Universality and quantum dimensions

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Vogel's plane

Vogel's plane is the projective plane P^2 (CP^2, RP^2) with homogeneous coordinates α, β, γ,

$$(\alpha, \beta, \gamma) \sim (\lambda\alpha, \lambda\beta, \lambda\gamma), \lambda \neq 0$$

factorized over group S_3 of permutations of α, β, γ:

$$P^2/S_3$$

It appears in Vogel's (1995) study of finite Vassiliev's knots invariants .

Vogel's parameterization.

Let $2t$ be an eigenvalue of second Casimir operator on adjoint representation: $C_2(ad) = 2t$. Then for all simple (super)algebras:

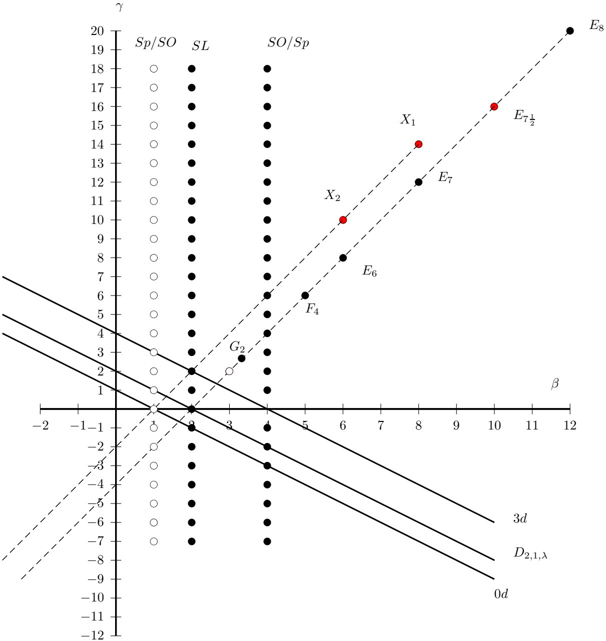
$$S^2 ad = \underline{1} + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma), \;\; \alpha + \beta + \gamma = t$$
$$C_2(Y_2(\alpha)) = 4t - 2\alpha$$
$$C_2(Y_2(\beta)) = 4t - 2\beta$$
$$C_2(Y_2(\gamma)) = 4t - 2\gamma$$

Table: Vogel's parameters for simple Lie algebras

Algebra/Parameters	α	β	γ	t	Line
$\mathfrak{sl}_N$	-2	2	N	N	$\alpha + \beta = 0$
$\mathfrak{so}_N$	-2	4	$N - 4$	$N - 2$	$2\alpha + \beta = 0$
$\mathfrak{sp}_N$	-2	1	$N/2 + 2$	$N/2 + 1$	$\alpha + 2\beta = 0$
$Exc(n)$	-2	$2n + 4$	$n + 4$	$3n + 6$	$\gamma = 2(\alpha + \beta)$

For the exceptional line $n = -2/3, 0, 1, 2, 4, 8$ for $\mathfrak{g}_2, \mathfrak{so}_8, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, respectively.

Parameters α, β, γ belong to Vogel's plane P^2/S_3. This table is on agreement with $SO(N) \leftrightarrow Sp(N)$ under $N \leftrightarrow -N$ duality (King 1971, Cvitanovich 1981, RM 1981).


γ
β
Sp/SO
SL
SO/Sp
E8
E7½
X1
E7
X2
E6
F4
G2
3d
D2,1,λ
0d

Dimension formulae

Dimension of simple Lie algebras in universal form:

$$dim\, \mathfrak{g} \quad = \quad \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}, \quad t=\alpha+\beta+\gamma$$

Further dimension formulae (Vogel 1995-2011, Landsberg, Manivel, 2002,2006)

$$dim\, Y_2(\alpha) \quad = \quad -\frac{(3\,\alpha-2\,t)\,(\beta-2\,t)\,(\gamma-2\,t)\,t\,(\beta+t)\,(\gamma+t)}{\alpha^2\,\beta\,\gamma\,(\alpha-\beta)\,(\alpha-\gamma)}$$

$$dim\, Y_3(\alpha) \quad = \quad \frac{(5\,\alpha-2\,t)\,(\alpha-2\,t)\,(\beta-2\,t)\,(\gamma-2\,t)\,t\,(\beta-\alpha+t)\,(\beta+t)\,(\gamma+t)}{\alpha^3\,\beta\,\gamma\,(\alpha-\beta)\,(\alpha-\gamma)\,(2\alpha-\beta)\,(2\alpha-\gamma)}$$

(For $Y_i(\beta)$, $Y_i(\gamma)$, $i=2,3$ permute α,β,γ)

Quantum dimension of adjoint representation. Generalized Freudenthal-DeVriese strange formula

Universal form of character of adjoint representation at line $x\rho$, i.e. quantum dimension of adjoint (Westbury 2004; Veselov, RM 2012):

$$\chi_{ad}(x\rho) = r + \sum_{\mu \in R} e^{x(\mu,\rho)} \equiv f(x)$$

$$f(x) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})}\frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})}\frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})}$$

Expansion over x: zeroth order - dimension formula for adjoint, then $O(x)$:

$$\sum_{\mu \in R_+} (\mu,\rho)^2 = \frac{t^2}{12} dim\, \mathfrak{g},$$

(Freudenthal-De Vriese strange formula), then $O(x^2)$:

$$\sum_{\mu \in R_+} (\mu,\rho)^4 = \frac{t(18t^3 - 3tt_2 + t_3)}{480} dim\, \mathfrak{g},$$

(generalized F-DV strange relation), etc.

Generating function for Casimirs' eigenvalues

Define Casimir operators for any simple Lie algebra $\mathfrak{g}$ as the following elements of the centre of the corresponding universal enveloping algebra $U\mathfrak{g}$

$$C_p = g^{\mu_1 \cdots \mu_p} X_{\mu_1} \dots X_{\mu_p},\ p = 0, 1, 2, \dots$$

where X_μ are the generators of $\mathfrak{g}$,

$$g_{\mu_1 \dots \mu_n} = Tr(\hat{X}_{\mu_1} \dots \hat{X}_{\mu_n}),$$

where the trace is taken in the adjoint representation of $\mathfrak{g}$ and the indices are lifted using the *canonical Cartan-Killing form*

$$g_{\mu\nu} = Tr(\hat{X}_\mu \hat{X}_\nu)$$

Then generating function $C(z) = \sum_{p=0}^{\infty} C_p z^p$ of their eigenvalues on adjoint has the universal form (Sergeev, Veselov, RM 2011):

$$C(z) = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma} + z^2 \frac{96t^3 + 168t^3 z + 6(14t^3 + tt_2 - t_3)z^2 + (13t^3 + 3tt_2 - 4t_3)z^3}{6(2t + \alpha z)(2t + \beta z)(2t + \gamma z)(2 + z)(1 + z)}.$$

where $t_2 = \alpha^2 + \beta^2 + \gamma^2$, $t_3 = \alpha^3 + \beta^3 + \gamma^3$. Derivation: Okubo expression for higher Casimirs in terms of dimensions and C_2 of irreps in square of a given irrep.

Invariant volume of compact simple Lie groups

Universal form (RM, 2013; Khudaverdian, RM, 2016) of invariant volume of compact simple Lie groups (Macdonald,1980; Kac,Peterson,1984), in an arbitrary normalization of invariant bilinear form:

$$-\ln(Vol) = \frac{1}{2} dim \ln\left(\frac{t}{4\pi^2}\right) + \int_0^\infty \frac{dx}{x} \frac{f(\frac{x}{t}) - dim}{(e^x - 1)} \tag{1}$$

$$f(x) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})} \frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})} \frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})}$$

$$dim = \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}, \quad t = \alpha+\beta+\gamma$$

This is the particular case of more general expression for partition function of Chern-Simons theory (RM, 2013).

Universality in Chern-Simons theory

Chern-Simons theory:

$$S(A) = \frac{\kappa}{4\pi} \int_M Tr \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$

A is $\mathfrak{g}$-valued 1-form on a 3-dimensional manifold M (S^3 below). Tr is some invariant bilinear form on a simple Lie algebra $\mathfrak{g}$. We will not fix the choice of such a form allowing coupling constant κ to change inversely under rescaling the form. $\kappa = k$ (level) in minimal normalization $\alpha = -2$. Theory depends on 4 parameters $\alpha, \beta, \gamma, \kappa$ defined up to a common multiplier, where α, β, γ are Vogel's parameters. Replace κ by

$$\delta = \kappa + t = \kappa + \alpha + \beta + \gamma.$$

An example of universal quantity in Chern-Simons theory - central charge (RM, Veselov, 2012):

$$c = k \frac{dim(g)}{k + h^\vee} = \frac{(\delta - t)(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma\delta}.$$

Universal partition function of Chern-Simons theory on S^3

Partition function Z can be expressed as product of perturbative Z_2 (Veselov, RM 2012) and non-perturbative Z_1 (RM 2013) parts:

$$
\begin{aligned}
Z &= Z_1 Z_2 \\
-\ln Z_2 &= \int_0^\infty \frac{dx}{x} \frac{f(x/\delta) - dim}{(e^x - 1)} \\
-\ln Z_1 &= (dim/2)\ln(\delta/t) - \int_0^\infty \frac{dx}{x} \frac{f(x/t) - dim}{(e^x - 1)} \\
-\ln Z &= (dim/2)\ln(\delta/t) + \int_0^\infty \frac{dx}{x} \frac{f(x/\delta) - f(x/t)}{(e^x - 1)}
\end{aligned}
$$

The road map of Vogel's plane (Diophantine classification of simple Lie algebras)

For simple Lie algebras character of adjoint representation is regular in finite x-plane. Let's find all points in Vogel's plane with this regularity feature (RM, 2012).

$$f(x) = \frac{\sinh(x\frac{\alpha-2t}{4})}{\sinh(x\frac{\alpha}{4})} \frac{\sinh(x\frac{\beta-2t}{4})}{\sinh(x\frac{\beta}{4})} \frac{\sinh(x\frac{\gamma-2t}{4})}{\sinh(x\frac{\gamma}{4})}$$

Each zero of sins in denominator at some value of x should be canceled by zero of nominator, which means that for each value of $\kappa = \alpha, \beta, \gamma$ at least one of the ratios $(2t-\alpha)/\kappa, (2t-\beta)/\kappa, (2t-\gamma)/\kappa$ should be integer.
The complete matrix of these ratios

$$R_{\kappa,\sigma} = (2t-\kappa)/\sigma,$$

where $\kappa, \sigma = \alpha, \beta, \gamma$, can be easily calculated for all simple Lie algebras, examples for SU(n) and G_2 are given below.

Table: Matrix R for SU(N)

-(N+1)	1-N	$-\frac{N}{2}$
N+1	N-1	$\frac{N}{2}$
$2+\frac{2}{N}$	$2-\frac{2}{N}$	1

Table: Matrix R for G_2

-5	$-\frac{7}{3}$	$-\frac{8}{3}$
3	$\frac{7}{5}$	$\frac{8}{5}$
$\frac{15}{3}$	$\frac{7}{4}$	2

There are seven patterns of poles' cancellation (=ways to put integers in each line) up to permutations. To list them we list three values of κ for which $R_{\kappa_1,\alpha}, R_{\kappa_2,\beta}, R_{\kappa_3,\gamma}$ are integers (denote them k, n, m respectively):

$$1)(\alpha,\alpha,\alpha), 2)(\alpha,\alpha,\beta), 3)(\alpha,\alpha,\gamma), 4)(\alpha,\beta,\gamma), 5)(\alpha,\gamma,\beta), 6)(\beta,\alpha,\alpha), 7)(\beta,\gamma,\alpha)$$

Call them 1aaa, 2aab, 3aag, 4abg, 5agb, 6baa and 7bga respectively.

Consider for example the fourth, most symmetric, pattern 4abg. One have

$$(2t-\alpha)=k\alpha, (2t-\beta)=n\beta, (2t-\gamma)=m\gamma$$

or in matrix form:

Table: Matrix form of equation

$$\begin{vmatrix} 1\text{-k} & 2 & 2 \\ 2 & 1\text{-n} & 2 \\ 2 & 2 & 1\text{-m} \end{vmatrix}\begin{vmatrix} \alpha \\ \beta \\ \gamma \end{vmatrix} = 0$$

This is a system of three linear equations on three variables α, β, γ, with zero variable-free terms, so non-trivial solution can exist only if corresponding determinant is zero. That determinant is a third order polynomial over k,n,m and equation is $knm = kn + nm + km + 3n + 3k + 3m + 5$. We shall call this equation a Diophantine equation (or condition) for a given pattern.

Diophantine condition for 4abg pattern and expressions for universal parameters in terms of k,n,m:

$$knm = kn + nm + km + 3n + 3k + 3m + 5$$
$$\frac{2}{k+1} + \frac{2}{n+1} + \frac{2}{m+1} = 1$$
$$\alpha = \frac{2t}{k+1}, \beta = \frac{2t}{n+1}, \gamma = \frac{2t}{m+1}$$

There are two types of solutions - with parameter, and isolated ones. Parametric solutions are:

$$SU(N) : (k, n, m) = (-N-1, N-1, 1), \ (\alpha, \beta, \gamma) = (2, -2, N)$$
$$D_{2,1,\lambda} : (k, n, m) = (-1, -1, -1), \ \alpha + \beta + \gamma = 0$$

Table: Isolated solutions of 4abg pattern

k n m	$\alpha\beta\gamma$	Dim	Rank	Algebra	Lines
4 2 -31	-6 -10 1	248	8	E_8	Exc, M
3 2 -13	-3 -4 1	78	6	E_6	Exc, F
2 2 -7	-2 -2 1	28	4	$SO(8)$	SO,Exc
0 -5 -5	4 -1 -1	0	0	$0d_3$	0d
0 -4 -7	6 -2 -1	0	0	$0d_4$	0d
5 5 5	1 1 1	-125	-19	Y_1	?
9 4 4	1 2 2	-144	-14	Y_{10}	K,M
7 7 3	1 1 2	-147	-17	Y_{11}	F
11 5 3	1 2 3	-165	-13	Y_{15}	F
19 4 3	1 4 5	-228	-10	Y_{29}	F,M
11 11 2	1 1 4	-242	-18	Y_{31}	Exc
14 9 2	2 3 10	-252	-8	Y_{35}	Exc
17 8 2	1 2 6	-272	-14	Y_{38}	Exc
23 7 2	1 3 8	-322	-12	Y_{43}	Exc
41 6 2	1 6 14	-492	-10	Y_{47}	Exc

Deligne's hypothesis on universal quantum dimensions

Let we have decomposition of product of representations

$$R \otimes R' = \sum R''$$

Deligne's hypothesis on universal quantum dimensions (2013) is that corresponding relation on quantum dimensions is fulfilled on entire Vogel's plane

$$f(R)f(R') = \sum f(R'')$$

provided all quantum dimensions $f(R), f(R'), f(R'')$ are in universal form.
This is proved for SL(N) line (Deligne, 2013).
Example of such relation:

$$\chi_{Sym^2(ad)}(x\rho) = \chi_{Y_2(\alpha)}(x\rho) + \chi_{Y_2(\beta)}(x\rho) + \chi_{Y_2(\gamma)}(x\rho) + 1$$

$$\chi_{Sym^2(ad)}(x\rho) = \frac{1}{2}\left(f^2(x) + f(2x)\right) \quad (2)$$

Deligne hypothesis is also checked for the cube of adjoint representation (RM 2017).

One-instanton contribution into Nekrasov's partition function for an arbitrary group in pure N=2 4d superYang-Mills theory, calculated for an arbitrary gauge group G in (Keller et al. 2011, Benvenuti et al. 2010). It is essentially given by $\sigma \to 0$ limit of character of representation

$$\sum_{n=1}^{\infty} V(-n\theta) \otimes T^{\otimes n}$$

of group $G \otimes U(1)^2$. Here $V(-n\theta)$ is the irrep with highest weight $n\lambda_{ad}$, i.e. n-th Cartan power of adjoint, $T = T_1 T_2$, $T_i = \exp(\sigma\epsilon_i)$, $i = 1, 2$, ϵ_i are Nekrasov's parameters. Universalization of this expression is possible, if vacuum expectation value of scalar field (argument of character) is restricted to Weyl line $x\rho$. Then in this formula appears the quantum dimension of n-th Cartan power of adjoint, calculated in (RM 2017) and we get a universal expression for 1-instanton contribution:

$$\sum_{n=1}^{\infty} e^{n\sigma(\epsilon_1+\epsilon_2)} \frac{\text{Sinh}\left[\frac{(2\gamma+2\beta-(-3+2n)\alpha)x}{4}\right]}{\text{Sinh}\left[\frac{(2\gamma+2\beta+3\alpha)x}{4}\right]} \times$$

$$\prod_{i=1}^{n} \frac{\text{Sinh}\left[\frac{(\gamma+2\beta-(-3+i)\alpha)x}{4}\right]}{\text{Sinh}\left[\frac{(\gamma-(-1+i)\alpha)x}{4}\right]} \frac{\text{Sinh}\left[\frac{(2\gamma+\beta-(-3+i)\alpha)x}{4}\right]}{\text{Sinh}\left[\frac{(\beta-(-1+i)\alpha)x}{4}\right]} \frac{\text{Sinh}\left[\frac{(2\gamma+2\beta-(-4+i)\alpha)x}{4}\right]}{\text{Sinh}\left[\frac{-i\alpha x}{4}\right]}$$

X_2 representation

$$\wedge^2 \mathfrak{g} = \mathfrak{g} + X_2$$

The representation X_2 is irreducible w.r.t. the semidirect product of simple Lie algebra on the automorphism of corresponding Dynkin diagram and has the highest weight in terms of fundamental ones given in table below. We assume the numeration of nodes of Dynkin diagram as in Mathematica program LieART.

	λ_{ad}	λ_{X_2}
G_2	ω_2	$3\omega_1$
F_4	ω_1	ω_2
E_6	ω_6	ω_3
E_7	ω_1	ω_2
E_8	ω_7	ω_6
A_1	2ω	0
$A_i,\ i > 1$	$\omega_1 + \omega_i$	$(2\omega_1 + \omega_{i-1}) \oplus (\omega_2 + 2\omega_i)$
B_2	$2\omega_2$	$\omega_1 + 2\omega_2$
B_3	ω_2	$\omega_1 + 2\omega_3$
$B_i,\ i > 3$	ω_2	$\omega_1 + \omega_3$
C_i	$2\omega_1$	$2\omega_1 + \omega_2$
D_4	ω_2	$\omega_1 + \omega_3 + \omega_4$
$D_i,\ i > 4$	ω_2	$\omega_1 + \omega_3$

The universal quantum dimension of X_2 is (Deligne 2013)

$$D_Q^{X_2} = \frac{\sinh\left(\frac{x}{4}(2t-\alpha)\right)\sinh\left(\frac{x}{4}(2t-\beta)\right)\sinh\left(\frac{x}{4}(2t-\gamma)\right)}{\sinh\left(\frac{\alpha x}{4}\right)\sinh\left(\frac{\beta x}{4}\right)\sinh\left(\frac{\gamma x}{4}\right)} \times$$
$$\frac{\sinh\left(\frac{x}{4}(t+\alpha)\right)\sinh\left(\frac{x}{4}(t+\beta)\right)\sinh\left(\frac{x}{4}(t+\gamma)\right)}{\sinh\left(\frac{\alpha x}{2}\right)\sinh\left(\frac{\beta x}{2}\right)\sinh\left(\frac{\gamma x}{2}\right)} \times$$
$$\frac{\sinh\left(\frac{x}{2}(t-\alpha)\right)\sinh\left(\frac{x}{2}(t-\beta)\right)\sinh\left(\frac{x}{2}(t-\gamma)\right)}{\sinh\left(\frac{x}{4}(t-\alpha)\right)\sinh\left(\frac{x}{4}(t-b)\right)\sinh\left(\frac{x}{4}(t-\gamma)\right)}$$

Quantum dimension of Cartan product $X_2^k ad^n$

Quantum dimension of irrep with highest weight $k\lambda_{X_2} + n\lambda_{ad}$

$$X(x,k,n,\alpha,\beta,\gamma) =$$
$$L_{31} \cdot L_{32} \cdot L_{21s1} \cdot L_{21s2} \cdot L_{21s3} \cdot L_{10s1} \cdot L_{10s2} \cdot L_{10s3} \cdot L_{11s1} \cdot L_{11s2} \cdot L_{11s3} \cdot L_{01} \cdot L_{c2}$$

$$L_{31} = \sinh\left[\frac{x}{4} : \frac{-2(\beta+\gamma)+\alpha(-4+3k+n)}{2(2\alpha+\beta+\gamma)}\right] = \frac{\sinh[\frac{x}{4}(-2(\beta+\gamma)+\alpha(-4+3k+n))]}{\sinh[\frac{x}{4}(2(2\alpha+\beta+\gamma))}$$

$$L_{32} = \sinh\left[\frac{x}{4} : \frac{-2(\beta+\gamma)+\alpha(-3+3k+2n)}{3\alpha+2(\beta+\gamma)}\right]$$

$$L_{21s1} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{2k+n} \frac{-2(\beta+\gamma)+\alpha(-5+i)}{-2\beta+\alpha(i-2)}\right]$$

$$L_{21s2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{2k+n} \frac{\beta+2\gamma-\alpha(-3+i)}{\beta+\gamma-\alpha(i-2)}\right]$$

Quantum dimension of Cartan product $X_2^k ad^n$

$$L_{31} = \sinh\left[\frac{x}{4} : \frac{-2(\beta+\gamma)+\alpha(-4+3k+n)}{2(2\alpha+\beta+\gamma)}\right.$$

$$L_{32} = \sinh\left[\frac{x}{4} : \frac{-2(\beta+\gamma)+\alpha(-3+3k+2n)}{3\alpha+2(\beta+\gamma)}\right.$$

$$L_{21s1} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{2k+n} \frac{-2(\beta+\gamma)+\alpha(-5+i)}{-2\beta+\alpha(i-2)}\right.$$

$$L_{21s2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{2k+n} \frac{\beta+2\gamma-\alpha(-3+i)}{\beta+\gamma-\alpha(i-2)}\right.$$

$$L_{21s3} = \sinh\left[\frac{x}{4} : \frac{2\beta+\gamma+\alpha(3-2k-n)}{3\alpha+2\beta+\gamma}\right.$$

$$L_{10s1} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k} \frac{2\gamma-\alpha(i-3)}{-\alpha i}\right.$$

$$L_{10s2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k} \frac{\beta+\gamma-\alpha(i-3)}{\beta-\alpha(i-2)}\right.$$

$$L_{10s3} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k} \frac{-2\beta+\alpha(i-3)}{\gamma-\alpha(i-2)}\right.$$

$$L_{11s1} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k+n} \frac{2\beta+\gamma-\alpha(i-4)}{\alpha(i+2)}\right.$$

$$L_{11s2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k+n} \frac{\beta+\gamma-\alpha(i-2)}{\beta-\alpha(i-1)}\right.$$

$$L_{11s3} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k+n} \frac{-2\beta+\alpha(i-2)}{\gamma+\alpha(1-i)}\right.$$

$$L_{01} = \sinh\left[\frac{x}{4} : \frac{\alpha(1+n)}{\alpha}\right.$$

$$L_{c2} = \sinh\left[\frac{x}{4} : \prod_{i=1}^{k} \frac{\gamma+2\beta-\alpha(i+k+n-4))}{\alpha(i+k+n-2)-2\gamma}\right.$$

Proposition.
The function $X(x,k,n,\alpha,\beta,\gamma)$ on the points from Vogel's table is equal to the quantum dimensions of representations of simple Lie algebras given in tables below.
((k,0): M.Avetisyan, RM 2018, (k,n): M.Avetisyan, RM 2019)

Table: $X(x,k,n,\alpha,\beta,\gamma)$ for classical algebras

k,n	$0,n$	$1,n$	$k,n\ (k>1)$
A_1	$n\lambda_{ad}$	0	0
$A_i, i\geq 2$	$n\lambda_{ad}$	$\lambda_{X_2}+n\lambda_{ad}$	$k\lambda_{X_2}+n\lambda_{ad}$
B_2	$n\lambda_{ad}$	$\lambda_{X_2}+n\lambda_{ad}$	0
$B_i,\ i>2$	$n\lambda_{ad}$	$\lambda_{X_2}+n\lambda_{ad}$	$k\lambda_{X_2}+n\lambda_{ad}$
$C_i,\ i>2$	$n\lambda_{ad}$	$\lambda_{X_2}+n\lambda_{ad}$	0
$D_i,\ i>3$	$n\lambda_{ad}$	$\lambda_{X_2}+n\lambda_{ad}$	$k\lambda_{X_2}+n\lambda_{ad}$

Table: $X(x,k,n,\alpha,\beta,\gamma)$ for exceptional algebras

k,n	k,n
L	$k\lambda_{X_2}+n\lambda_{ad}$

L is any of exceptional simple Lie algebras.

Permutations of parameters α, β, γ

Table: $X(x, k, n, \beta, \alpha, \gamma)$ for exceptional algebras

k, n	1,0	1,1	1,2	1,3	1,4	1,5	2,0	2,1	2,2
G_2	$3\omega_1$	$\omega_1 + \omega_2$	0	0	0	0	0	0	0
F_4	ω_2	$\omega_3 + \omega_4$	$\omega_1 + \omega_4$	0	0	0	$3\omega_4$	0	0
E_6	ω_3	E:$(\omega_1 + \omega_2)$ $\oplus(\omega_4 + \omega_5)$	ω_3	0	-1	0	E:$3\omega_1 \oplus 3\omega_5$	$-\omega_3$	$-\omega_6$
E_7	ω_2	$\omega_6 + \omega_7$	0	E:$2\omega_6$	0	0	0	E:$-\omega_6 - \omega_7$	$-\omega_5$
E_8	ω_6	ω_8	$-\omega_8$	$-\omega_6$	0	1	0	0	ω_6

Table: $X(x, k, n, \gamma, \alpha, \beta)$ for exceptional algebras

k, n	1,0	1,1	1,3	2,0	2,1
G_2	$3\omega_1$	$-3\omega_1$	1	$3\omega_1$	ω_2
F_4	ω_2	$-\omega_2$	1	ω_2	ω_1
E_6	ω_3	$-\omega_3$	1	ω_3	ω_6
E_7	ω_2	$-\omega_2$	1	ω_2	ω_1
E_8	ω_6	$-\omega_6$	1	ω_6	ω_7

Notation E: (and SL:, SO:) means restriction on exceptional (respectively special linear and orthosymplectic) line.
That is necessary at points in Vogel's plane, where function $X(x, k, n, \alpha, \beta, \gamma)$ has singularity.
E.g., from above tables the (k,n)=(1,3) case for E_7 is singular at point $\alpha = -2, \beta = 8, \gamma = 12$. To study that singularity, consider deviation $\alpha = -2, \beta = 8 + p, \gamma = 12 + q, p, q \to 0$. Then singular factor of function $X(x, k, n, \alpha, \beta, \gamma)$ is

$$\frac{p - q}{3p - 2q}$$

Depending on how p, q tend to 0, answer may be an arbitrary. However, if we restrict them on exceptional line $\gamma = 2(\alpha + \beta)$, i.e. $q = 2p$, we obtain unique and correct answer for E_7.
This rule should be derived in the framework of some regular approach.

Table: $X(x, k, n, \beta, \alpha, \gamma)$ for classical algebras. Data for A_i, C_i are valid for sufficiently large rank i (depending on k, n)

k, n	$1, n$	$k, n\ ,\ k \geq 2$
A_i	$(\omega_1 + \omega_{1+n} + \omega_{i-1-n}) \oplus (\omega_i + \omega_{i-n} + \omega_{n+2})$	$(\omega_k + \omega_{k+n} + \omega_{i+1-2k-n}) \oplus (\omega_{2k+n} + \omega_{i+1-k} + \omega_{i+1}$
B_i	$\omega_1 + \omega_{2n+3}$	0
C_i	$\omega_1 + \omega_{n+1} + \omega_{n+2}$	$\omega_k + \omega_{k+n} + \omega_{2k+n}$
D_i	$\omega_1 + \omega_{2n+3}$	0

For the small (i.e. not "sufficiently large") values of rank i the function $X(x, k, n, \beta, \alpha, \gamma)$ still gives (quantum) dimensions of irreducible representations of corresponding algebra. However, picture is chaotic.

Table: $X(x, k, n, \gamma, \alpha, \beta)$ for classical algebras

$k, n > 0$	1, 2
A_i	-1
L	0

L is any classical algebra, except A_i

According to this table, for classical algebras, $X(x, k, n, \gamma, \alpha, \beta)$ is non-zero (besides previously known case $(k, n) = (k, 0)$) for $(k, n) = (1, 2)$, A_i, only. In that case $X(x, k, n, \gamma, \alpha, \beta) = -1$.

Universal eigenvalues of Casimir operator

Eigenvalues of second Casimir operator on the Cartan product of k X_2 representations and n adjoint ones can be presented in universal form (M.Avetisyan 2019):

$$C_{k,n} = \alpha(3k - 3k^2 + n - n^2 - 3kn) + t(4k + 2n) \tag{3}$$

This coincides with eigenvalues, given in (Cohen, deMan 1996) for representations (Cartan products of) $X_2, X_2^2, gX_2, g^2 X_2$ (with Casimir's eigenvalues $C_{1,0}, C_{2,0}, C_{1,1}, C_{1,2}$, respectively). Permutations of parameters give eigenvalues for corresponding representations.

Problem: establish what objects in the theory of simple Lie algebras and their applications have the universal representation.
Thanks!