Theories with unfree gauge algebra: general properties, master equation, and quantization.

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The talk is based on the works:

D.Kaparulin, SL, Unfree gauge symmetry in the BV formalism, 2019;
D.Kaparulin, SL, A note on unfree gauge symmetry, 2019;

D.Francia, SL, A.Sharapov, On gauge symmetries of Maxwell-like higher-spin Lagrangians, 2014;
D.Kaparulin, SL., A.Sharapov, Consistent interactions and involution, 2013;
P.Kazinski, SL, A.Sharapov, Lagrange structure and quantization, 2005.
SL, A.Sharapov, Characteristic classess of gauge systems, 2004.

General Plan

1. **Introduction.**
   - Examples of the theories with unfree gauge symmetries: unimodular gravity, and the higher spin analogues;
   - Why naive BRST embedding and usual quantization schemes do not work for the unfree gauge algebra.

2. **Unfree gauge symmetry algebra.**
   - Completion functions, regularity conditions for the mass shell;
   - Noether identities and constraints on gauge parameters;
   - Structure relations of the unfree gauge algebra.

3. **BV master equation and quantization.**
   - Ghosts, anti-fields, gradings, master equation;
   - Koszul-Tate differential;
   - Gauge fixing, Faddeev-Popov integral for unfree gauge algebra;

4. **Concluding remarks.**

Example of unfree gauge symmetry: Unimodular gravity, T-diffs.

$$\det g_{\mu\nu} = -1, \qquad \delta_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu, \quad \nabla_\mu \epsilon^\mu = 0; \tag{1}$$

$$\delta_\omega g_{\mu\nu} = \nabla_\mu \epsilon^{(\omega)}_\nu + \nabla_\nu \epsilon^{(\omega)}_\mu, \qquad \epsilon^{(\omega)\mu} = \epsilon^{\mu\nu\lambda\rho} \partial_\nu \omega_{\lambda\rho}. \tag{2}$$

$$S = \int d^4 x R, \qquad \frac{\delta S}{\delta g^{\mu\nu}} \equiv R_{\mu\nu} - \frac{1}{4} g_{\mu\nu} R \approx 0; \tag{3}$$

$$\delta_\epsilon S = \int d^4 x (\nabla_\mu \epsilon^\mu) \cdot R, \quad \frac{\delta S}{\delta \epsilon^\mu} \equiv \nabla^\nu \frac{\delta S}{\delta g^{\mu\nu}} \equiv -\frac{1}{2} \partial_\mu R \not\equiv 0; \tag{4}$$

$$\frac{\delta S}{\delta g^{\mu\nu}} \approx 0 \quad \Rightarrow \quad \partial_\mu R \approx 0 \quad \Rightarrow \quad \tau \equiv R - \Lambda \approx 0, \quad \Lambda = const. \tag{5}$$

$$\tau \neq D^{\mu\nu} \frac{\delta S}{\delta g^{\mu\nu}}, \tag{6}$$

with $D^{\mu\nu}$ being differential operator. $\tau = 0$ is NOT a differential consequence of (3), while it imposes no restrictions on the mass shell. We term such quantities completion functions.

Examples of unfree gauge symmetry: massless symmetric tensors.

$$\frac{\delta S}{\delta h}=\partial\partial\cdot h-\Box h\approx 0,\ \ \partial\cdot\frac{\delta S}{\delta h}=\partial\tau,\ \ \tau\approx 0,\ \ \tau=\partial\cdot\partial\cdot h-\Lambda,\ \ \Lambda=const;$$

$$\delta_\epsilon h=\partial\epsilon,\quad \delta_\epsilon S=\int dx(\partial\cdot\partial\cdot h)(\partial\cdot\epsilon)\not\equiv 0,\quad \partial\cdot\epsilon=0. \tag{7}$$

Traceless tensor $h_{\mu\nu}$ corresponds to linearized unimodular gravity.
Traceless rank s symmetric tensor h describes irreducible spin s (E.Skvortsov, M.Vasiliev, 2008).
Tracefull rank s tensor describes a spectrum of spins: s, s-2, s-4, ... (A.Campoleoni, D.Francia, 2013).
No auxiliary fields are involved, unlike the Fronsdal Lagrangian.
Unconstrained parametrization of (7) (D.Francia, SL, A.Sharapov):

$$\delta_\omega h=\partial\epsilon^{(\omega)},\quad \epsilon^{(\omega)}=(\partial\cdot)^{s-1}\omega,\quad \delta_\omega S\equiv 0,\ \forall\omega,$$

where ω is a two-row tableau of the length $s-1$.

Generalities of unfree gauge symmetry noted in examples:

- Local functions τ exist such that $\tau \approx 0$, while τ's are not spanned by the Lagrange equations with local coefficients;
- The gauge parameters are unfree being constrained by PDEs;
- The identities exist between Lagrange equations and τ's;
- The gauge variation of action is the product of τ's and the constraints imposed onto the gauge parameters;
- The gauge symmetry admits reducible unconstrained parametrization of a higher order.

As we shall see, all these facts follow from the first one.

Implications of the unfree gauge algebra for dynamics.

Why unfree gauge algebra is inconsistent with naive Noether procedures, BV embedding, and usual quantization schemes:

- The generators of unfree gauge algebra do not necessarily form on-shell integrable distribution. The disclosure can involve the operators of gauge parameter constraints;
- If the usual BRST-BV formalism is applied to the systems with unfree gauge algebra, the on shell vanishing local function(al)s would not correspond to the BRST-exact quantities.
- The usual Noether procedure for inclusion of interactions can lead to the fictitious vertices.
  Example of formally admissible, though fictitious vertex noted by D.Francia, G.Monaco, K.Mktrchyan, JHEP 2017;
- Once the gauge parameters are unfree, the ghosts cannot be independent, they have to be constrained like the parameters.

Unfree gauge algebra. Completion functions and modified Noether identities.

Let M be a configuration space of fields ϕ^i, and Σ be a mass shell $\Sigma = \{\phi \in M \mid \partial_i S(\phi) = 0\}$. Let R be the ring of local functions on M, and $I \subset R$ be the ideal of the on shell vanishing functions

$$I = \{ T \in R \mid T \approx 0 \} . \tag{8}$$

The generating set for I includes completion functions $\boldsymbol{\tau}_a$:

$$T(\phi) \approx 0 \quad \Leftrightarrow \quad T = T^i \partial_i S + T^a \boldsymbol{\tau}_a \ , \quad \boldsymbol{\tau}_a(\phi) \neq D^i_a \partial_i S \ . \tag{9}$$

The completion functions are assumed independent, while identities are possible between $\boldsymbol{\tau}$'s and EoM's:

$$\Gamma^i_{\boldsymbol{\alpha}} \partial_i S + \Gamma^a_{\boldsymbol{\alpha}} \boldsymbol{\tau}_a \equiv 0 . \tag{10}$$

The identities define the unfree gauge symmetry of the action

$$\delta_\epsilon \phi^i = \Gamma^i_{\boldsymbol{\alpha}}(\phi) \epsilon^{\boldsymbol{\alpha}}, \quad \delta_\epsilon S(\phi) \equiv - \epsilon^{\boldsymbol{\alpha}} \Gamma^a_{\boldsymbol{\alpha}} \boldsymbol{\tau}_a = 0 \quad \Leftrightarrow \quad \epsilon^{\boldsymbol{\alpha}} \Gamma^a_{\boldsymbol{\alpha}} = 0 \ . \tag{11}$$

Unfree gauge algebra: gauge invariants.

The local function(al) $O(\phi)$ is gauge invariant if the unfree gauge variation vanishes on shell,

$$\delta_\epsilon O(\phi)=\epsilon^\alpha \Gamma^i_\alpha(\phi)\partial_i O(\phi)\approx 0, \quad \epsilon^\alpha \Gamma^a_\alpha=0 \,. \qquad (12)$$

The gauge invariants form algebra G. Given the completeness conditions, off shell $\delta_\epsilon O(\phi)\approx 0$ reads

$$\Gamma^i_\alpha \partial_i O(\phi)+V^i_\alpha(\phi)\partial_i S(\phi)+V^a_\alpha(\phi)\tau_a(\phi)+W_a(\phi)\Gamma^a_\alpha(\phi)\equiv 0. \qquad (13)$$

The ideal I is also gauge invariant,

$$T(\phi)\approx 0 \quad \Rightarrow \quad \delta_\epsilon T(\phi)\approx 0 \,. \qquad (14)$$

The algebra of physical observables is understood as G/I.
For the completion functions, (13) reads

$$\Gamma^i_\alpha(\phi)\partial_i \tau_a(\phi)=R^i_{\alpha a}(\phi)\partial_i S(\phi)+R^b_{\alpha a}(\phi)\tau_b(\phi)+W_{ab}(\phi)\Gamma^b_\alpha(\phi) \,,$$

where the last structure function is symmetric $W_{ab}\approx W_{ba}$.

Unfree gauge algebra. Structure relations.

Proceeding from the modified Noether identities $\Gamma^i_\alpha \partial_i S + \Gamma^a_\alpha \tau_a \equiv 0$ and accounting for the regularity and completeness conditions for the mass shell and identity generators, one can deduce the structure relations of the unfree gauge algebra:

$$\begin{aligned}
\Gamma^i_\alpha(\phi)\partial_i\Gamma^j_\beta(\phi) - \Gamma^i_\beta(\phi)\partial_i\Gamma^j_\alpha(\phi) &= U^\gamma_{\alpha\beta}(\phi)\Gamma^j_\gamma(\phi) + \\
E^{ij}_{\alpha\beta}(\phi)\partial_i S(\phi) + E^{aj}_{\alpha\beta}(\phi)\tau_a(\phi) &+ R^j_{\alpha a}(\phi)\Gamma^a_\beta(\phi) - R^j_{\beta a}(\phi)\Gamma^a_\alpha(\phi) \,.
\end{aligned}$$

$$\begin{aligned}
\Gamma^i_\alpha(\phi)\partial_i\Gamma^a_\beta(\phi) - \Gamma^i_\beta(\phi)\partial_i\Gamma^a_\alpha(\phi) &= U^\gamma_{\alpha\beta}(\phi)\Gamma^a_\gamma(\phi) + \\
R_{\alpha}{}^a_b(\phi)\Gamma^b_\beta(\phi) - R_{\beta}{}^a_b(\phi)\Gamma^b_\alpha(\phi) &+ E^{ab}_{\alpha\beta}(\phi)\tau_b(\phi) - E^{ai}_{\alpha\beta}(\phi)\partial_i S(\phi) \,,
\end{aligned}$$

where E's are antisymmetric, $E^{ij}_{\alpha\beta} = -E^{ji}_{\alpha\beta}, E^{ab}_{\alpha\beta} = -E^{ba}_{\alpha\beta}$.

Remark: unconstrained parametrization of gauge transformations.

$$\exists \Lambda^{\alpha}_{A}(\phi): \quad \epsilon^{\alpha}\Gamma^{a}_{\alpha}(\phi)\approx 0 \quad \Leftrightarrow \quad \epsilon^{\alpha}\approx\Lambda^{\alpha}_{A}(\phi)\omega^{A}, \tag{15}$$

The on-shell equality can be extended off shell,

$$\Lambda^{\alpha}_{A}\Gamma^{a}_{\alpha}=E^{ab}_{A}\tau_{b}+E^{ai}_{A}\partial_{i}S, \quad E^{ab}_{A}=-E^{ba}_{A}\ . \tag{16}$$

Introduce the new generators of gauge symmetry, being linear combinations of the original ones modulo on-shell vanishing terms:

$$G^{i}_{A}=\Lambda^{\alpha}_{A}\Gamma^{i}_{\alpha}+E^{ai}_{A}\tau_{a}\ , \quad \delta_{\omega}S=\omega^{A}G^{i}_{A}\partial_{i}S\equiv 0, \ \forall\omega^{A}. \tag{17}$$

The gauge transformations $\delta_{\omega}\phi^{i}=G^{i}_{A}\omega^{A}$ can be reducible, the symmetry of symmetry can occur for ω^{A}.
In the linear PDE systems, the sequence of symmetry for symmetry is always finite due to Hilbert's syzygy theorem (D.Francia, SL, A.Sharapov, 2014)

BV formalism for unfree gauge algebra: introduction of anti-fields.

Main distinctions of the theory with unfree gauge algebra:

- The generating set for the ideal I of on-shell vanishing local function(al)s includes $\partial_i S$ *and* completion functions τ_a;
- The gauge parameters are constrained by the equations, hence the ghosts C^α are to obey the same equations $\Gamma^a_\alpha C^\alpha = 0$;
- The constraints on gauge parameters are paired with completion functions. The gauge identities $\Gamma^i_\alpha \partial_i S + \Gamma^a_\alpha \tau_a \equiv 0$ are paired with unfree gauge symmetries $\delta_\epsilon \phi^i = \Gamma^i_\alpha \epsilon^\alpha$.

The BRST embedding procedure of the general not necessarily Lagrangian PDE systems (P.Kazinski, SL, A.Sharapov, 2005) implies to assign anti-field to every EoM and to every gauge identity between the equations, while the ghosts are assigned to gauge symmetries.

BV formalism for unfree gauge algebra: introduction of anti-fields.

The anti-field content for unfree gauge symmetry:

- EoM's $\partial_i S$ and completion functions τ_a are considered on equal footing. The anti-fields ϕ^*_i, ξ^*_a are assigned to all;
- The constraints imposed on the ghosts $\Gamma^a_\alpha C^\alpha$ are considered as equations and the anti-fields ξ^a are assigned to;
- The anti-fields C^*_α are assigned to the gauge identities ;

Notation and grading of the fields and anti-fields

grading/variable	ϕ^i	ξ^a	C^α	ϕ^*_i	ξ^*_a	C^*_α
Grassmann parity ε	0	0	1	1	1	0
ghost number gh	0	0	1	-1	-1	-2
resolution degree deg	0	**1**	0	1	1	2

Ghost number assignment principle:
gh(*antifield*)=−gh(*equation*)−1;
gh(*identity*−*antifield*)=−gh(*antifield*)−1.

Ghost-dependent trivial functions. K-T differential.

Consider algebra $\mathcal{A}$ of local functions of $\varphi=(\phi,\xi,C)$, $\varphi^*=(\phi^*,\xi^*,C^*)$. Zero resolution degree subalgebra $\mathcal{A}_0$ is formed by the functions (ϕ,C).

The function $\bar{T}\in\mathcal{A}_0$ is considered trivial if it vanishes on shell *and* on the ghost constraints. The trivial quantities form an ideal $\bar{I}\subset\mathcal{A}_0$.

Koszul-Tate differential:

$$\delta A=-\frac{\partial^R A}{\partial\phi^*_i}\partial_i S-\frac{\partial^R A}{\partial\xi^*_a}\tau_a+\frac{\partial^R A}{\partial C^*_\alpha}(\phi^*_i\Gamma^i_\alpha+\xi^*_a\Gamma^a_\alpha)+\frac{\partial^R A}{\partial\xi^a}\Gamma^a_\alpha C^\alpha\ .$$

$$\delta^2 A\equiv-\frac{\partial^R A}{\partial C^*_\alpha}(\Gamma^j_\alpha\partial_j S+\Gamma^a_\alpha\tau_a)\equiv 0\ ,\quad \deg\delta=-1,\quad \mathrm{gh}\,\delta=1\ .$$

The differential is acyclic in strictly positive resolution degrees. The ideal $\bar{I}$ is δ-exact, so δ is a resolution for $\bar{I}$.

Antibracket

$$(A,B)=\frac{\partial^R A}{\partial\varphi^I}\frac{\partial^L B}{\partial\varphi^*_I}-\frac{\partial^R A}{\partial\varphi^*_I}\frac{\partial^L B}{\partial\varphi^I}\,,\quad \varphi^I=(\phi^i,\xi^a,C^\alpha)\,,\quad \varphi^*_I=(\phi^*_i,\xi^*_a,C^*_\alpha)\,.$$

The antibracket is odd, and it shifts the ghost number by one:

$$\mathrm{gh}((A,B))=\mathrm{gh}(A)+\mathrm{gh}(B)+1,\qquad \varepsilon((A,B))=\varepsilon(A)+\varepsilon(B)+1. \quad (18)$$

Master action is defined as an expansion w.r.t. deg-grading

$$S(\varphi,\varphi^*)=\sum_{k=0}S_k,\quad \mathrm{gh}(S_k)=\varepsilon(S_k)=0,\quad \deg(S_k)=k,\quad S_0=S(\phi).$$

The most general first and second resolution degree terms read

$$S_1 = \tau_a\xi^a+(\phi^*_i\Gamma^i_\alpha+\xi^*_a\Gamma^a_\alpha)C^\alpha\ ; \quad (19)$$

$$\begin{aligned} S_2 &= \frac{1}{2}(C^*_\gamma U^\gamma_{\alpha\beta}+\phi^*_j\phi^*_i E^{ij}_{\alpha\beta}+2\xi^*_a\phi^*_i E^{ia}_{\alpha\beta}+\xi^*_b\xi^*_a E^{ab}_{\alpha\beta})C^\alpha C^\beta \\ &- \xi^b(\phi^*_i R^i_{b\alpha}+\xi^*_a R^a_{b\alpha})C^\alpha-\frac{1}{2}\xi^b\xi^a W_{ab}\ . \end{aligned} \quad (20)$$

Master equation

$$(S,S)=0\ ,\qquad Q=(\cdot,S),\quad Q^2=0. \tag{21}$$

Substituting expansion of S into (21) and expanding in deg

$$(S,S)_0=2(\Gamma^a_\alpha\partial_i S+\Gamma^a_\alpha\tau_a)C^\alpha=0,$$

$$(S,S)_1=2\xi^a(\Gamma^i_\alpha\partial_i\tau_a-R^i_{\alpha a}\partial_i S-R^b_{\alpha a}\tau_b-W_{ab}\Gamma^b_\alpha)C^\alpha-$$
$$C^\alpha C^\beta(\phi^*_i(\Gamma^j_\alpha\partial_j\Gamma^i_\beta-\Gamma^j_\beta\partial_j\Gamma^i_\alpha-U^\gamma_{\alpha\beta}\Gamma^i_\gamma-R^i_{\alpha a}\Gamma^a_\beta+R^i_{\beta a}\Gamma^a_\alpha-E^{ji}_{\alpha\beta}\partial_j S-E^{ia}_{\alpha\beta}\tau_a)$$
$$-\xi^*_a(\Gamma^j_\alpha\partial_j\Gamma^a_\beta-\Gamma^j_\beta\partial_j\Gamma^a_\alpha-U^\gamma_{\alpha\beta}\Gamma^a_\gamma-R^a_{\alpha b}\Gamma^b_\beta+R^a_{\beta b}\Gamma^b_\alpha+E^{ja}_{\alpha\beta}\partial_j S-E^{ab}_{\alpha\beta}\tau_b))=0,$$

we reproduce the basic structure relations of unfree gauge algebra. The BRST differential has the usual decomposition in deg

$$Q=\delta+\gamma+\overset{(1)}{s}+\ldots\ ,\quad \deg\gamma=0,\ \ \deg\overset{(1)}{s}=1,\quad \ldots\ , \tag{22}$$

that provides the well defined homological perturbation theory, given the acyclicity of δ.

Unfree gauge algebra: gauge fixing, Faddeev-Popov action.

Gauge fixing is the choice of the gauge Fermion which defines the Lagrange surface in the odd cotangent bundle:

$$\varphi^*_I = \frac{\partial \Psi}{\partial \varphi^I}, \qquad \mathrm{gh}(\Psi) = -1, \quad \varepsilon(\Psi) = 1 \, . \tag{23}$$

The simplest option for choosing Ψ:

$$\Psi = \bar{C}_A \chi^A(\phi) + \bar{C}_a \xi^a . \tag{24}$$

The non-minimal sector ghosts and action read

$$\mathrm{gh}\pi_A = \mathrm{gh}\pi_a = 0 \quad \mathrm{gh}\,\bar{C}_A = \mathrm{gh}\,\bar{C}_a = -1 \, , \quad \mathrm{gh}\,\bar{C}^{*A} = \mathrm{gh}\,\bar{C}^{*a} = 0 \, . \tag{25}$$

$$S_{non-min} = S + \bar{C}^{*A} \pi_A + \bar{C}^{*a} \pi_a \, . \tag{26}$$

The gauge fixed action reads

$$S_\chi(\phi^i, C^\alpha, \bar{C}_A, \pi^A) = S(\phi) + \pi_A \chi^A(\phi) + \bar{C}_A \Gamma^i_\alpha \frac{\partial \chi^A}{\partial \phi^i} C^\alpha + \bar{C}_a \Gamma^a_\alpha C^\alpha + \ldots .$$

Remark: re-interpretation of ξ^a as compensator/Stückelberg fields for unfree gauge symmetry.

Introduce collective notation $u^I{=}(\phi^i,\xi^a)$, $u^*_I{=}(\phi^*_i,\xi^*_a)$, assign $\deg' u{=}0$. The same master action can be expanded w.r.t. $\deg'$:

$$S{=}S'(u){+}C^\alpha R^I_\alpha(u)u^*_I{+}\dots\ ,$$

$$S'(\phi,\xi){=}S(\phi){+}\tau_a(\phi)\xi^a{+}\frac{1}{2}W_{ab}(\phi)\xi^a\xi^b{+}\dots\ ;$$

$$R^i_\alpha(\phi,\xi){=}\Gamma^i_\alpha(\phi){+}R^i_{\alpha a}(\phi)\xi^a{+}\dots,\quad R^a_\alpha(\phi,\xi){=}\Gamma^a_\alpha(\phi){+}R^a_{\alpha b}\xi^b{+}\dots\ .$$

By virtue of the master equation, the action $S'(u)$ is invariant under the unconstrained gauge transformations of the fields u,

$$\delta_\epsilon u^I{=}R^I_\alpha\epsilon^\alpha,\quad \delta_\epsilon S{\equiv}0,\ \forall\epsilon.$$

Hence, the antifields ξ can be re-interpreted as compensator fields. Further re-interpretation: if τ_a are replaced by lower order differential consequences, then ξ^a would be Stückelberg fields.

Concluding remarks

Conclusions.

- The unfree gauge algebra is generated by four constituents: the action, the gauge generators, the completion functions, the gauge parameter constraints;
- The BV embedding involves extra anti-fields paired with completion functions and constraints on gauge parameters;
- Once the proper solution exists for the master equation, the unfree gauge theory can be consistently deformed by interactions and quantized.

Open problems.

- Unfree gauge symmetry in terms of Hamiltonian constrained systems and corresponding Hamiltonian BFV-BRST complex;
- Classical and quantum dynamics of modular parameters;
- Unification of unfree gauge symmetry with gauge algebra of non-involutive Lagrangian systems.

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WITH UNFREE
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THANK YOU!