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Pseudo-Plactic Algebra and Snake's Numbers

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- V.I. Arnold Snake's Numbers

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- $\mathbb{C}[GL_q(V)]$ via Schur functors

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- Pre-Plactic algebra

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- Proof of the Krob and Thibon conjecture

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- Pre-Plactic algebra
- Proof of the Krob and Thibon conjecture
- Snake's numbers and Pre-Plactic algebra

Snake's numbers

Definition

An alternating permutation or *snake* is a permutation σ whose word satisfies

$$\sigma_1 > \sigma_2 < \sigma_3 > \dots .$$

alternating permutations are counted by Désiré André

$$y(x) := \sum_{n \geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

Bernoulli-Euler numbers E_n (V.I. Arnold)

n	0	1	2	3	4	5	6	7
E_n	1	1	1	2	5	16	61	272

Diagonal versus Pseudo-Plactic algebra

Definition

The quantum diagonal algebra $\mathbb{C}_q[\mathbb{T}]$ is the subalgebra of the quantum matrix algebra $\mathbb{C}[GL_q(V)]$ generated by the diagonal elements $x_1^1, x_2^2, \ldots, x_n^n$.

Definition

The pseudo-plactic algebra is the quotient

$$\mathfrak{PP}_q(\mathbb{T}) := \mathbb{C}\langle\mathbb{T}\rangle/(\mathfrak{L}_q(\mathbb{T}))$$

of the free associative algebra $\mathbb{C}\langle\mathbb{T}\rangle$ by the cubic ideal $(\mathfrak{L}_q(\mathbb{T}))$

$$\begin{array}{lclcl} \mathfrak{L}_q(\mathbb{T})_{i_3}^{i_1 i_2} & := & [[x_{i_1}^{i_1}, x_{i_3}^{i_3}], x_{i_2}^{i_2}] & \text{with} & i_1 < i_2 < i_3 \\ \mathfrak{L}_q(\mathbb{T})_{i_2}^{i_1 i_1} & := & [[x_{i_1}^{i_1}, x_{i_2}^{i_2}], x_{i_1}^{i_1}]_{q^2} & \text{with} & i_1 < i_2 \\ \mathfrak{L}_q(\mathbb{T})_{i_2}^{i_1 i_2} & := & [x_{i_2}^{i_2}, [x_{i_1}^{i_1}, x_{i_2}^{i_2}]]_{q^2} & \text{with} & i_1 < i_2 \end{array}.$$

Krob and Thibon Conjecture

q is generic, $q \neq 0$, $q^N \neq 1$

$$\mathbb{C}_q[\mathbb{T}] = \mathbb{C}[GL_q(V)]^{\Delta} = \mathbb{C}[GL_q(V)]|_{\mathbb{T}}$$

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Motivation: Theory of characters of $\mathbb{C}[GL_q(V)]$-comodules

$$qdet = \sum_{\sigma \in S_n} q^{sgn(\sigma)} x^1_{\sigma_1} \dots x^n_{\sigma_n} = \frac{1}{1-q^n}[\dots[x^n_n, x^{n-1}_{n-1}]_{q^2}, \dots, x^1_1]_{q^2}$$

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Theorem

The diagonal algebra is isomorphic to the pseudo-plactic algebra

$$\mathbb{C}_q[\mathbb{T}] \cong \mathfrak{PP}_q(\mathbb{T}) \ .$$

KROB, Daniel; THIBON, Jean-Yves. Noncommutative symmetric functions IV: Journal of Algebraic Combinatorics, 1997

The Hecke algebra $\mathcal{H}_r(q)$

a deformation of the group algebra $\mathbb{C}[\mathfrak{S}_r]$

$$\begin{array}{rcll} T_{s_i} T_{s_{i+1}} T_{s_i} & = & T_{s_{i+1}} T_{s_i} T_{s_{i+1}} & i = 1, \ldots, r-1 \\ T_{s_i} T_{s_j} & = & T_{s_j} T_{s_i} & |i-j| \geq 2 \\ T_{s_i}^2 & = & \mathbb{1} + (q - q^{-1}) T_{s_i} & i = 1, \ldots, r-1 \end{array} . \quad (1)$$

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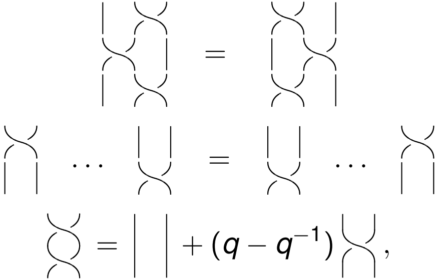

Drinfeld-Jimbo R-matrix

$$\hat{R} = \sum_{i,j \in I} q^{\delta_{ij}} e_j^i \otimes e_i^j + (q - q^{-1}) \sum_{i,j \in I : i<j} e_j^i \otimes e_j^i \qquad e_j^i \in \mathfrak{gl}(V) \; .$$

Hecke algebra $\mathcal{H}_r(q)$ Representation
$\pi(T_{s_i}) = (\hat{R}_q)_{ii+1} = id^{\otimes(i-1)} \otimes \hat{R}_q \otimes id^{\otimes(r-i-1)}$
Braid (Yang-Baxter) relation

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}$$

Hecke relation

$$\hat{R}^2 = 1\!\!1 + (q - q^{-1})\hat{R}$$

Faddeev-Reshetikhin-Takhtajan $W = x^j_k e^k_j = \mathrm{End}(V)$

$$\hat{R}\ W \otimes W = W \otimes W\ \hat{R} \qquad\qquad \hat{R} \in \mathrm{End}(V \otimes V)$$

Coordinate ring $\mathbb{C}[GL_q(V)]$

$$\begin{array}{lll} x^j_k x^i_k = q x^i_k x^j_k & x^k_j x^k_i = q x^k_i x^k_j & j > i \\ x^j_l x^i_k = x^i_k x^j_l + (q - q^{-1}) x^i_l x^j_k & x^j_k x^i_l = x^i_l x^j_k & j > i \quad l > k \end{array}$$

$$\mathbb{C}[GL_q(V)] \cong \mathbb{C}\langle W \rangle / (\hat{R} W \otimes W - W \otimes W \hat{R}) \qquad (2)$$

Hecke algebra projectors $e_{(2)}$ and $e_{(1^2)}$

q-(anti)-Symmetrizers $e_{(2)} = e_{\square\square}$ and $e_{(1^2)} = e_{\substack{\square\\\square}}$

$$e_{(2)} := \frac{1}{[2]}\left(q^{-1}\mathbb{1} + T_{s_1}\right) \qquad e_{(1^2)} := \frac{1}{[2]}\left(q\mathbb{1} - T_{s_1}\right)$$

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Schur functor: maps $\mathcal{H}_2(q)$-module to left $U_q\mathfrak{gl}(V))$-module

$$S_{(2)}(V) = V^{\otimes 2} e_{(2)} = V^{\otimes 2} \otimes_{\mathcal{H}_2(q)} \mathcal{H}_2(q) e_{(2)} = V^{\otimes 2} \otimes_{\mathcal{H}_2(q)} S_{(2)}$$

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$$S_{(1^2)}(V) = V^{\otimes 2}e_{(1^2)} = V^{\otimes 2} \otimes_{\mathcal{H}_2(q)} \mathcal{H}_2(q)e_{(1^2)} = V^{\otimes 2} \otimes_{\mathcal{H}_2(q)} S_{(1^2)}$$

Orthogonality $=$ Factorization of the Hecke relation

$$e_{(2)}e_{(1^2)} = 0 = (q - T_{s_1})(q^{-1} + T_{s_1})$$

Functorial form of $\mathbb{C}[GL_q(V)]$ ideal

$$W = \mathrm{End}(V) \cong V \otimes V^*$$

$$h_q(W) := \hat{R} W \otimes W - W \otimes W \hat{R} \quad \Leftrightarrow \begin{cases} h_q^+(W) := e_{(2)} W^{\otimes 2} e_{(1^2)} \\ h_q^-(W) := e_{(1^2)} W^{\otimes 2} e_{(2)} \end{cases} .$$

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Semi-group relations $h_q(W) := h_q^+(W) + h_q^-(W)$

$$h_q^+(W) := V^{*\otimes 2} \otimes_{\mathcal{H}_2(q)} \left(e_{(2)} \otimes e_{(1^2)} \right) \otimes_{\mathcal{H}_2(q)} V^{\otimes 2}$$

$$h_q^+(W) := S_{(2)}(V^*) \otimes S^{(1^2)}(V)$$

$$h_q^-(W) := S_{(1^2)}(V^*) \otimes S^{(2)}(V)$$

Peter-Weyl theorem for $\mathbb{C}[GL_q(V)]$

left and right $U_q\mathfrak{gl}(V)$)-modules decomposition

$$\mathbb{C}[GL_q(V)]_r \cong (W^{\otimes r})^{\mathcal{H}_r(q)} = (V^*)^{\otimes r} \otimes_{\mathcal{H}_r(q)} V^{\otimes r}$$

$$(V^{*\otimes r})^A \otimes_{\mathcal{H}_r(q)} (V^{\otimes r})_B \cong \bigoplus_{\lambda \vdash r} S^A_\lambda(V^*) \otimes_{\mathcal{H}_r(q)} S^\lambda_B(V) \qquad A, B \in I^r \ .$$

left Schur $U_q\mathfrak{gl}(V)$)-module

$$S_\lambda(V^*) = (V^*)^{\otimes r} \otimes_{\mathcal{H}_r(q)} \mathcal{H}_r(q) e_\lambda(T) = (V^*)^{\otimes r} \otimes_{\mathcal{H}_r(q)} S_\lambda$$

Orthogonal idempotents

$$e_\lambda(T) e_\mu(T') = e_\lambda(T) \delta_{\lambda\mu} \delta_{TT'}$$

$$\sum_{\lambda \vdash r} \sum_{sh(T) = \lambda} e_\lambda(T) = 1\!\!1_{\mathcal{H}_r(q)}$$

Schur-Weyl duality

$$\mathcal{H}(q) = \bigcup_{r \geq 0} \mathcal{H}_r(q)$$

$$\mathcal{H}_r(q) \cong \mathcal{H}_r(q) \otimes_{\mathcal{H}_r(q)} \mathcal{H}_r(q)$$

$$S_\lambda = \mathcal{H}_r(q) e_\lambda(T) \qquad S^\lambda = e_\lambda(T) \mathcal{H}_r(q)$$

$$\mathcal{H}_r(q) \cong \bigoplus_{\lambda \vdash r} S_\lambda \otimes_{\mathcal{H}_r(q)} S^\lambda$$

$$\mathcal{H}_r(q)^\alpha_\beta \cong \bigoplus_{\lambda \vdash r} S^\alpha_\lambda \otimes_{\mathcal{H}_r(q)} S^\lambda_\beta \quad .$$

$$\begin{pmatrix} \alpha_1 & \dots & \alpha_r \\ \beta_1 & \dots & \beta_r \end{pmatrix} = \begin{pmatrix} \alpha_1 & \dots & \alpha_r \\ 1 & \dots & r \end{pmatrix} \otimes_{\mathcal{H}_r(q)} \begin{pmatrix} 1 & \dots & r \\ \beta_1 & \dots & \beta_r \end{pmatrix}$$

$$\bar{T}^\alpha_\beta := T_{\alpha^{-1}} \otimes_{\mathcal{H}_r(q)} T_\beta = (T^\alpha_\beta)^{\mathcal{H}_r(q)}$$

Polarization of the ideal $h_q(W)$

$$x_2^2 x_1^1 - x_1^1 x_2^2 = (q - q^{-1}) x_2^1 x_1^2 \qquad x_2^1 x_1^2 = x_1^2 x_2^1$$

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$$T_{21}^{21} - T_{12}^{12} = (q - q^{-1}) T_{21}^{12} \qquad T_{21}^{12} = T_{12}^{21} \tag{3}$$

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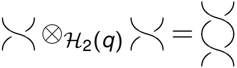
$$T_{21}^{21} = T_{12}^{21} \otimes_{\mathcal{H}_2(q)} T_{21}^{12} = T_{s_1} \otimes_{\mathcal{H}_2(q)} T_{s_1} =$$



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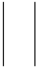

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$\otimes_{\mathcal{H}_2(q)}$

$=$


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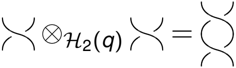

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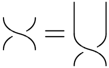
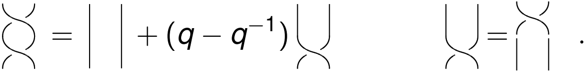


These relations have the pictorial representation



Hecke relation is equivalent to $e_{(2)} \otimes_{\mathcal{H}_2(q)} e_{(1^2)} = 0$

$$\left(q^{-1}\,|\;\;| + \times\right) \otimes_{\mathcal{H}_2(q)} \left(q\,|\;\;| - \times\right) = 0$$

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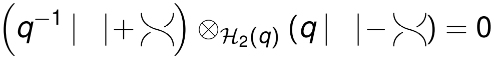

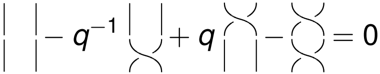

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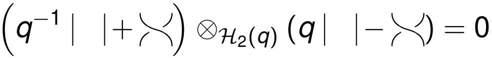

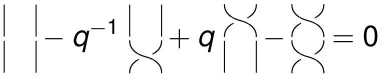

Dressing $e_{(2)} \otimes_{\mathcal{H}_2(q)} e_{(1^2)}$ with $(V^*)^{\otimes 2}$ and $V^{\otimes 2}$ yields

$$(V^*)^{\otimes 2} e_{(2)} \otimes_{\mathcal{H}_2(q)} e_{(1^2)} V^{\otimes 2} = 0$$

Lemma (O. Ogievetsky)

$$h_q(W) := S_{(1^2)}(V^*) \otimes_{\mathcal{H}_2(q)} S^{(2)}(V) \cong S_{(2)}(V^*) \otimes_{\mathcal{H}_2(q)} S^{(1^2)}(V)$$

$$\mathbb{C}[GL_q(V)] \cong \mathbb{C}\langle W \rangle / (h_q(W))$$

Functorial pseudo-plactic relations $\mathfrak{L}_q(\mathbb{T})$

central idempotent $E^{(2,1)}$ in $\mathcal{H}_3(q)$ attached to $\lambda = (2,1) =$ ,

$$E^{(2,1)} = e^+(q) + e^-(q) \qquad e^+(q)e^-(q) = 0$$

q-Eulerian idempotent $e^+(q)$ (with Jean-Louis Loday)
left and right $U_q\mathfrak{gl}(V)$-modules by the Schur functors

$$S^{\pm}_{(2,1)}(V^*) := V^{*\otimes 3}e^{\pm}(q) \qquad S^{(2,1)}_{\pm}(V) = e^{\pm}(q)V^{\otimes 3}$$

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Lemma

Pseudo-plactic relations stem from the restriction to the diagonal of a $U_q\mathfrak{gl}(V)$-bimodule $\mathfrak{L}^{+}_q(W) = e^{+}(q) W^{\otimes 3} e^{-}(q)$

$$\mathfrak{L}_q(\mathbb{T}) := S^{+}_{(2,1)}(V^*) \otimes_{\mathcal{H}_3(q)} S^{(2,1)}_{-}(V)|_{\mathbb{T}^*}$$

Pre-plactic algebra $\mathfrak{PP}_q$

$$[[x^{i_1}_{i_1}, x^{i_3}_{i_3}], x^{i_2}_{i_2}] \in \mathfrak{L}_q(\mathbb{T}) \qquad \text{when} \qquad i_1 < i_2 < i_3$$

$$[[x^1_1, x^3_3], x^2_2] = x^1_1 x^3_3 x^2_2 - x^3_3 x^1_1 x^2_2 - x^2_2 x^1_1 x^3_3 + x^2_2 x^3_3 x^1_1$$

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Definition

The pre-plactic algebra $\mathfrak{PP}_q$ is the factor of the $\mathcal{H}(q)$-module $(\mathcal{H}(q) \otimes \mathcal{H}(q))^\Delta$ and the $\mathcal{H}(q)$-submodule ideal $(\mathfrak{L}_q)$

$$\mathfrak{PP}_q = (\mathcal{H}(q) \otimes \mathcal{H}(q))^\Delta / (\mathfrak{L}_q)$$

$$\mathfrak{L}_q = [[13]2] := T^{132}_{132} - T^{312}_{312} - T^{213}_{213} + T^{231}_{231} \,.$$

Pre-plactic algebra $\mathfrak{PP}_q$

$$[[x_{i_1}^{i_1}, x_{i_3}^{i_3}], x_{i_2}^{i_2}] \in \mathfrak{L}_q(\mathbb{T}) \qquad \text{when} \qquad i_1 < i_2 < i_3$$
$$[[x_1^1, x_3^3], x_2^2] = x_1^1 x_3^3 x_2^2 - x_3^3 x_1^1 x_2^2 - x_2^2 x_1^1 x_3^3 + x_2^2 x_3^3 x_1^1$$

Definition

The pre-plactic algebra $\mathfrak{PP}_q$ is the factor of the $\mathcal{H}(q)$-module $(\mathcal{H}(q) \otimes \mathcal{H}(q))^{\Delta}$ and the $\mathcal{H}(q)$-submodule ideal $(\mathfrak{L}_q)$

$$\mathfrak{PP}_q = (\mathcal{H}(q) \otimes \mathcal{H}(q))^{\Delta} / (\mathfrak{L}_q)$$

$$\mathfrak{L}_q = [[13]2] := T_{132}^{132} - T_{312}^{312} - T_{213}^{213} + T_{231}^{231} \; .$$

Theorem

$$\mathcal{H}^{\Delta}(q) \cong \mathfrak{PP}_q .$$

symmetric braidings $T_{\alpha_1 \ldots \alpha_r}^{\alpha_1 \ldots \alpha_r} = T^{\alpha} \otimes T_{\alpha} = T_{\alpha^{-1}} \otimes T_{\alpha} \in \mathcal{H}_r^{\Delta}(q)$

Pictorial Proof

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 1 & 2 & 3 \end{pmatrix} \otimes_{\mathcal{H}_r(q)} \begin{pmatrix} 1 & 2 & 3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix}$$

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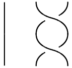
$$T^{132}_{132} \quad = \quad T^{132}_{123} \otimes T^{123}_{132} \quad = \quad T_{s_2} \otimes T_{s_2} \quad =$$



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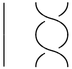
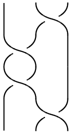
$$T^{132}_{132} = T^{132}_{123} \otimes T^{123}_{132} = T_{s_2} \otimes T_{s_2} =$$



$$T^{312}_{312} = T^{312}_{123} \otimes T^{123}_{312} = T_{s_2 s_1} \otimes T_{s_1 s_2} =$$



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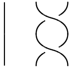
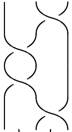
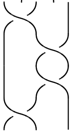


$$T^{312}_{312} = T^{312}_{123} \otimes T^{123}_{312} = T_{s_2 s_1} \otimes T_{s_1 s_2} =$$



$$T^{231}_{231} = T^{231}_{123} \otimes T^{123}_{231} = T_{s_1 s_2} \otimes T_{s_2 s_1} =$$



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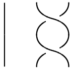
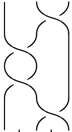


$$T^{312}_{312} = T^{312}_{123} \otimes T^{123}_{312} = T_{s_2 s_1} \otimes T_{s_1 s_2} =$$



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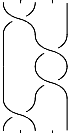
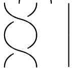
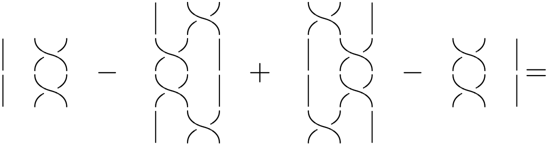


$$T^{213}_{213} = T^{213}_{123} \otimes T^{123}_{213} = T_{s_1} \otimes T_{s_1} =$$



Reducing $\mathfrak{L}_q$

$$T^{132}_{132} - T^{312}_{312} - T^{213}_{213} + T^{231}_{231} =$$



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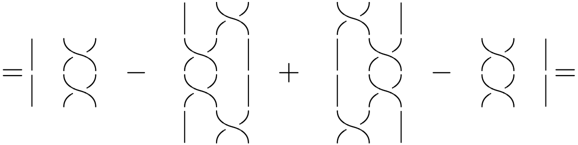
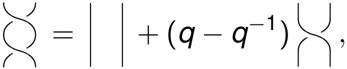
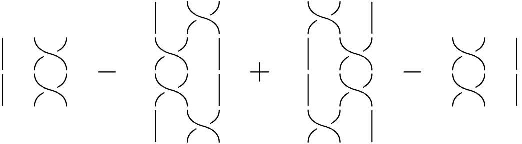


We can reduce braids resolving the bubbles



$T^{132}_{132} - T^{312}_{312} - T^{213}_{213} + T^{231}_{231} =$  $=$

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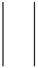
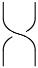
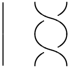
 $=$  $+ (q - q^{-1})$  ,



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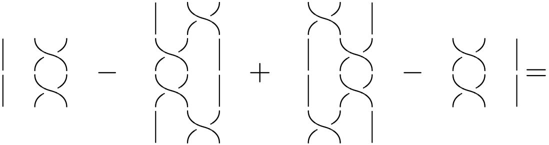

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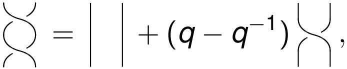

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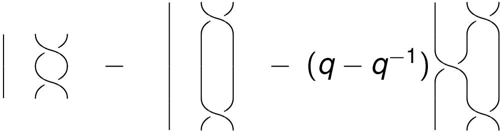

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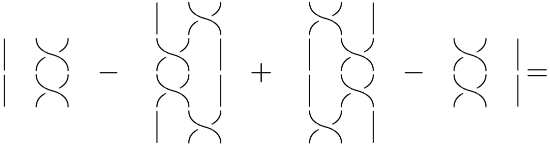

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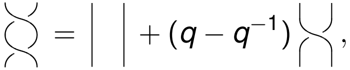

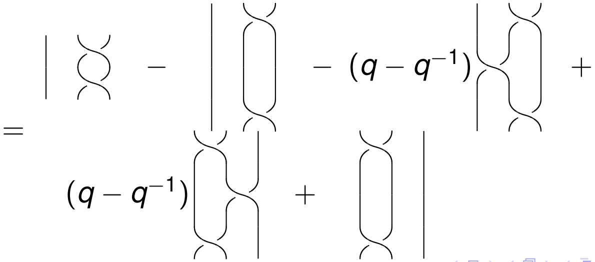

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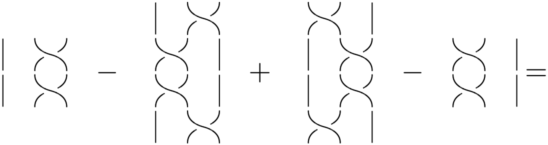

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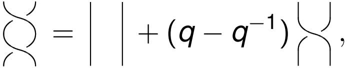
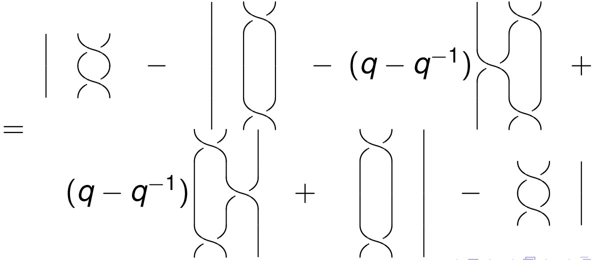
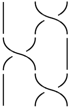




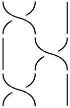
$\mathfrak{L}_q$ and braid relations

Terms with bubbles cancel hence

$$T^{132}_{132} - T^{312}_{312} - T^{213}_{213} + T^{231}_{231} = -\omega \Bigg(\quad$$



$$-$$



$$\Bigg) = 0$$

where $\omega := (q - q^{-1})$

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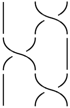
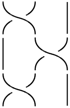
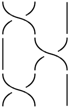
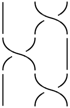




where $\omega := (q - q^{-1})$

$$[[13]2] := \mathfrak{L}_q = \omega \quad - \quad \omega \quad = \omega T_{s_1} T_{s_2} T_{s_1} - \omega T_{s_2} T_{s_1} T_{s_2} \ .$$





$$\mathfrak{L}_q \sim e^{+}(q) \otimes_{\mathcal{H}_3(q)} e^{-}(q) = 0 = \omega T_{s_1} T_{s_2} T_{s_1} - \omega T_{s_2} T_{s_1} T_{s_2}$$

Pre-plactic algebra and Krob-Thibon conjecture

$$[[13]2]4 \sim \omega T_{s_1 s_2 s_1} - \omega T_{s_2 s_1 s_2} = \mathfrak{L}_q \otimes 1\!\!1$$

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$$1\!\!1^{\otimes(i-1)} \otimes \mathfrak{L}_q \otimes 1\!\!1^{\otimes(n-2-i)} = T_{s_i s_{i+1} s_i} - T_{s_{i+1} s_i s_{i+1}}$$

Theorem

$$\mathcal{H}^{\Delta}(q) \cong \mathfrak{PP}_q.$$

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$$\mathcal{H}^{\Delta}(q) \cong \mathfrak{PP}_q.$$

Corollary

$$\mathbb{C}[GL_q(V)]^{\Delta} = \mathbb{C}_q[\mathbb{T}] \cong \mathfrak{PP}_q(\mathbb{T}).$$

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By functoriality $\mathfrak{L}_q(\mathbb{T}) := S^{+}_{(2,1)}(V^*) \otimes_{\mathcal{H}_3(q)} S^{(2,1)}_{-}(V)|_{\mathbb{T}^*}$

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Theorem

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Corollary

$$\mathbb{C}[GL_q(V)]^{\Delta} = \mathbb{C}_q[\mathbb{T}] \cong \mathfrak{PP}_q(\mathbb{T}).$$

By functoriality $\mathfrak{L}_q(\mathbb{T}) := S^{+}_{(2,1)}(V^*) \otimes_{\mathcal{H}_3(q)} S_{-}^{(2,1)}(V)|_{\mathbb{T}^*}$

$$(h_q)^{\Delta} \cong (\mathfrak{L}_q) \quad \Rightarrow \quad (h_q(W))^{\Delta} \cong (\mathfrak{L}_q(\mathbb{T}))$$

Snake's numbers

Definition

An alternating permutation or *snake* is a permutation σ whose word satisfies

$$\sigma_1 > \sigma_2 < \sigma_3 > \dots .$$

alternating permutations are counted by Désiré André

$$y(x) := \sum_{n\geq 0} E_n \frac{x^n}{n!} = \sec x + \tan x$$

Bernoulli-Euler numbers E_n (V.I. Arnold)

n	0	1	2	3	4	5	6	7
E_n	1	1	1	2	5	16	61	272

$$y'(x) = \left(\frac{1}{\cos x} + \tan x\right)' = \frac{\sin x}{\cos^2 x} + \frac{1}{\cos^2 x} = \frac{1}{1 - \sin x}$$

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$\dim \mathfrak{P}\mathfrak{P}_q(n) = E_{n+1}$

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$$\frac{E_{n+1}}{n!} = \sum_{k>0} \sum_{\Lambda \vdash n} \frac{(-1)^{(|\Lambda|-k)/2} k!}{\prod_{i=1}^{k} \Lambda_i! \prod_{i=1}^{k} m_i(\Lambda)!}$$

sum over Young Diagrams Λ with odd parts $\Lambda_i = 2\lambda_i + 1$

$\dim \mathfrak{PP}_q(n) = E_{n+1}$

$$y'(x) = \left(\frac{1}{\cos x} + \tan x\right)' = \frac{\sin x}{\cos^2 x} + \frac{1}{\cos^2 x} = \frac{1}{1-\sin x}$$

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sum over Young Diagrams Λ with odd parts $\Lambda_i = 2\lambda_i + 1$

$$\mathfrak{PP}_q = (\mathcal{H}(q)\otimes\mathcal{H}(q))^\Delta/(\mathfrak{L}_q)$$

$\dim \mathfrak{P}\mathfrak{P}_q(n) = E_{n+1}$

$$\sum_{n\geq 0} \dim \mathfrak{P}\mathfrak{P}_q(n) \frac{x^n}{n!} = \sum_{n=0}^{\infty} E_{n+1} \frac{x^n}{n!} .$$

Lemma

The snake numbers satisfy the identity

$$\sum_{n\geq 0} E_{n+1} \frac{x^n}{n!} = \exp \sum_{n\geq 1} E_{n-1} \frac{x^n}{n!} . \tag{4}$$

$$2y' = y^2 + 1 \qquad y(0) = 1 , \qquad y(x) = \sec x + \tan x \tag{5}$$

$$y'' = yy' \qquad \text{or} \qquad (\ln y')' = y$$

$$y'(x) = \exp \int_0^x y(t)dt . \tag{6}$$

$$g_{\mathfrak{PP}_q}(x) = \sum_{n\geq 0} \dim \mathfrak{PP}_q(n) \frac{x^n}{n!} = \exp\left(\sum_{n\geq 1} \dim Prim(\mathfrak{PP}_q(n)) \frac{x^n}{n!} \right) .$$

PBW-basis of $\mathfrak{PP}_q(\mathbb{T})$ with primitives generators $Prim(\mathfrak{PP}_q(n))$

$$H_{\mathfrak{PP}_q(\mathbb{T})}(t) = \prod_{n=1}^{\dim V} (1 - t^n)^{-E_{n-1}\binom{\dim V}{n}} \qquad (7)$$

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$$g_{\mathfrak{S}}(x) = \frac{1}{1-x} = \sum_{n\geq 0} n! \frac{x^n}{n!} = \exp \ln \frac{1}{(1-x)} = \exp \sum_{k\geq 1} (k-1)! \frac{x^k}{k!} \tag{8}$$

$(k-1)! = \dim Lie(k)$ of primitive Lie elements $Lie(k) \subset \mathbb{C}[S_k]$