Geometric model of particles

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Outline

Introduction

Geometrical description of elementary particles attracted interest of physicists at different stages of physics history [1,2] . With the advent of general relativity all attempts in that direction focussed on the curvatures and singularities of spacetimes as potential sites to represent particles. In his description Geometrodynamics [3] , John Wheeler advocated the view of concentrated field points - the Geons - as particle-like structures in spacetimes. More recently, the trend of constructing particle models from geometry oriented towards non-singular spacetimes such as the de Sitter core with the cosmological constant.

[1] A. V. Vilenkin, and P. I. Fomin, ITP Report No. ITP-74-78R (1974).
[2] A. V. Vilenkin, and P. I. Fomin, Nuovo Cimento Soc. Ital. Fis. A **45**, 59 (1978).
[3]J. A. Wheeler, Geometrodynamics (Academic Press, the University of California, 1962).

Since a particle can also have charge besides mass, the electromagnetic spacetimes such as the non-singular Bertotti-Robinson geometry also attracted attentions [4,5], in this regard. Having no interior singularity and possessing both mass and electric charge became basic criteria in search for a geometrical model of particles. The method has been to consider a spherical shell as the representative surface of the particle with inner/outer regions satisfying certain junction conditions [6] . Among those conditions we cite continuity of the first fundamental form (the induced metric) and the second fundamental form across the surface of the particle.

[4] O. B. Zaslavskii, Phys. Rev. D **70**, 104017 (2004).
[5] S. H. Mazharimousavi and M. Halilsoy, Int. J. Geom. Meth. Mod. Phys. **16**, 1950121 (2019).
[6] G. Darmois (1927) Mémorial de Sciences Mathématiques, Fascicule XXV, "Les equations de la gravitation einsteinienne", Chapitre V. W. Israel, Thin shells in general relativity, *Il Nuovo Cim.* **66** (1966) 1.

In the first part of this study, we show that geometrical model of a particle is possible within the context of third-order Lovelock gravity (LG) combined with the thin-shell formalism. LG [7], is known to have the most general combination of curvature invariants that still maintains the second-order field equations. Our model will cover up to the third-order terms, supplemented by the Maxwell Lagrangian, apt for the dimension of $n+1$-dimensional spacetimes with $n \geq 6$, in which the theory admits non-trivial solutions. We made choice amongst available solutions of the theory that suits our purpose. [7] D. Lovelock, J. Math. Phys. **12**, 498 (1971).

A spherical thin-shell is assumed as surface of the particle, whose inside is the flat Minkowski spacetime with a suitable Lovelock solution to represent the outer region. *Let us add that in Einstein's gravity a constant potential inside* (Φ =*constant*$\neq 0$) *the shell amounts to the presence of a global monopole at the origin. To avoid any physical inconvenience invited by such a monopole we can choose the interior geometry of our particle to be represented by the flat Minkowski metric, i.e., the potential* $\Phi = 0$, *which is the simplest case to be chosen.* Deliberate choice of the Lovelock's coupling constants in the action, renders physical boundary conditions possible for a surface energy-momentum on the shell. As the final step, we set the pressure and surface energy density to zero and search for viable geometrical criteria. Those conditions determine the mass and charge of the particle constructed, entirely from the geometrical parameters of the third-order LG in $6 + 1$-dimensions, as an example.

In the second part of the paper we extend our particle model to a rather more interesting era of modified theory of gravity which is called *pure* LG. This terminology has been used first by David Kastor and Robert Mann in [8] and has been developed by Rong-Gen Cai, et al in [9], N. Dadhich, et al in [10,11].

[8] D. Kastor and R. Mann, JHEP 04, 048 (2006).
[9] R.-G. Cai and N. Ohta,Phys. Rev. D **74**, 064001 (2006)
[10] S. Chakraborty and N. Dadhich, Eur. Phys. J. C **78**, 81 (2018).
[11] N. Dadhich and J. M. Pons, J. Math. Phys. **54**, 102501 (2013).

While in the Lovelock theory, the action is given by

$$I = \int d^{n+1}x\sqrt{-g}\left(\sum_{i=0}^{\left[\frac{n}{2}\right]}\alpha_i\mathcal{L}_i + \mathcal{L}_{matter}\right) \tag{1}$$

in which $\left[\frac{n}{2}\right]$ stands for the integral part of the $\frac{n}{2}$, α_i are Lovelock parameters,

$$\mathcal{L}_i = \frac{1}{2}\delta^{\alpha_1\beta_1\ldots\alpha_i\beta_i}_{\mu_1\nu_1\ldots\mu_i\nu_i}R^{\mu_1\nu_1}{}_{\alpha_1\beta_1}\ldots R^{\mu_i\nu_i}{}_{\alpha_i\beta_i} \tag{2}$$

are the Euler densities of a $2i$-dimensional manifold, in pure LG the action is expressed as

$$I = \int d^{n+1}x\sqrt{-g}\left(\alpha_0 + \alpha_p\mathcal{L}_p + \mathcal{L}_{matter}\right) \tag{3}$$

in which α_0 represents the cosmological constant and $1 \leq p \leq \left[\frac{n}{2}\right]$. For instance, the Einstein R-gravity has $p = 1$, $\alpha_1 = 1$ and $\mathcal{L}_1 = R$ which is the particular case of the pure LG applicable in all dimensions. For $p = 2$, one finds the pure Gauss-Bonnet (GB) gravity applicable in $n + 1 \geq 5$. Finally, in this paper, $p = 3$ represents pure third order LG which is valid with $n + 1 \geq 7$. Unlike the particle model in third order Lovalock gravity, in this second part we give a general formalism in the three different pure Lovelock theories mentioned above, i.e., the pure Einstein, GB and third order LG.

It will be shown that in specific case where the inner and outer spacetimes admit identical Lovelock parameters i.e., $\alpha_p^+ = \alpha_p^-$ the junction conditions result in the same in terms of the metric functions and their first derivatives, irrespective of the order of the Lovelock term. These junction conditions are simply the continuity of the bulk's metric function and its first derivatives across the surface of the particle. These are equivalent to the continuity of the first and second fundamental forms of the surface.

Throughout the article, the unit convention $4\pi\varepsilon_{0(n+1)} = 8\pi G_{(n+1)} = c = 1$ is applied.

Thin-Shell Formalism

Consider a spherically symmetric Riemannian manifold in $n+1$ dimensions, with two distinct regions, say $(\Sigma, g)^{\pm} = \{x^{\mu}_{\pm} | r_{+} \geq a > r_e, r_{-} \leq a\}$ (where r_e is a (probable) event horizon), distinguished by their common timelike hypersurface $\partial\Sigma = \{\xi^{\mu}_{\pm} | r = a\}$. The hypersurface $\partial\Sigma$ is therefore a thin-shell separating the two regions with different line elements and probably different coordinates $x^{\mu}_{\pm}$. The inner spacetime (marked with $(-)$) is (preferably) non-singular and the outer spacetime (marked with $(+)$) can have an event horizon behind the thin–shell (if there is any at all). For the purpose of this study, we would like to have for both inner and outer spacetimes, the solutions to the third-order LG. Expectedly, the two spacetimes cannot be matched randomly at the thin-shell location, and it takes some proper junction conditions which will be discussed below.

In the $n+1$-dimensional third-order LG ($n \geq 6$) coupled with linear electrodynamics, the action is given by

$$I = \int d^{n+1}x\sqrt{-g}\left(\mathcal{L}_1 + \alpha_2\mathcal{L}_2 + \alpha_3\mathcal{L}_3 - F^{\mu\nu}F_{\mu\nu}\right), \tag{4}$$

in the presence of an electromagnetic field with the anti-symmetric field tensor $F_{\mu\nu}$. Here,

$$\mathcal{L}_1 = R, \tag{5}$$

$$\mathcal{L}_2 = \mathcal{L}_{GB} = R^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} - 4R^{\mu\nu}R_{\mu\nu} + R^2, \tag{6}$$

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and

$$\begin{aligned}\mathcal{L}_3 &= 2R^{\kappa\lambda\rho\sigma}R_{\rho\sigma\mu\nu}R^{\mu\nu}{}_{\kappa\lambda} + 8R^{\mu\nu}{}_{\kappa\lambda}R^{\kappa\sigma}{}_{\nu\rho}R^{\lambda\rho}{}_{\mu\sigma} + 24R^{\kappa\lambda\mu\nu}R_{\mu\nu\lambda\rho}R^{\rho}{}_{\kappa} \\ &+3RR^{\kappa\lambda\mu\nu}R_{\kappa\lambda\mu\nu} + 24R^{\kappa\lambda\mu\nu}R_{\mu\kappa}R_{\nu\lambda} + 16R^{\mu\nu}R_{\nu\sigma}R^{\sigma}{}_{\mu} - 12RR^{\mu\nu}R_{\mu\nu} + R^3\end{aligned} \tag{7}$$

are the first (Einstein-Hilbert), the second (GB), and the third-order Lovelock Lagrangians, respectively, accompanied with their respective constant coefficients α_2, and α_3. The spherically symmetric line element suitable for this action is given by

$$ds^2 = -f(r)\,dt^2 + f^{-1}(r)\,dr^2 + r^2 d\Omega_{n-1}^2, \tag{8}$$

where $d\Omega_{n-1}^2$ is the line element of the $n-1$-dimensional unit sphere. Although there are three general solutions for $f(r)$, the one we consider here is

$$f(r) = 1 - r^2 \left(-\frac{\tilde{\alpha}_2}{3\tilde{\alpha}_3} + \delta + u\delta^{-1} \right), \tag{9}$$

where

$$\begin{cases} \tilde{\alpha}_2 = (n-3)(n-2)\,\alpha_2 \\ \tilde{\alpha}_3 = (n-5)(n-4)(n-3)(n-2)\,\alpha_3 \\ \delta = \left(v + \sqrt{v^2 - u^3} \right)^{1/3} \end{cases}, \tag{10}$$

and

$$\begin{cases} u = \dfrac{\tilde{\alpha}_2^2 - 3\tilde{\alpha}_3}{9\tilde{\alpha}_3^2} \\ v = \dfrac{9\tilde{\alpha}_2\tilde{\alpha}_3 - 2\tilde{\alpha}_2^3}{54\tilde{\alpha}_3^3} + \dfrac{1}{2\tilde{\alpha}_3}\left[\dfrac{m}{r^n} - \dfrac{q^2}{r^{2(n-1)}} \right] \end{cases}. \tag{11}$$

In the special case where $\tilde{\alpha}_2^2 = 3\tilde{\alpha}_3 = \beta^2$, one finds $u = 0$ and $\delta = (2v)^{1/3}$, with which the solution in Eq. (9) reduces to

$$f(r) = 1 + \frac{r^2}{\beta}\left\{1 - \left[1 + 3\beta\left(\frac{m}{r^n} - \frac{q^2}{r^{2(n-1)}}\right)\right]^{\frac{1}{3}}\right\}. \qquad (12)$$

Depending on the values of the mass m, the charge q, and the number of spatial dimensions n, this particular solution could represent a black hole with two horizons, an extremal black hole with a single horizon, or a non-black hole solution with a naked singularity. In this study, we assign to the inner spacetime an $n+1$-dimensional Minkowski geometry, which amounts to choosing $m_{-}=q_{-}=0$, hence, $f_{-}(r_{-})=1$. Furthermore, we consider $m_{+}=m$, $q_{+}=q$, and that $a>r_e$ for the outer spacetime, where r_e is the event horizon of $f_{+}(r_{+})$ (if there is any), and $f_{+}(r_{+})$ has the general form in Eq. (9). Note that, although we exploited the special choice of $\tilde{\alpha}_2^2=3\tilde{\alpha}_3=\beta^2$ to obtain Eq. (12), we distinguish β_{+} and β_{-} for the outer and inner spacetimes, respectively. With the choice $f_{-}(r_{-})=1$, however, the parameter β_{-} remains arbitrary, which will play role in determining the mass and the charge of the particle.

The Junction Conditions

As it is already mentioned, the matching at $r_{\pm} = a$ follows certain junction conditions. In general relativity these are known as Darmois-Israel junction conditions [6] which are, however, inapplicable when it comes to modified theories of gravity. In the case of the third-order LG the proper junction conditions are given by Dehghani *et.al* in [12,13] (Also see [14,15]). These junction conditions firstly demand the continuity of the first fundamental form at the thin-shell's surface, so that one has a smooth transition while passing across the shell.

[12] M. H. Dehghani, and R. B. Mann, Phys. Rev. D **73**, 104003 (2006).
[13] M. H. Dehghani, N. Bostani, and A. Sheykhi, Phys. Rev. D **73**, 104013 (2006).
[14] M. H. Dehghani, and M. R. Mehdizadeh, Phys. Rev. D **85**, 024024 (2012).
[15] M. R. Mehdizadeh, M. Kord Zangeneh, and F. S. N. Lobo, Phys. Rev. D **92**, 044022 (2015).

The line element of the thin-shell can be stated as

$$ds^2_{\partial\Sigma} = \gamma^{\pm}_{ab} d\xi^a d\xi^b = -d\tau^2 + a^2 d\Omega^2_{n-1}, \tag{13}$$

where τ is the proper time on the shell and $\gamma^{\pm}_{ab} = \frac{\partial x^{\mu}_{\pm}}{\partial \xi^a} \frac{\partial x^{\nu}_{\pm}}{\partial \xi^b} g^{\pm}_{\mu\nu}$. The first junction condition with the assumption $\frac{dt_+}{d\tau} = \frac{dt_-}{d\tau}$, implies

$$f_+(a) = f_-(a), \tag{14}$$

for a static shell, which for our choice of outer and inner metric functions immediately leads to

$$a = \left(q^2/m\right)^{1/(n-2)}. \tag{15}$$

This static radius, is therefore, where the two spacetimes could join smoothly. Note that, in three spatial dimensions ($n = 3$) this radius is interestingly in full agreement with the classical radius of a charged particle, remarking the unit convention used here. Consequently, this expression for the radius of the particle in this model can be regarded as the classical radius of a charged particle in higher dimensional third-order LG.

Secondly, there exists a non-zero surface energy-momentum tensor on the shell S_{ab}, with their components given in orthonormal coordinates for a static thin-shell by

$$-S_b^a = \left[K_b^a - K\delta_b^a + 2\alpha_2\left(3J_b^a - J\delta_b^a\right) + 3\alpha_3\left(5P_b^a - P\delta_b^a + \mathcal{L}_2\left(K_b^a + K\delta_b^a\right)\right)\right]_-^+, \tag{16}$$

where $[\ ]_-^+$ denotes a jump in the expression inside the brackets, i.e. $[\Psi]_-^+ \equiv \Psi_+ - \Psi_-$. Herein, K_b^a are the mixed tensor components of the extrinsic curvature tensor of the shell given by

$$K^{\pm}_{ab} = -n^{\pm}_{\lambda}\left(\frac{\partial^2 x^{\lambda}_{\pm}}{\partial\xi^a\partial\xi^b} + \Gamma^{\lambda\pm}_{\alpha\beta}\frac{\partial x^{\alpha}_{\pm}}{\partial\xi^a}\frac{\partial x^{\beta}_{\pm}}{\partial\xi^b}\right), \tag{17}$$

where $n^{\pm}_{\lambda}$ are the unit spacelike normals to the surface identified by $n^{\pm}_{\mu}\frac{\partial x^{\mu}_{\pm}}{\partial\xi^i} = 0$, $n^{\pm}_{\mu}n^{\mu}_{\pm} = 1$, and $\Gamma^{\lambda\pm}_{\alpha\beta}$ are the Christoffel symbols of the outer and inner spacetimes, compatible with $g^{\pm}_{\alpha\beta}$. Note that, since the metric of our bulks are diagonal, for our radially symmetric static timelike shell the extrinsic curvature tensor will be diagonal, as well. Also, δ^a_b are the Kronecker symbol, and J and P are the respective traces of J_{ab} and P_{ab}, with their corresponding mixed tensors for a diagonal metric, such as the one in Eq. (12), are specified by

$$\begin{cases} J_b^a = diag\left(-\frac{2!}{3}\left\{\sum_{s=0}^{2}\frac{(-1)^s}{n}\binom{n}{s}\left[sK_\tau^\tau+(n-s)K_\theta^\theta\right]\left(K_\theta^\theta\right)^{s-1}\left(K_b^a\right)^{5-s}\right\}\right) \\ P_b^a = diag\left(\frac{4!}{5}\left\{\sum_{s=0}^{4}\frac{(-1)^s}{n}\binom{n}{s}\left[sK_\tau^\tau+(n-s)K_\theta^\theta\right]\left(K_\theta^\theta\right)^{s-1}\left(K_b^a\right)^{5-s}\right\}\right) \end{cases} \tag{18}$$

Here, K_τ^τ and K_θ^θ are the components associated with the time and angular coordinates of the thin-shell, since $K_\theta^\theta = K_{\theta_1}^{\theta_1} = K_{\theta_2}^{\theta_2} = \ldots = K_{\theta_{n-1}}^{\theta_{n-1}}$. In what follows we take $n = 6$, noting that the same argument can be applied to higher dimensions, at equal ease.

Having everything done on the second junction condition, we arrive at

$$\begin{cases} \sigma = -\frac{5}{9a^5} \sum_{i=+,-} i\sqrt{f_i} \left\{ \left[3a^2 + \beta_i \left(3 - f_i\right)\right]^2 + \frac{4}{5}\beta_i^2 f_i^2 \right\} \\ p = \frac{1}{2a^4} \sum_{i=+,-} \frac{1}{i\sqrt{f_i}} \left\{ f_i' \left[a^2 + \beta_i \left(1 - f_i\right)\right]^2 + 8af_i \left[a^2 + \beta_i \left(1 - f_i/3\right)\right] \right\} \end{cases} \tag{19}$$

which are the surface energy density and angular pressure of the fluid on the shell, in accordance with the energy-momentum tensor $S_b^a = diag\left(-\sigma, p, p, ..., p\right)$, respectively. By applying the radius of the particle acquired from the first junction condition (Eq. (15)), we set both σ and p to zero [2,4] to obtain the one and only solution for m and q as

$$\{m, q\} = \left\{ \frac{8\left(\beta_-^2 - \beta_+^2\right)}{15}, \pm \frac{8\left(\beta_- + \beta_+\right)\sqrt{\beta_-^2 - \beta_+^2}}{5\sqrt{30}} \right\}. \tag{20}$$

This in turn yields

$$a = \sqrt{\frac{2}{5}\left|\beta_- + \beta_+\right|}, \tag{21}$$

for the radius of the particle, according to Eq. (8).

To have physically meaningful quantities, then, we must impose $|\beta_-| > |\beta_+|$. This condition is specially surprising in the sense that the inner spacetime is flat, and one may naively think that the value of β_- would not affect the results in a great deal; say, it could be set to zero from the beginning. However, as can be perceived from the condition $|\beta_-| > |\beta_+|$, setting $\beta_- = 0$ (which directly assigns a usual Minkowski geometry to the inner spacetime), will not lead to a physically sensible mass for the particle.

Moreover, note that the steps leading to Eq. (20) could have been looked at in a different way. Theoretically, having β_- and β_+ as the constants of the theory, one can find the mass, the charge and the radius of the particle using Eqs. (20) and (21). However, an experimentalist would rather find β_- and β_+ by fine-tuning them such that the exact values of the mass and the charge of the fundamental particles come out of the theory. In other words, we could have solved the system of equations $\sigma = 0$ and $p = 0$ for β_- and β_+ instead of m and q. This would result in

$$\beta_{\pm} = \pm \frac{(3m^2 \mp 10q^2)}{8\sqrt{mq^2}}. \tag{22}$$

Consequently, taking into account the positivity of the mass, β_- will always be a negative constant. On the contrary, depending on the mass and the charge of the particle, β_+ can be either positive or negative as long as it satisfies $|\beta_-| > |\beta_+|$.

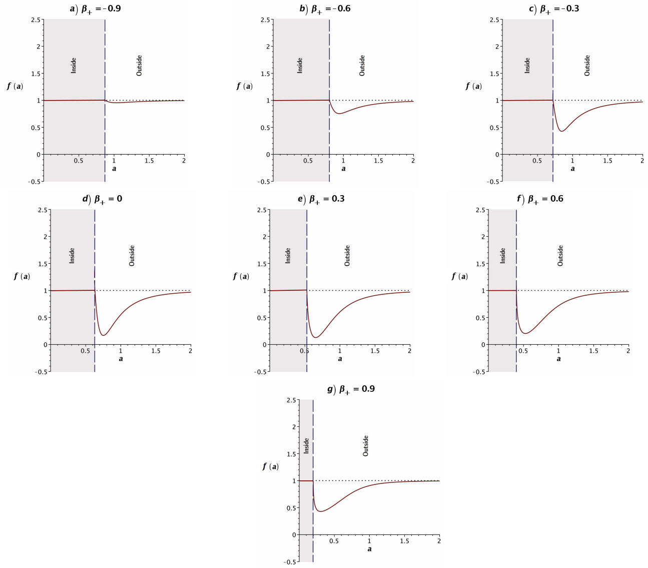

Figure: The graphs illustrate the metric function $f(a)$ ($f_+(a)$ of the exterior and $f_-(a) = 1$ of the interior spacetimes) versus the equilibrium radius a, for five different values of β_+ where $\beta_- = -1$. The results suggest that for the admissible domain of β_+ the particle model is feasible. The vertical line is the location of the particle's radius.

Fig. 1 is plotted for $f(a)$ against a for different values of β_+, where we have applied $\beta_- = -1$ and hence $-1 < \beta_+ < 1$. Inside the particle we have $f_-(a) = 1$ and outside the particle we have $f_+(a)$. Under these conditions the solution is real and non-black hole for all the values of β_+, post-particle's radius, and therefore, can be considered as a consistent particle model.

Here are some remarks. From Eq. (20) it is evident that the charge q is zero either when $\beta_- = \beta_+$ or $\beta_- = -\beta_+$. However, these two choices are banned since they also lead to a null mass (and a null equilibrium radius in the latter case). Consequently, at least for $n = 6$ and the particular solution that we considered here (Eq. (12)), the existence of the Maxwell Lagrangian in the action is essential. In other words, for $n = 6$ and the metric function in Eq. (12) as the exterior region, the charge must be non-zero. Let us also note that we narrowed down our degrees of freedom by choosing the special case $\tilde{\alpha}_2^2 = 3\tilde{\alpha}_3 = \beta^2$ and $n = 6$.

Particle model in pure LG

So far we have introduced a particle model in third order LG in $7-$dimensional spacetime. Such study can be done in any other specetime with $n+1 \geq 7$ too. In the sequel we introduce higher-dimensional particle model in the so called pure LG.

$n+1-$dimensional particle model in pure Einstein gravity

In this section we construct a $n+1-$dimensional chargeless particle model in pure Einstein R-gravity. The basic principles are the same as in the general approach presented in the previous sections. We assume a timelike spherical shell of radius $r=a$ such that its inside and outside spacetimes are $f_-(r)$ and $f_+(r)$, respectively. The standard Israel junction conditions imply that

$$\sigma := -\frac{5}{a}\left(\sqrt{f_+}-\sqrt{f_-}\right)_\Sigma = 0 \tag{23}$$

and

$$p := \frac{1}{2}\left(\left(\frac{f_+'}{\sqrt{f_+}}+\frac{8\sqrt{f_+}}{a}\right)-\left(\frac{f_-'}{\sqrt{f_-}}+\frac{8\sqrt{f_-}}{a}\right)\right)_\Sigma = 0. \tag{24}$$

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It is clear that imposing $\sigma = p = 0$ simultaneously results in the bulk's metric function and its first derivative remain continuous across the surface of the particle i.e., $(f_+ = f_-)_\Sigma$ and $\left(f'_+ = f'_-\right)_\Sigma$. In other words, continuity of the first and second fundamental forms across the thin-shell correspond to having the metric function and its first derivative continuous on the surface of the particle. Hence, we are very much restricted in choosing the spacetimes inside and outside. For instance, a Minkowski flat and a Schwarzschild spacetimes do not satisfy the two conditions. Furthermore, an inner Minkowski flat and an outer Reissner-Nordström can not be matched, whereas a global monopole spacetime of the form

$$f_{-}=1-\frac{2\eta}{\left(n-1\right)r^{n-3}} \qquad (25)$$

can be matched to the Reissner-Nordström

$$f_{+}=1-\frac{2m}{r^{n-2}}+\frac{q^{2}}{r^{2(n-2)}}. \qquad (26)$$

Herein, η is the global monopole parameter, m is a mass parameter (proportional to the physical mass) and q is the charge parameter (proportional to the physical charge). Applying $\left(f_{+}=f_{-}\right)_{\Sigma}$ and $\left(f_{+}^{\prime}=f_{-}^{\prime}\right)_{\Sigma}$ yield

$$\frac{m}{q^{2}}=\frac{n-1}{2a^{n-2}} \qquad (27)$$

and

$$\frac{\eta}{q^{2}}=\frac{\left(n-1\right)\left(n-2\right)}{2a^{n-1}}. \qquad (28)$$

Specifically, in 4-dimensional spacetime ($n = 3$) one finds $a = \frac{q^2}{m}$ and $\eta = \left(\frac{m}{q}\right)^2$. Furthermore, in 4−dimensional spacetime the Newton's potential inside the shell becomes a constant i.e., $f_- = 1 + 2\Phi_- = 1 - \eta$ which leads to $\Phi_- = -\frac{\eta}{2}$. In addition, the potential outside the particle is given by $f_+ = 1 + 2\Phi_+ = 1 - \frac{2m}{r} + \frac{q^2}{r^2}$ and explicitly $\Phi_+ = -\frac{m}{r} + \frac{q^2}{2r^2}$. On the surface trivially $\Phi_- = \Phi_+ = -\frac{\eta}{2}$ which is in agreement with our understanding of Newton's potential outside and inside a spherical shell in classical mechanics. We should admit, however, that although the constant Newton's potential seems a better choice inside the particle than a flat Minkowski, such a spacetime is singular with the energy-momentum

$$T_{\mu}^{\nu} = diag\left[\frac{\eta}{r^2}, \frac{\eta}{r^2}, 0, 0\right]. \quad (29)$$

In higher-dimensional particle model without singularity at the center one may think of a Anti/de Sitter spacetime inside the particle of the form

$$f_{-} = 1 - \frac{r^2}{\ell^2} \quad (30)$$

and outside the Reissner-Nordström given in (26). Matching the two metrics, we obtain

$$\frac{m}{q^2} = \frac{n-1}{na^{n-2}} \quad (31)$$

and

$$q^2 \ell^2 = \frac{n}{(n-2)} a^{2(n-1)}. \quad (32)$$

Again in the specific 4-dimensions we find $a = \frac{2}{3}\left(\frac{q^2}{m}\right)$ and $\ell^2 = \frac{16}{27}\frac{q^6}{m^2}$. By reversing these expressions we obtain the mass and charge in terms of the geometric parameters of the theory. We must add that, in all cases a should be grater than the radius of the event horizon of the outside metric and smaller than the possible horizons of inside metric. These are necessary to avoid any horizon or singularity inside or outside of the particle. For instance, again in 4-dimensions, in order to avoid any horizon within outer spacetime one has to assume $a > \frac{2}{3}q$.

$n+1-$dimensional pure GB gravity

In pure GB gravity one writes

$$I_{GB}=\frac{1}{2}\int dx^{n+1}\sqrt{-g}\left(\alpha_0+\alpha_2\mathcal{L}_{GB}\right),\ \ n\geq 4 \tag{33}$$

in which α_0 is the cosmological constant, α_2 is the GB free parameter and $\mathcal{L}_{GB}$ is the GB Lagrangian (6). In [9] a spherically symmetric solution with the line element (8) is considered such that the metric function is found to be

$$f(r)=1\pm\frac{r^2}{\sqrt{\tilde{\alpha}_2}}\sqrt{\frac{2M}{(n-1)\Sigma_{n-1}r^n}+\frac{1}{\ell^2}}. \tag{34}$$

Here M is the mass of the solution, $\Sigma_{n-1}=\frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$ is the surface area of $(d-2)$-sphere,

$$\frac{\alpha_0}{n(n-1)}=-\frac{1}{\ell^2} \tag{35}$$

is the cosmological constant and

$$\tilde{\alpha}_2=(n-2)(n-3)\alpha_2. \tag{36}$$

In particular for $n+1=7$-dimensions one finds

$$f(r)=1\pm\frac{r^2}{\sqrt{\tilde{\alpha}_2}}\sqrt{\frac{2M}{5\pi^3r^6}+\frac{1}{\ell^2}} \tag{37}$$

in which the parameters, upon choosing the $(+)$ sign are connected by $\alpha_0=-\frac{30}{\ell^2}$ and $\tilde{\alpha}_2=12\alpha_2$. Note that an interesting property of the 7-dimensional pure GB theory is that, its potential from (37), gives the same fall-off as in the 4-dimensional Einstein gravity.

Next, let's consider a particle of radius $r = a$ whose inner and outer spacetimes are the solutions in pure GB. The generalized Israel junction condition must be applied at the surface of the particle where the two incomplete spacetimes are glued. Based on the generalized Israel junction conditions without assuming $\frac{dt_+}{d\tau} = \frac{dt_-}{d\tau}$ one simply finds the induced metric on the shell to be given by (13) and the surface energy momentum tensor $S^a_b = \left(-\sigma, \underbrace{p, ..., p}_{n-times}\right)$ is expressed as

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$$-S_b^a = 2\left[\alpha_2\left(3J_b^a - J\delta_b^a\right)\right]_-^+ \tag{38}$$

in which J_b^a and J are defined in section III. An explicit calculation reveals

$$\sigma = -\frac{2(n-1)}{3a^3}\left[\tilde{\alpha}_2\sqrt{f}\,(3-f)\right]_-^+ \tag{39}$$

and

$$p = -\frac{2}{3a^2}\left[\tilde{\alpha}_2\left(\frac{3f'}{2\sqrt{f}}(f-1) + \frac{(n-4)}{a}(f-3)\sqrt{f}\right)\right]_-^+ . \tag{40}$$

Having considered specific metric functions, $f_\pm$ and introducing $\tilde{\alpha}_2^- = \tilde{\alpha}_{in}$ and $\tilde{\alpha}_2^+ = \tilde{\alpha}_{out}$ for inside and outside respectively, the conditions $\sigma = 0$ and $p = 0$ yield

$$\tilde{\alpha}_{out}\sqrt{f_+}\,(3 - f_+) - \tilde{\alpha}_{in}\sqrt{f_-}\,(3 - f_-) = 0 \tag{41}$$

and

$$\tilde{\alpha}_{out}\left(\frac{3f_+'}{2\sqrt{f_+}}(f_+ - 1) + \frac{(n-4)}{a}(f_+ - 3)\sqrt{f_+}\right) - \tag{42}$$
$$\tilde{\alpha}_{in}\left(\frac{3f_-'}{2\sqrt{f_-}}(f_- - 1) + \frac{(n-4)}{a}(f_- - 3)\sqrt{f_-}\right) = 0.$$

For $\tilde{\alpha}_{out} = \tilde{\alpha}_{in}$ (41) implies $(f_+ = f_-)_\Sigma$ and consequently (42) yields $(f'_+ = f'_-)_\Sigma$. For $\tilde{\alpha}_{out} \neq \tilde{\alpha}_{in}$ one finds from (41)

$$\tilde{\alpha}_{in} = \frac{\sqrt{f_+}\,(3 - f_+)}{\sqrt{f_-}\,(3 - f_-)}\tilde{\alpha}_{out} \tag{43}$$

and inserting this into (42) gives

$$f'_+ (f_+ - 1) f_-^2 + f_- \left(-2f_+^2 f'_- + 3f_+ \left(f'_- - f'_+\right) + 3f'_+\right) + f_+ f'_- (f_+ - 3) = 0 \tag{44}$$

which is the only constraint on the metric functions and their first derivatives to be satisfied on Σ. For an example let's consider $f_{-}=1$ and f_{+} (the positive branch) as is given in (34). Upon considering $\tilde{\alpha}_{out}\neq\tilde{\alpha}_{in}$, from (44) and (43) we obtain

$$M=\frac{2\left(n-1\right)\Sigma_{n-1}a^{n}}{\ell^{2}\left(n-4\right)} \tag{45}$$

and

$$\alpha_{in}=\frac{\alpha_{out}}{2}\left(2-\lambda\right)\sqrt{1+\lambda} \tag{46}$$

in which

$$\lambda=\frac{a^{2}}{\alpha_{out}}\sqrt{\frac{1}{\ell^{2}}\frac{n}{n-4}}. \tag{47}$$

To complete our arguments, we would like to add that, in the case where $\alpha_{in} = \alpha_{out}$ the junction conditions simply reduces to the one in R-gravity, i.e., the continuity of the bulk's metric function and its first derivative.

Particle model in pure third order LG

Similar to what we have studied in the previous section, in this section we shall give a particle model in pure third order LG. The corresponding action is given by

$$I_3 = \frac{1}{2}\int dx^{n+1}\sqrt{-g}\left(\alpha_0 + \alpha_3\mathcal{L}_3\right), \;\; n \geq 6 \tag{48}$$

in which α_0, α_3 and $\mathcal{L}_3$ are as described in the previous sections. A $(n+1)$-dimensional static spherically symmetric solution with electric charge is found after supplementing the Lagrangian with a Maxwell term whose line element is given by (34) with the metric function written as

$$f(r) = 1 + \frac{r^2}{|\tilde{\alpha}_3|^{1/3}} \sqrt[3]{\frac{2M}{(n-1)\Sigma_{n-1} r^n} - \frac{q^2}{r^{2(n-1)}} + \frac{1}{\ell^2}} \qquad (49)$$

in which the integration constants M and q are the usual mass and electric charge parameters, $\ell^2 > 0$ represents the cosmological constant and $\tilde{\alpha}_3 = -|\tilde{\alpha}_3| < 0$. This solution contains two theory parameters i.e., $\tilde{\alpha}_3$ and ℓ^2 and two solution parameters which are the integration constants i.e., M and q.

Next, we consider the spherically symmetric particle to consist of inner and outer spacetimes as of (34) where the inside the and outside metric functions are some solutions in pure third order LG such as given in (49). The associated junction conditions extracted from (16) are expressed as

$$-S_b^a = \left[3\alpha_3\left(5P_b^a - P\delta_b^a + \mathcal{L}_2\left(K_b^a + K\delta_b^a\right)\right)\right]_-^+$$

in which S_b^a, P_b^a, K_b^a and K are as before. Within explicit calculations we obtain

$$-S_\tau^\tau = \sigma = -\frac{(n-1)}{5a^5}\left[\tilde{\alpha}_3^+\sqrt{f_+}\left(15 - 10f_+ + 3f_+^2\right) - \tilde{\alpha}_3^-\sqrt{f_-}\left(15 - 10f_- + 3f_-^2\right)\right] \tag{50}$$

and

$$S_{\theta_i}^{\theta_i} = p = \frac{3}{2a^4}\left[\tilde{\alpha}_3^+\frac{f_+'}{\sqrt{f_+}}\left(1 - f_+\right)^2 - \tilde{\alpha}_3^-\frac{f_-'}{\sqrt{f_-}}\left(1 - f_-\right)^2\right] - \frac{n-6}{n-1}\sigma. \tag{51}$$

As of the boundary conditions on Σ, the surface of the particle, the first fundamental form h_{ab} should be continuous as well as $\sigma = 0$ and $p = 0$ simultaneously. Similar to the pure GB case, there are two distinct possibilities: i) $\tilde{\alpha}_3^+ = \tilde{\alpha}_3^-$ which implies $f_+ = f_-$ and $f'_+ = f'_-$ on Σ and ii) $\tilde{\alpha}_3^+ \neq \tilde{\alpha}_3^-$ which results in the following relation between $\tilde{\alpha}_3^+$ and $\tilde{\alpha}_3^-$

$$\tilde{\alpha}_3^- = \frac{\sqrt{f_+}\left(15 - 10f_+ + 3f_+^2\right)}{\sqrt{f_-}\left(15 - 10f_- + 3f_-^2\right)}\tilde{\alpha}_3^+ \tag{52}$$

and a constraint on the metric functions

$$3F_+'F_+^2F_-^3 + F_-^2\left[-3F_-'F_+^3 + F_+^2\left(F_-' - F_+'\right) - 4F_-'F_+ - 8F_-'\right]F_-^2 + \\ 4F_+'F_+^2\left(2 + F_-\right) = 0 \tag{53}$$

in which $F_\pm = f_\pm - 1$. Therefore, for $\tilde{\alpha}_3^+ \neq \tilde{\alpha}_3^-$ whatever solutions in pure LG are considered for the inner and outer spacetimes, this condition i.e., Eq. (53) has to be satisfied on the surface Σ. Moreover with $\tilde{\alpha}_{in} = \tilde{\alpha}_{out}$ the junction conditions rediucesto the one in R-gravity and also in pure GB gravity with $\tilde{\alpha}_{in} = \tilde{\alpha}_{out}$.

As an example, since we are interested to construct a singularity free particle model, let's set $f_- = 1$ and consequently $f'_- = 0$. In addition to that let's also consider $\tilde{\alpha}_3^+ = \tilde{\alpha}_{out}$ and $\tilde{\alpha}_3^- = \tilde{\alpha}_{in}$. The first condition, i.e., $\sigma = 0$ implies

$$\tilde{\alpha}_{in} = \frac{\tilde{\alpha}_{out}}{8} \sqrt{f_+} \left(15 - 10f_+ + 3f_+^2\right). \tag{54}$$

Considering the second condition i.e., $p = 0$ yields

$$f'_+ \left(1 - f_+\right)^2 = 0 \tag{55}$$

which admits two possibilities, either $f_+ = 1$ on the surface which consequently implies $\tilde{\alpha}_{in} = \tilde{\alpha}_{out}$ or $f'_+ = 0$ or $f'_+ = 0$. If we consider f'_+ to be the general charged solution given in (49) without the cosmological constant, $f'_+ = 0$ admits the relation

$$\frac{M}{q^2} = \frac{(n-1)(n-4)\Sigma_{n-1}}{(n-6)\,a^{n-2}} \tag{56}$$

between the mass and the charge of the particle in terms of its radius and

$$\tilde{\alpha}_{in} = \frac{\tilde{\alpha}_{out}}{8}\sqrt{1-\lambda}\left(8+4\lambda+3\lambda^2\right) \tag{57}$$

in which

$$\lambda = \frac{a^2}{|\tilde{\alpha}_{out}|^{1/3}}\left(\frac{(d-2)\,q^2}{(d-6)\,a^{2(d-2)}}\right)^{1/3}. \tag{58}$$

In 7-dimensional spacetime where $n = 6$, however, such a particle model fails to work and one needs to consider additional theory parameters such as a cosmological constant. In short, having the third order parameters involved in this specific case, provides us a particle model with mass and charge which could not be made in R-gravity. This shows the rich structure of the LG in any form in constructing particle models. Once more we add that to avoid nonphysical particle models, the spacetime of the particle including inside, on, and outside the shell have to be singularity / horizon free. These are the additional conditions to be imposed on the radius of any physical particle.

Conclusion

In a satisfactory geometric description of a particle the physical properties are expressible in terms of the parameters of the theory. For a charged spherical model, for instance in $3+1$ dimensions, the mass m and the charge q are constrained to satisfy the condition $a =$radius$= q^2/m$. In search for an analogous model in higher dimensions, we employ the third-order LG as a useful model. The reason for this choice relies on the existence of enough free parameters to define the particle properties. The existence of exact solutions, suitable to define thin, spherical shells, provides enough motivating factors towards construction of a particle model.

For $n+1$ dimensions ($n \geq 6$), we consider a spherical shell as representative of a particle whose inside is a flat vacuum to be connected through the junction conditions to a curved, asymptotically flat outside region. In the process, the emergent fluid's energy-momentum components on the shell are required to vanish. This yields the mass and the charge of the shell (or particle) in terms of the parameters of the theory, which are to be tuned-finely. The dimensionality naturally reflects in the radius of the particle, i.e. $a = (q^2/m)^{1/(n-2)}$. Finally, let us add that further tuning of the parameters (without the Maxwell Lagrangian) renders the construction of a chargeless massive particle model also possible in higher dimensions and the mass generated in this manner will not be of electromagnetic origin. *We have tested these cases as examples in the pure Einstein-Hilbert and GB Lagrangian. Conclusion to be drawn is that the higher-order LG provides a fruitful ground to construct particles of geometrodynamics.*