# CONSISTENT TRUNCATIONS WITH DIFFERENT AMOUNT OF SUPERSYMMETRY

[D. CASSANI, G. JOSSE, M.P, C., D. WALDRAM, arXiv:1907.06730]

# INTRODUCTION

- A major question in string theory/sugra is how to derive lower dimensional effective actions
  - effective low-dimensional theories in string compactifications
  - gravity solutions in the gauge/gravity duality

- Look for 10D solutions of the form

$$M_{10} = X_{10-d} \times M_d$$

maximally symmetric $\hookleftarrow$ $\hookrightarrow$ compact

  - expand the 10d field in harmonics on $M_d$
  - obtain a $10 - d$ action with infinite tower of KK modes
  - truncate the theory to a finite set of fields in a consistent way
    - no dependence on the internal manifold in the eom and susy variations
    - all 10 - $d$ solutions lift to solutions of the higher dimensional ones

- Typically consistent truncations are based on the geometric properties of the internal manifold
  - cohomologies on Calabi-Yau manifolds $\rightarrow$ massless modes
  - symmetries of group-, coset- and G-structure manifolds $\rightarrow$ invariant modes

and thus they used to be rare and non-trivial

- Things have changed thanks to new formulations of type II and M-theory
  - DFT/EFT: Double and/or Exceptional Field Theories [hull, zwiebach 09; hohm, hull, zwiebach 10; hohm, samtleben 13; ...]
  - GCG/EGG: Generalised Complex and/or Exceptional Generalised Geometry

[hitchin 02; gualtieri 04; hull 07; pacheco, waldram 08, ...]

where

- isometries and p-form gauge symmetries have a geometrical interpretation
- role of the U-duality groups in the higher-dimensional theory

- Sphere reductions are a good example
  - consistency is not guaranteed by symmetry
  - only $S^1$, $S^3$, anf $S^7$ are parallelisable
  - understanding of such reduction requires explicit use of the U-duality symmetry
    - reformulation with manifest SU(8) symmetry [de Wit and Nicolai 87;...]
    - generalised Scherck-Schwarz reductions in EGG or EFT [lee, strickland-constable, waldram 14; hohm, samtleben 14, ...]

- In this talk I will focus on Generalised Geometry
  - link between G-structures and consistent truncations in ordinary geometry
  - summary of the main feature of Generalised Geometry
  - generalised structures as a unified framework to study truncations with different amount of supersymmetry

# CONSISTENT TRUNCATIONS AND ORDINARY G-STRUCTURES

- Conventional Scherk-Schwarz reductions
  - consider a group manifold $M_d = G$
  - decompose all higher-dimensional fields into representations of $G$
  - keeping only the singlet representations gives a consistent truncation
    - products of singlet representations never source non-singlet representations

- This extends to $G$-structure manifolds
  - two ingredients: G-structure and intrinsic torsion

- A $d$-dimensional manifold $M_d$ has a $G_S$-structure if (equivalently)
  - its structure group is reduced to $G_S \subset GL(d)$
  - $\exists$ $G_S$-invariant, no-where-vanishing tensors, $\Xi^i$
  - Ex: a $G_S = O(d)$ structure defines

    a set of orthonormal frames $\Longleftrightarrow$ an invariant metric $g$

- Intrinsic torsion for a $G_S \subset O(d)$ structure
  - consider the action of the Levi-Civita connection on the $G_S$-invariant tensors

$$\begin{aligned}\nabla_m \Xi_i{}^{n_1 \dots n_r}{}_{p_1 \dots p_s} = K_m{}^{n_1}{}_q \Xi_i{}^{q \dots n_r}{}_{p_1 \dots p_s} + \dots + K_m{}^{n_r}{}_q \Xi_i{}^{n_1 \dots q}{}_{p_1 \dots p_s} \\ - K_m{}^q{}_{p_1} \Xi_i{}^{n_1 \dots n_r}{}_{q \dots p_s} + \dots - K_m{}^q{}_{p_s} \Xi_i{}^{n_1 \dots n_r}{}_{p_1 \dots q},\end{aligned}$$

  - the tensor $K$ is a section of $T^*M \otimes \mathfrak{g}^\perp$ with $\Lambda^2 T^*M \simeq so(d) = \mathfrak{g} \oplus \mathfrak{g}^\perp$
  - the intrinsic torsion is

$$(T_{int})_{mn}{}^p = K_n{}^p{}_m - K_m{}^p{}_n$$

    and it decomposes in representations of $G_S$

- Any field theory on a manifold $M_d$ with a G-structure with constant, singlet intrinsic torsion admits a consistent truncation
  - decompose all fields into $G_S$ representations
  - keep only the $G_S$ singlets
  - if the intrinsic torsion has only singlet components the derivatives of any field are expanded only in singlets
  - the equations of motion only contain singlets representations

- The $G_S$-structure also determines field content and gaugings of the consistent truncations

| | | | |
|---|---|---|---|
| • | scalar manifold | $H \in \frac{C_{GL(d)}(G_S)}{C_{SO(d)}(G_S)}$ | $C_G(G_S) \to$ commutant of $G_S$ in $G$ |
| • | vector fields | $A^a k_a$ | $k_a \to$ globally defined vectors on $TM$ |
| • | gauge group | $[k_a, k_b] = f^c_{ab} k_c$ | $f^c_{ab} \to$ K-singlets of the intrinsic torsion |

# Examples

- Scherk-Schwarz reduction on. $M = G$
  - $M_d$ admits globally defined left-invariant vector fields $\{\hat{e}_a\}$
  - the co-frame $e^a$ defines an identity structure (parallelisation)

$$G_S = \mathbb{I} \subset GL(d)$$

  - the fields of the truncated theory are

| | | |
|---|---|---|
| scalars | $\Leftrightarrow$ | $h_{ab} \in \frac{GL(d)}{SO(d)}$ |
| $d$ gauge fields | $\Leftrightarrow$ | $\mathcal{A}^a \hat{e}_a$ |

  - the gauge algebra is

$$\mathcal{L}_{\hat{e}_a} \hat{e}_b = f_{ab}^{\ \ c} \hat{e}_c \qquad\qquad f_{ab}^{\ \ c} \text{ constant}$$

  - the truncation ansatz for the metric is

$$\mathrm{d}s^2 = g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + h_{ab}(e^a + \mathcal{A}^a)(e^b + \mathcal{A}^b)$$

- Reduction on a Sasaki-Einstein manifold $M$ of $d = 2n + 1$
  - $M_d$ admits a $G_S = SU(n) \subset GL(d, R)$ structure

$$(\eta\,,\ \omega\,,\ \Omega)$$

real 1-form $\hookleftarrow$ $\downarrow$ $\hookrightarrow$ complex $n$-form

real 2-form

with constant singlet intrinsic torsion

$$\mathrm{d}\eta = 2\omega \qquad \mathrm{d}\omega = 0 \qquad \mathrm{d}\Omega = i(n+1)\eta \wedge \Omega$$

  - the fields of the truncated theory are

| | | |
|---|---|---|
| scalars | $\Leftrightarrow$ | $h_{ab} \in \frac{\mathbb{R}^+ \times \mathbb{C}}{U(1)} = \mathbb{R}^+ \times \mathbb{R}^+$ |
| 1 gauge fields | $\Leftrightarrow$ | $\mathcal{A}\xi$ |

where $\xi$ is the Reeb vector $\xi \lrcorner \eta = 1$.

  - the truncation ansatz for the metric is

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2U}\mathrm{d}s^2_{2n} + e^{2V}(\eta + \mathcal{A})$$

# EXCEPTIONAL GENERALISED GEOMETRY

[hitchin 02; gualtieri 04; hull 07; pacheco, waldram 08, ...]

- Geometrise the gauge symmetries of RR and NS potentials by enlarging the tangent space $\longrightarrow$ generalised tangent bundle $E$
  - the transition functions involve RR and NS potentials as generalised diffeomorphims
  - the structure group is the duality group on the internal manifold
- Ex: IIB on $X_5 \times M_5$

| | Riemannian | E G G |
|---|---|---|
| tangent b. | $TM$ | $T \oplus T^* \oplus \Lambda^- \oplus \Lambda^5 T^* \oplus (T^* \otimes \Lambda^6 T^*)$ |
| structure group | $SO(5)$ | $E_{6(6)}$ U-duality |

- Generalise ordinary notions
  - generallised vectors and

$$v \in TM \longrightarrow V \in E \qquad \Xi_i \longrightarrow Q_i$$

relevant $E_{d(d)}$ groups and representations

| $D$ | $E_{d(d)}$ | $E$ | $\mathrm{ad}F \subset E \otimes E^*$ | $N \subset S^2 E$ | $\tilde{H}_d$ | $\mathcal{S}$ |
|---|---|---|---|---|---|---|
| 4 | $E_{7(7)}$ | $\mathbf{56}$ | $\mathbf{133}$ | $\mathbf{133}$ | $SU(8)$ | $\mathbf{8} \oplus \bar{\mathbf{8}}$ |
| 5 | $E_{66}$ | $\mathbf{27}$ | $\mathbf{78}$ | $\mathbf{27}'$ | $USp(8)$ | $\mathbf{8}$ |
| 6 | $Spin(5,5)$ | $\mathbf{16}^s$ | $\mathbf{45}$ | $\mathbf{10}$ | $USp(4) \times USp(4)$ | $(\mathbf{4},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4})$ |
| 7 | $SL(5,\mathbb{R})$ | $\mathbf{10}$ | $\mathbf{24}$ | $\mathbf{5}'$ | $USp(4)$ | $\mathbf{4}$ |

  - generalised metric

$$g \in \frac{GL(d)}{SO(d)} \longrightarrow G \in \frac{E_{d(d)}}{H_d}$$

$H_d \subset E_{d(d)}$ maximal compact subgroup (R-symmetry)

- Dorfman derivative

$$\mathcal{L}_v v'^m = v^n \partial_n v'^m - (\partial \times_{\rm ad} v)^m{}_n v'^n \longrightarrow L_V V'^M = V^N \partial_N V'^M - (\partial \times_{\rm ad} V)^M{}_N V'^N$$

- Generalised Levi-Civita connection

$$D_M V^A = \partial_M V^A + \Omega_M^A{}_B V^B \qquad V \in E$$

such that $DG = 0$ and torsion free

- Exceptional $G_S$ structure
    - $\exists$ a set of generalised tensors $Q_i$ that are invariant under $G_S \subset E_{d(d)}$

      Ex: $G$ defines an $H_d$ structure
    - with torsion $D_M Q_i = \Sigma_M Q_i$, with $\Sigma_M \in \Gamma(E^* \otimes \mathrm{ad} F_{H_d})$

## CONSISTENT TRUNCATIONS AND EXCEPTIONAL G-STRUCTURES

- Consistent truncations of M-theory or Type II on

$$X_D \times M \qquad \begin{cases} \dim M = d & \text{M-theory} \\ \dim M = d-1 & \text{Type II} \end{cases}$$

are pretty common

- any exceptional $G_S$-structure with constant, singlet intrinsic torsion gives a consistent truncation
- the exceptional $G_S$-structure does not necessarily reduce to an ordinary $G_S$-structure
- the geometrical data of the $G_S$-structure and its singlet intrinsic torsion completely determine the truncated theory

## The truncation procedure

- The gen. $G_S$ - structure is defined by a set of $G_S$-invariant tensors

$$Q_i = \{\mathcal{K}_A \in \Gamma(E)\,, J_\Sigma \in \Gamma(N)\}$$

such that $D_M Q_i = \Sigma_M Q_i$ only contains singlet torsion

- Arrange the bosonic fields into generalised tensors in representations of $GL(D;R) \times E_{d(d)}$

| | | |
|---|---|---|
| D-dim scalars: | $G_{MN}(x,y) \in \Gamma(S^2 E^*)$ | (gen. metric) |
| D-dim vectors: | ${\mathcal{A}_\mu}^M(x,y) \in \Gamma(T^*X \otimes E)$ | (gen vector) |
| D-dim two forms: | ${\mathcal{B}_{\mu\nu}}^{MN} \in \Gamma(\Lambda^2 T^*X \otimes N)$ | |

- Ex: IIB on $X_5 \times M_5$

| | |
|---|---|
| scalars: | $\{g_{mn},\, B_{mn},\, C_0,\, C_{mn},\, C_{mnpq}\}$ |
| vectors: | $\{{h_\mu}^m,\, B_{\mu n},\, C_{\mu n},\, C_{\mu npq}\}$ |
| two forms: | $\{B_{\mu\nu},\, C_{\mu\nu},\, C_{\mu\nu pq}\}$ |

- Expand the fields terms of the invariant tensors

$$\text{scalars:} \qquad h^I(x) \in \mathcal{M}_{scal} = \frac{C_{E_{d(d)}}(G_S)}{C_{H_d}(G_S)} := \frac{\mathcal{G}}{\mathcal{H}},$$

$$\text{vectors:} \qquad \mathcal{A}^{\mathcal{A}}_{\mu}(x)\, K_{\mathcal{A}} \in \Gamma(T^*X) \otimes \mathcal{V}$$

$$\text{two-forms:} \qquad \mathcal{B}^{\Sigma}_{\mu\nu}(x)\, J_{\Sigma} \in \Gamma(\Lambda^2 T^*X) \otimes \mathcal{B}$$

with $\mathcal{V} \subset \Gamma(E) \rightarrow$ vector space spanned by $\mathcal{K}_A$

$\mathcal{B} \subset \Gamma(N) \rightarrow$ vector space spanned by $J_\Sigma$

  - since the intrinsic torsion has only singlet components, the derivatives of any field are expanded only in singlets
  - the e.o.m are expanded in invariant tensors and only contain singlets representations
  - the truncation is consistent

- The singlet intrinsic torsion gives the embedding tensor of the gauged sugra

[coimbra, strickland-constable, waldram 11, lee, strickland-constable, waldram 14]

$$L_{\mathcal{K}_A} Q_i = -T_{int}(\mathcal{K}_A) \cdot Q_i \qquad \begin{array}{l} T_{int} \ : \Gamma(E) \to \mathrm{ad}F \\ T_{int}(\mathcal{K}_A) \ G_S \text{ singlet} \end{array}$$

then $T_{int}$ is the algebra of the commutant group $\mathcal{G} = C_{E_{d(d)}}(G_S)$

$$-T_{int} = \Theta : \mathcal{V} \to \mathrm{Lie}\mathcal{G}$$

- Gauge group and generators

$$L_{K_\mathcal{A}} K_\mathcal{B} = \Theta_\mathcal{A} \cdot K_\mathcal{B} = \Theta_\mathcal{A}{}^{\hat{\alpha}} (t_{\hat{\alpha}})_\mathcal{B}{}^\mathcal{C} K_\mathcal{C} = X_{\mathcal{A}\mathcal{B}}{}^\mathcal{C} K_\mathcal{C}$$

generators of $\mathrm{Lie}\mathcal{G}$ acting on $\mathcal{V}$ $\hookleftarrow$

with

$$[X_\mathcal{A}, X_\mathcal{B}] = -X_{\mathcal{A}\mathcal{B}}{}^\mathcal{C} X_\mathcal{C} \qquad (X_A)_B^C = X_{AB}^C$$

Then

- $\mathcal{K}_A$ generate the Lie algebra $\mathrm{Lie}G_{gauge} = \Theta(\mathcal{V}) \subseteq \mathrm{Lie}\mathcal{G}$
- $X_A$ define adjoint representation
- $\Theta$ gives the embedding of the gauge group in $\mathcal{G}$.

- Scalar covariant derivatives

$$\hat{D}_\mu h^I = \partial_\mu h^I - \mathcal{A}_\mu^{\mathcal{A}}\, \Theta_{\mathcal{A}}{}^{\hat{\alpha}} k_{\hat{\alpha}}{}^I$$

with $k_{\hat{\alpha}}$ Killing vectors on $\mathcal{M}_{scal}$ generating the action of the $\mathrm{Lie}\mathcal{G}$

- Gauge transformations of the vectors and two-forms

$$\delta \mathcal{A}_\mu^{\mathcal{A}} = \partial_\mu \Lambda^{\mathcal{A}} + X_{\mathcal{BC}}{}^{\mathcal{A}} \big( \mathcal{A}_\mu^{\mathcal{B}}\, \Lambda^{\mathcal{C}} - \Xi_\mu^{\mathcal{BC}} \big)$$

$$\delta \mathcal{B}_{\mu\nu}^{\Sigma} = 2 d_{\mathcal{AB}}{}^{\Sigma} \Big( \partial_{[\mu} \Xi_{\nu]}^{\mathcal{AB}} + 2 X_{\mathcal{CD}}{}^{\mathcal{A}} \mathcal{A}_{[\mu}^{\mathcal{C}}\, \Xi_{\nu]}^{\mathcal{DB}} - \Lambda^{\mathcal{A}} \mathcal{H}_{\mu\nu}^{\mathcal{B}} - \mathcal{A}_{[\mu}^{\mathcal{A}}\, \delta \mathcal{A}_{\nu]}^{\mathcal{B}} \Big)$$

with $\Xi_\mu^{\mathcal{AB}} = \Xi_\mu{}^{\Sigma} \tilde{d}_{\Sigma}{}^{\mathcal{AB}}$

$$\mathcal{H}^{\mathcal{A}} = \mathrm{d}\mathcal{A}^{\mathcal{A}} + X_{\mathcal{BC}}{}^{\mathcal{A}} (\mathcal{A}^{\mathcal{B}} \wedge \mathcal{A}^{\mathcal{C}} + \mathcal{B}^{\Sigma} \tilde{d}_{\Sigma}{}^{\mathcal{BC}})$$

- The number of supersymmetries preserved by the truncation is the number of $G_S$-singlets in the generalised spinor bundle $\mathcal{S}$

## HALF-MAXIMAL SUSY: generalised G-structures

- Compactifications on $M_5$: a half-maximal structure is an $SO(5) \subset E_{6(6)}$ structure

  [see malek 17, for the EFT version]

  - half-maximal susy corresponds to the breaking

$$USp(8) \supset USp(4)_R \times USp(4)_S \qquad \begin{cases} USp(4)_R \text{ R-symmetry} \\ USp(4)_S \text{ structure group} \end{cases}$$

  - the half-maximal structure is defined by the singlets of $USp(4)_S$

$$E_{6(6)} \supset SO(1,1) \times SO(5,5) \supset SO(1,1) \times SO(5)_R \times SO(5)_S$$

$$\mathbf{27} \to \mathbf{10}_2 \oplus \mathbf{16}_{-1} \oplus \mathbf{1}_{-4} \to (\mathbf{5},\mathbf{1})_2 \oplus (\mathbf{1},\mathbf{5})_2 \oplus (\mathbf{4},\mathbf{4})_{-1} \oplus (\mathbf{1},\mathbf{1})_{-4}$$

- it is defined by six generalised vectors

$$K_0, K_a \in \Gamma(E) \qquad a = 1, \dots, 5$$

such that

$$c(K_0, K_0, V) = 0, \qquad \forall V \in \Gamma(E)$$
$$c(K_0, K_a, K_b) = \delta_{ab}\,\mathrm{vol}_5$$

where $c(V, V', V'')$ is the $E_{6(6)}$ cubic invariant,

$$c(V, V', V'') = -\tfrac{1}{2}\left(\iota_v \rho' \wedge \rho'' + \epsilon_{\alpha\beta}\,\rho \wedge \lambda'^{\alpha} \wedge \lambda''^{\beta} - 2\epsilon_{\alpha\beta}\,\iota_v \lambda'^{\alpha} \sigma''^{\beta}\right) + \text{symm. perm.}\,.$$

- we also need

$$\left.\begin{aligned} K_0^* &= -\tfrac{1}{5}\delta^{ab} c(K_a, K_b, \cdot\,) \\ K_a^* &= c(K_0, K_a, \cdot\,) \end{aligned}\right\} \in \Gamma(\det T^*M \otimes E^*) \qquad \begin{aligned} \langle K_0^*, K_0\rangle &= 1 \\ \langle K_a^*, K_b\rangle &= \eta_{ab} \\ \langle K_0^*, K_a\rangle &= 0 \end{aligned}$$

- The generalised metric is computed using the $SO(5,5)$ structure defined by $\{K_0, K_0^*\}$

$$SO(5) \subset SO(5) \times SO(5) \subset SO(5,5) \subset E_{6(6)}$$

- the $SO(5,5)$ structure gives a decomposition of the generalised tangent bundle

$$V = V_0 + \tilde{V} + \Psi \quad \in \quad E = E_0 + E_{10} + E_{16}$$

$$\mathbf{27} = \mathbf{1} + \mathbf{10} + \mathbf{16}$$

- the generalised metric on $E$ splits into metrics on $E_0$, $E_{10}$ and $E_{16}$

$$\begin{aligned} G &= G_0 + G_{10} + G_{16} \\ &= \langle K_0^*, V\rangle^2 + \delta^{ab}\langle K_a^*, V\rangle\langle K_b^*, V\rangle + \frac{c(K_0, V, V)}{\text{vol}} - 4\sqrt{2}\langle V, K_1 \cdots K_5 \cdot V\rangle \end{aligned}$$

where

$$\langle Z, V\rangle = \hat{v}_m v^m + \hat{\lambda}^m_\alpha \lambda^\alpha_m + \tfrac{1}{3!}\,\hat{\rho}^{mnp}\rho_{mnp} + \tfrac{1}{5!}\,\hat{\sigma}^{mnpqr}_\alpha \sigma^\alpha_{mnpqr}$$

- $G_0$ : projection onto the singlet,
- $G_{10}$ : an $SO(5) \times SO(5) \subset SO(5,5)$ structure splits $E_{10}$ into positive- and negative-definite eigenspaces

$$E_{10} = C_+ \oplus C_- \qquad \Leftrightarrow \qquad \begin{array}{l} \eta = G_+ - G_- \\ G_{10} = G_+ + G_- \end{array}$$

with ($K_a$ form a basis for $C_-$)

$$\eta(\tilde{V}, \tilde{V}) = \frac{c(K_0, \tilde{V}, \tilde{V})}{\mathbf{vol}_5} \qquad G_-(V, V) = \delta^{ab} \langle K_a^*, V \rangle \langle K_b^*, V \rangle$$

- $G_{16}$ : inner product between $SO(d,d)$ spinors

$$\langle \Psi, \Gamma^{(+)} \Psi \rangle \qquad \Gamma^+ = \Gamma_1^+ \cdots \Gamma_5^+ \text{ chirality matrix}$$

$$\Updownarrow$$

$$\langle V, K_1 \cdots K_5 \cdot V \rangle$$

- Generalised $SO(5-n)$ structures
  - the structure is further reduced to $SO(5-n) \subset SO(5)$ by $n$ globally-defined generalised vectors in the $\mathbf{27}$

$$(K_0, K_A) \quad A = 1, \dots, n+5 \qquad \begin{aligned} c(K_0, K_0, V) &= 0, \quad \forall V \in \Gamma(E), \\ c(K_0, K_A, K_B) &= \eta_{AB} \mathrm{vol}_5 \\ c(K_A, K_B, K_C) &= 0 \end{aligned}$$

with $\eta_{MN} = \mathrm{diag}(-1, -1, -1, -1, -1, +1, \dots, +1)$ flat $SO(5, n)$ metric.

  - dual generalised vectors

$$\begin{aligned} K_0^* &= \frac{1}{5+n} \eta^{AB} c(K_A, K_B, \cdot) \\ K_A^* &= c(K_0, K_A, \cdot) \end{aligned}$$

with $\langle K_0^*, K_0 \rangle = 1$, $\langle K_A^*, K_B \rangle = \eta_{AB}$ and $\langle K_0^*, K_a \rangle = 0$

  - the generalised metric is computed as before

$$G = \langle K_0^*, V \rangle^2 + \delta^{ab} \langle K_a^*, V \rangle \langle K_b^*, V \rangle + \frac{c(K_0, V, V)}{\mathrm{vol}} - 4\sqrt{2} \langle V, K_1 \cdots K_5 \cdot V \rangle$$

- truncation ansatz for the scalars
  - the scalar manifold is given by the coset

$$H \in \frac{C_{E_{6(6)}}(SO(5-n))}{C_{SU(8)}(SO(5-n))} = O(1,1) \times \frac{SO(5,n)}{SO(5)\times SO(n)}$$

  - the generalised metric is built out of dressed generalised vectors

$$\tilde{K}_0 = \Sigma^2 K_0. \qquad \tilde{K}_a = \Sigma^{-1}\,\mathcal{V}_a{}^A K_A \qquad \tilde{K}_{\hat{a}} = \Sigma^{-1}\,\mathcal{V}_{\hat{a}}{}^A K_A$$
$$\tilde{K}_0^* = \Sigma^{-2} K_0^* \qquad \tilde{K}_a^* = \Sigma\,\mathcal{V}_a{}^A K_A^* \qquad \tilde{K}_{\hat{a}}^* = \Sigma\,\mathcal{V}_{\hat{a}}{}^A K_A^*$$

where $\Sigma \in O(1,1)$, $\{\mathcal{V}_A{}^a, \mathcal{V}_A{}^{\hat{a}}\} \in SO(5,n)$, $a = 1,\ldots,5, \hat{a} = 1,\ldots,n$

$$\eta_{AB} = \delta_{ab}\,\mathcal{V}_A{}^a\mathcal{V}_B{}^b - \delta_{\hat{a}\hat{b}}\,\mathcal{V}_A{}^{\hat{a}}\mathcal{V}_B{}^{\hat{b}} \quad M_{AB} = \delta_{ab}\,\mathcal{V}_A{}^a\mathcal{V}_B{}^b + \delta_{\hat{a}\hat{b}}\,\mathcal{V}_A{}^{\hat{a}}\mathcal{V}_B{}^{\hat{b}}$$

  - the generalised metric is

$$\begin{aligned} G &= G_0 + G_{10} + G_{16} \\ &= \Sigma^{-4}\,\langle V K_0^*\rangle^2 + \Sigma^2\left(2\;\delta^{ab}\mathcal{V}_a{}^C\mathcal{V}_b{}^D\,\langle V, K_A^*\rangle\langle V, K_B^*\rangle + \frac{c(K_0,V,V)}{\text{vol}}\right) \\ &\quad -\tfrac{4\sqrt{2}}{5!}\,\Sigma^{-1}\,\epsilon^{abcde}\mathcal{V}_a{}^A\mathcal{V}_b{}^B\mathcal{V}_c{}^C\mathcal{V}_d{}^D\mathcal{V}_e{}^E\,\langle V, K_A\cdots K_E^*\cdot V\rangle \end{aligned}$$

- Example: Sasaki Einstein reduction in type IIB
  - truncation on squashed SE manifolds in 5 d [cassani, faedo, dall'agata 10]
    - the theory is $\mathcal{N}=4$ 5d sugra with two vector multiplets and with $Heis_3 \times U(1)$ gauging and scalars parameterise the coset

$$\mathcal{M}_{\mathrm{scal}} = SO(1,1) \times \frac{SO(5,2)}{SO(5)\times SO(2)}$$

  - SE geometry
    - $U(1)$ fibration over a KE base

$$\mathrm{d}s^2_{SE} = \mathrm{d}s^2_{KE} + \eta^2 = \sum_{i=2}^{5}(e^i)^2 + (e^1)^2 \qquad F_5 = \kappa \mathrm{vol}_5$$

    - 5d SE are $SU(2)$ structure manifolds

$$\begin{aligned} J_1 &= e^2\wedge e^5 - e^3\wedge e^4 \\ J_2 &= e^2\wedge e^4 + e^3\wedge e^5 \\ J_3 &= e^2\wedge e^3 - e^4\wedge e^5 \end{aligned} \qquad \Longleftrightarrow \qquad \begin{aligned} \Omega &= J_1 + iJ_2 \\ \omega &= J_3 \\ \eta &= -e^1 \end{aligned}$$

- the $SU(2)$ structure extend to a generalised $SU(2) \subset SO(5)$ structure
  - generalised vectors

$$
\begin{aligned}
K_0 &= \xi \\
K_{1,2,3} &= \tfrac{1}{\sqrt{2}}\,\eta \wedge J_{1,2,3}
\end{aligned}
\qquad
\begin{aligned}
K_4 &= \tfrac{1}{\sqrt{2}}(n\eta - r\mathrm{vol}) \\
K_5 &= \tfrac{1}{\sqrt{2}}(-r\eta - n\mathrm{vol}) \\
K_6 &= \tfrac{1}{\sqrt{2}}(n\eta + r\mathrm{vol}) \\
K_7 &= \tfrac{1}{\sqrt{2}}(-r\eta + n\mathrm{vol})
\end{aligned}
$$

    where $n = (1,0)$ and $r = (0,1)$.

  - under the commutant $SO(1,1) \times SO(5,2)$ of $SU(2)$ in $E_{6(6)}$

$$K_0 \in \mathbf{1}_{-1} \qquad K_A \in \mathbf{7}_{1/2}$$

- scalar ansatz
  - the scalar parameterise the coset

$$O(1,1) \times \frac{SO(5,2)}{SO(5) \times SO(2)}$$

  - generalised metric is built using the dressed vectors

$$\begin{pmatrix} \tilde{K}_a \\ \tilde{K}_{\underline{a}} \end{pmatrix} = e^{-(B^+ + B^- + C)} \cdot m \cdot r \cdot e^{-l} \cdot \begin{pmatrix} K_a \\ K_{\underline{a}} \end{pmatrix} = \Sigma^{-1} \begin{pmatrix} \mathcal{V}_a{}^B \\ \mathcal{V}_{\underline{a}}{}^B K_B \end{pmatrix}$$

with

$$B^\alpha = (n^\alpha b_i + r^\alpha c_i) J_i \,, \qquad C = -a J_3 \wedge J_3$$

$$m^\alpha{}_\beta = \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ e^{\frac{\phi}{2}} C_0 & e^{-\frac{\phi}{2}} \end{pmatrix} \,, \quad r = \mathrm{diag}\left(e^V, e^U, e^U, e^U, e^U\right) \qquad l = \frac{\mathrm{tr}(r)}{3} = \frac{4U + V}{3}$$

  - a lengthy but straightforward computation reproduce the scalars of SE truncation

[cassani, dall'agata, faedo 10]

- vector ansatz

$$
\begin{aligned}
h_\mu &= \mathcal{A}^0_\mu\, \xi\,, \\
B^+_{\mu,1} &= \tfrac{1}{\sqrt{2}}\left(\mathcal{A}^4_\mu + \mathcal{A}^6_\mu\right)\eta\,, \\
B^-_{\mu,1} &= -\tfrac{1}{\sqrt{2}}\left(\mathcal{A}^5_\mu + \mathcal{A}^7_\mu\right)\eta\,, \\
C_{\mu,3} &= \tfrac{1}{\sqrt{2}}\,\mathcal{A}^i_\mu\, j_i \wedge \eta\,, \\
\tilde{B}^+_{\mu,5} &= -\tfrac{1}{\sqrt{2}}\left(\mathcal{A}^5_\mu - \mathcal{A}^7_\mu\right)\mathrm{vol} + \tfrac{1}{\sqrt{2}}\left(\mathcal{A}^4_\mu + \mathcal{A}^6_\mu\right)\wedge\eta\,, \\
\tilde{B}^-_{\mu,5} &= -\tfrac{1}{\sqrt{2}}\left(\mathcal{A}^4_\mu - \mathcal{A}^6_\mu\right)\mathrm{vol} + \tfrac{1}{\sqrt{2}}\left(\mathcal{A}^5_\mu + \mathcal{A}^7_\mu\right)\wedge\eta\,.
\end{aligned}
$$

- two-form ansatz

$$
\begin{aligned}
B_{\mu\nu,0\,+} &= \tfrac{1}{\sqrt{2}}\left(\mathcal{B}_{\mu\nu\,7} - \mathcal{B}_{\mu\nu\,5}\right)\,, \\
B_{\mu\nu,0\,-} &= \tfrac{1}{\sqrt{2}}\left(\mathcal{B}_{\mu\nu\,6} - \mathcal{B}_{\mu\nu\,4}\right)\,, \\
C_{\mu\nu,2} &= \tfrac{1}{\sqrt{2}}\,\mathcal{B}_{\mu\nu\,i}\, j_i\,, \\
\tilde{B}_{\mu\nu,4\,+} &= \tfrac{1}{\sqrt{2}}\left(\mathcal{B}_{\mu\nu\,4} + \mathcal{B}_{\mu\nu\,6}\right)\mathrm{vol}_4 + \tfrac{1}{\sqrt{2}}\left(\mathcal{B}_{\mu\nu\,7} - \mathcal{B}_{\mu\nu\,5}\right)\,, \\
\tilde{B}_{\mu\nu,4\,-} &= -\tfrac{1}{\sqrt{2}}\left(\mathcal{B}_{\mu\nu\,5} + \mathcal{B}_{\mu\nu\,7}\right)\mathrm{vol}_4 + \tfrac{1}{\sqrt{2}}\left(\mathcal{B}_{\mu\nu\,6} - \mathcal{B}_{\mu\nu\,4}\right)\,.
\end{aligned}
$$

- gauge group $\to$ Dorfman derivative
  - the embedding tensor of $\mathcal{N} = 4$ sugra [schon, weidner 06]

$$(\xi_A, \xi_{AB} = \xi_{[AB]}, f_{ABC} = f_{ABC]}) \quad \longrightarrow \quad \begin{cases} f_{[AB}{}^E f_{CD]E} = 0 \\ \xi_A{}^D f_{DBC} = 0 \\ \xi_A = 0 \end{cases}$$

  - Dorfman derivative

$$L_{K_\mathcal{A}} K_\mathcal{B} = X_{\mathcal{A}\mathcal{B}}{}^\mathcal{C} K_\mathcal{C} \quad \underset{\mathcal{A}=(0,A)}{\Longrightarrow} \quad \begin{cases} X_{AB}{}^C = -f_{AB}{}^C \\ X_{0A}{}^B = -\xi_A{}^B \end{cases}$$

- in this case

$$X_{01}{}^{2} = -X_{02}{}^{1} = 3\,,$$

$$X_{04}{}^{5} = -X_{05}{}^{4} = -X_{04}{}^{7} = -X_{07}{}^{4} = X_{05}{}^{6} = X_{06}{}^{5} = -X_{06}{}^{7} = X_{07}{}^{6} = \frac{\kappa}{2},$$

$$X_{34}{}^{5} = -X_{34}{}^{7} = -X_{35}{}^{4} = X_{35}{}^{6} = X_{36}{}^{5} = -X_{36}{}^{7} = -X_{37}{}^{4} = X_{37}{}^{6} = \sqrt{2}\,,$$

$$X_{45}{}^{3} = X_{47}{}^{3} = -X_{56}{}^{3} = X_{67}{}^{3} = \sqrt{2}\,,$$

- the embedding tensor components are

$$\xi_{12} = 3\,, \qquad \xi_{45} = \xi_{47} = -\xi_{56} = \xi_{67} = \frac{\kappa}{2}\,,$$

$$f_{345} = f_{347} = -f_{356} = f_{367} = \sqrt{2}\,.$$

- the gauge algebra is $Heis_3 \times U(1)$

# SUMMARY AND OUTLOOK

- Generalised Geometry is a powerful framework to study consistent truncations with different amount of susy
  - the truncation ansatze are associated to generalised structures
    - the intrinsic torsion of the G-structure must contains only singlets
  - the geometry determines all the features of the lower-dimensional gauged supergravity
    - amount of supersymmetry
    - scalar coset manifold
    - number of gauge and tensor fields, and the gauging
- Examples
  - consistency of sphere reductions [lee, strickland-constable, waldram 14; hohm, samtleben 14) , ...]
  - consistent truncations for massive IIA [ ciceri, guarino, inverso 16; cassani, de felice, m.p. strickland constable, waldram 16]
  - half-maximal truncations of M-theory on $S^4$ and of type IIB on $\beta$-deformed backgrounds [ cassani, josse, m.p., waldram 19]

- Consistent reductions are not a mathematical curiosity
  - establish a map between sugra theories in different dimensions
  - insight on the higher dimensional origin of the lower dimensional gauge symmetries
  - powerful tool in AdS/CFT
  - embed into string theory AdS vacua, black holes, domain walls, and non-relativistic backgrounds