0,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 \end{bmatrix}_1^1 = C_{NS}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_{NS}(\alpha_t)}{\pi W_{NS}(Q-\alpha_1)W_{NS}(Q-\alpha_3)}. \tag{4.17}$$

$$F_{0,\alpha_t}\begin{bmatrix} \alpha_3 & \alpha_1 \\ \alpha_3 & \alpha_1 \end{bmatrix}_1^2 = \tilde{C}_{NS}(\alpha_t,\alpha_1,\alpha_3)\frac{W_{NS}(Q)W_{NS}(\alpha_t)}{\pi W_{NS}(Q-\alpha_1)W_{NS}(Q-\alpha_3)}. \tag{4.18}$$

Fusion matrix in NS sector

$$\tilde{F}_{\alpha_s,0}\left[\begin{array}{cc}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{array}\right]_1^1 = \lim_{\epsilon\to 0}\epsilon^2 F_{\alpha_s,\epsilon}\left[\begin{array}{cc}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{array}\right]_1^1 = (4.19)$$

$$\frac{4}{\pi C_{NS}(\alpha_s,\alpha_2,\alpha_1)}\frac{W_{NS}(0)W_{NS}(Q-\alpha_s)}{W_{NS}(\alpha_1)W_{NS}(\alpha_2)}.$$

$$\tilde{F}_{\alpha_s,0}\left[\begin{array}{cc}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{array}\right]_2^1 = \lim_{\epsilon\to 0}\epsilon^2 F_{\alpha_s,\epsilon}\left[\begin{array}{cc}\alpha_2 & \alpha_2\\ \alpha_1 & \alpha_1\end{array}\right]_2^1 = (4.20)$$

$$\frac{4}{\pi \tilde{C}_{NS}(\alpha_s,\alpha_2,\alpha_1)}\frac{W_{NS}(0)W_{NS}(Q-\alpha_s)}{W_{NS}(\alpha_1)W_{NS}(\alpha_2)}.$$

Parafermionic hyperbolic gamma functions

Consider now the functions $\Lambda_b(y;m)$, $0 \leq m \leq r$, for arbitrary r:

$$\Lambda_b(y,m) = \prod_{k=0}^{m-1} S_b\left(\frac{y}{r} + b\left(1 - \frac{m}{r}\right) + Q\frac{k}{r}\right) \times \prod_{k=0}^{r-m-1} S_b\left(\frac{y}{r} + \frac{m}{r}b^{-1} + Q\frac{k}{r}\right). \quad (4.21)$$

Compare them with the $\Upsilon_m^{(r)}(y)$ functions defined in [M. A. Bershtein, V. A. Fateev and A. V. Litvinov, Nucl. Phys. B **847** (2011) 413] for the purpose of calculation of three-point functions in the parafermionic Liouville field theory:

$$\Upsilon_m^{(r)}(y) = \prod_{j=1}^{r-m} \Upsilon_b\left(\frac{y+mb^{-1}+(j-1)Q}{r}\right) \tag{4.22}$$
$$\times \prod_{j=r-m+1}^{r} \Upsilon_b\left(\frac{y+(m-r)b^{-1}+(j-1)Q}{r}\right).$$

Then the substitution $j = k+1$ in its first product yields precisely the second product in (4.21). Similarly, the substitution $j = k+r-m+1$ converts its second product to the first one in (4.21) because $b^{-1} - Q = -b$. So, we have intriguing exact structural correspondence between the functions (4.22) and (4.21). For this reason we call $\Lambda_b(y;m)$ the parafermionic hyperbolic gamma function. It should play the same role in the construction of parafermionic fusion matrices as $\Upsilon_m^{(r)}(y)$ serves the correlation functions.

S-t relation for Parafermionic hyperbolic gamma functions

$$\Lambda_b(y, m+kr) = (-1)^{mk+r\frac{k(k-1)}{2}}\Lambda_b(y,m) \quad (4.23)$$

$$\int_{-i\infty}^{i\infty}\sum_{m=0}^{r-1}(-1)^m\prod_{a=1}^{3}\Lambda_b(y+f_a, n_a+m+\epsilon)\Lambda_b(-y+g_a, l_a-m)\frac{dy}{i}$$

$$=(-1)^{\epsilon+n_1+n_2+n_3}r\prod_{a,c=1}^{3}\Lambda_b(f_a+g_c, n_a+l_c+\epsilon). \quad (4.24)$$

$$\sum_{a=1}^{3}(f_a+g_a) = Q \quad (4.25)$$

and

$$\sum_{a=1}^{3}(n_a+l_a) = -3\epsilon. \quad \epsilon = 0,1 \quad (4.26)$$