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QUANTUM DEFORMATIONS OF TWISTORS

1. **Introduction: twistors versus space-time geometry and their quantization**
2. **Twist-deformed quantum-mechanical twistors and inhomogeneous conformal symmetries: towards Palatial twistors**
3. **Quantum deformations of SU(2,2), conformal symmetry breaking and NC twistors**
4. **Quantum deformations of complex Minkowski spaces in six-dimensional approach and Grassmanian framework**
5. **Final remarks**

– Research done together with Mariusz Woronowicz

Some references linked with quantum twistors:

1.) R. Penrose "Twistor quantization and curved space-time, Int. J. Theor. Phys. 1 (1968), 61

2.) A. Kapustin, A. Kuznetzov, D. Orlov, "Noncommutative instantons and twistor transforms", hep-th/0002193; Comm. Math. Phys. 221 (2001), 385

3.) K.C. Hannabus, "Noncommutative twistor space", hep-th/0108228; Lett. Math. Phys. 58 (2001), 153166

4.) S. Brain, S. Majid, " Quantization of twistor theory by cocycle twist", math/0701893 [math.QA], Comm. Math. Phys. 284 (2008), 713

5.) R. Penrose, "Palatial twistor theory and the twistor google problem", Phil. Trans. Roy. Soc. Lond. A373 (2015) 20140237

6.) J. Lukierski, M. Woronowicz, "Noncommutative space-time from quantized twistors", arXiv:1311.7498, Proc. of the Conf. in Honour of 90^{th} Birthday of Freeman Dyson (26-29.08.2013, Singapore), ed. K.K. Phua et all. World Scientific (2014) p.407

1. INTRODUCTION

Twistor theory (Penrose 1967 ...) - the proposal of more fundamental (elementary) geometric level under the space-time geometry. For flat space-time

Twistor space coordinates $t_A \in C^4 = T^4$ (or CP(3)) described by conformal (projective) D=4 spinors parametrize light-like lines in space-time

$\longrightarrow$

Space-time points $x^\mu \in M^{3,1}$(real Minkowski space) or $Z^\mu \in CM(4)$ (complex Minkowski space) parametrize twistor 2-planes

$\downarrow$

Twistor 2-plane (α-plane) given by a pair of twistors
$t^i_A = (\pi^i_\alpha, \omega^{\dot\alpha i}) \quad i = 1, 2$

$\longleftrightarrow$

Cartan-Penrose incidence relation
$\omega^{\dot\alpha i} = i z^{\dot\alpha\beta}\pi^i_\beta$ or
$\Omega = iZ\Pi \leftarrow 2 \times 2$ matrix eq.
with solution

$\downarrow$

Bilocal (nonlocal) functions in twistor space provide the Minkowski space-time points:

$$z^{\dot\alpha\beta} = -\frac{i}{\pi^{1\alpha}\pi^2_\alpha}(\omega^{1\dot\alpha}\pi^{2\beta} - \omega^{2\dot\alpha}\pi^{1\beta})$$

$\longrightarrow$ nonlocal relation between twistor and space-time geometries

$\longrightarrow$ fundamental geometry is spinorial-conformal (mass parameters are not present ab initio)

In Penrose program twistors (besides describing conformal - invariant fields) should in its nonflat version encode the information about the curved space-time structure, provides new geometric description of gravity (GR)

However: Penrose program has been only partially fulfilled: we are not able to encode in twistor language all structures of Einstein space-time, in particular with arbitrary Weyl curvature occurring in GR

Recent hope: New conjecture that NC twistors after generalizing QM quantization should be helpful in deriving full twistorial description of GR (!!!???)

Geometrization of incidence relations linking twistors and complex Minkowski space-time can be derived from the coset decomposition of conformal group $SU(2,2)$:

$$CP(3) = \frac{SU(2,2)}{U(2,1)} \qquad \longrightarrow \qquad G_{SU(2,2)}(4;2) = \frac{SU(2,2)}{S(U(2)\otimes U(2))} \ni CM(4)$$

single projective twistors — composite Minkowski coordinates

CM(4) parametrizes (modulo scaling) complexified D=6 light cone:

- $O(4,2) = \overline{SU(2,2)} \longrightarrow O(4,2)$ 6-vectors $\equiv SU(2,2)$ bispinors $Z_{[AB]}$
- O(4,2) metric is $\epsilon^{ABCD} \longrightarrow Z_{AB}\epsilon^{ABCD} Z_{CD} = 0 \Rightarrow Z_{[AB]}$ simple

$$Z^{[AB]}Z_{[AB]} = 0 \quad \text{A,B=1,2,3,4} \qquad \longleftrightarrow \qquad Z_{[AB]} \sim t^1_{[A} \cdot t^2_{B]} \in CP(5)$$

(six-dimensional light cone) — (composite space-time)

Two levels of noncommutativity of twistors - in analogy with two levels of noncommutativity of standard phase space variables (x^μ, p^ν) in standard framework:

classical phase space $\xrightarrow{\text{QM}}$ quantum-mechanical phase-space $(x_\mu^{(QM)}, p_\mu^{(QM)})$

$\xrightarrow{\text{QM}+\text{QG}}$ quantum-deformed phase space with QG effects $(\hat{x}_\mu, \hat{p}_\mu)$

Only at the second QM+QG stage we get $\hat{x}_\mu = x_\mu^{QM} + x_\mu^{QG}$

$$[x_\mu, x_\nu] = 0 \qquad \overset{\text{QG}}{\Longrightarrow} \qquad [\hat{x}_\mu, \hat{x}_\nu] \neq 0 \qquad \text{(NC space-time)}$$

Symmetries of NC space-time: quantum-deformed Poincaré-Hopf algebra H and quantum-deformed dual Poincaré-Hopf quantum group $\tilde{H}$:

$H \longrightarrow$ generalization of momenta sector : $(P_\mu, M_{\mu\nu})$

$\tilde{H} \longrightarrow$ generalization of coordinate sector: $(X^\mu, \Lambda^{\mu\nu})$

Hopf-algebraic action of H on $\tilde{H}$:

$$h \triangleright (a \cdot b) = h_{(1)} a \cdot h_{(2)} b \qquad \begin{pmatrix} h \in H \\ a, b \in \tilde{H} \end{pmatrix}$$

$\mathcal{H} = H \triangleright \tilde{H}$ – Heisenberg double $\Rightarrow$ generalized phase space $(X^\mu, P_\nu; M_{\mu\nu}, \Lambda^{\mu\nu})$ - Hopf algebroid

Two levels of noncommutativity in twistor theory: $\hat{t}_A = t_A^{QM} + t_A^{QG}$

① Quantum - mechanical twistors:

$$\left[t_A^{QM}, t_B^{QM}\right] = \left[\bar{t}_A^{QM}, \bar{t}_B^{QM}\right] = 0$$

$$\left[t_A^{QM}, \bar{t}_B^{QM}\right] = \hbar\eta_{AB} \leftarrow \text{SU(2,2) metric}$$

$\xrightarrow{\text{QG}}$

② Quantum - deformed twistors with QG effects

$$\left[\hat{t}_A, \hat{t}_B\right] \neq 0 \quad \left[\hat{\bar{t}}_A, \hat{\bar{t}}_B\right] \neq 0$$

$$\left[\hat{t}_A, \hat{\bar{t}}_B\right] \neq 0$$

At level ② we employ in analogous way for space-time and for twistors the quantum Hopf-algebraic symmetries:

Quantum deformations of Poincaré-Hopf algebra H (quantum - deformed vectorial space-time) $\rightsquigarrow$ Quantum deformations of inhomogeneous conformal Hopf algebra (quantum - deformed spinorial twistors)

$$(M_{\mu\nu} \ltimes P_\mu) \rightsquigarrow (M_{\mu\nu}, D, K_\mu, P_\mu) \ltimes \bar{T}^4 = su(2,2) \ltimes \bar{T}^4$$

(Also possible alternative choice, obtained by partial duality map

$(M_{\mu\nu} \ltimes X_\mu)$ (flat limit of Snyder model) $\rightsquigarrow$ $(M_{\mu\nu}, D, K_\mu, P_\mu) \ltimes T^4$)

Comment: In twistor theory already on first QM level the composite space-time is becoming noncommutative. One gets the extension of Heisenberg algebra by necessary addition of (composite) Pauli-Lubanski four-vector coordinates W_μ

fourlinear in $t, \bar{t}$ $\rightarrow$ $W_\mu = s^r e_\mu^{(r)}$ $\qquad (\tau^r)_i{}^j s^r + \delta_i{}^j s^o = \bar{t}_i^A t_A^j \qquad i,j = 1,2 \quad r = 1,2,3$

where s^r, s^o are Lorentz-invariant spin projections ($s^r = \frac{1}{2} tr(\bar{t}\tau^r t)$) and

$e_\mu^{(r)} = \frac{1}{2}(\sigma_\mu)^{\dot{\alpha}\beta} \bar{\pi}_{\dot{\alpha}}^i (\tau^r)_i{}^j \pi_{\alpha j}$ "soldering" of space-time and internal symmetries

↑ internal su(2) Pauli matrices

We get

$$W_\mu W^\mu = p^2(S_1^2 + S_2^2 + S_3^2) \xrightarrow{\text{O(3) quantization}} \sum_{r=1}^{3} S_r^2 = s(s+1) \quad s = 0, \frac{1}{2}, 1, \ldots$$

Extended Heisenberg algebra of Poisson brackets, which can be quantized

$$\{x_\mu, p_\nu\} = \eta_{\mu\nu} \qquad \{x_\mu, x_\nu\} = -\frac{1}{(p^2)^2} \epsilon_{\mu\nu\rho\sigma} W^\rho p^\sigma$$

$$\{W_\mu, x_\nu\} = -\frac{1}{p^2} W_{[\mu} p_{\nu]} \qquad \{W_\mu, W_\nu\} = 0$$

$$\{W_\mu, W_\nu\} = \epsilon_{\mu\nu\rho\tau} W^\rho p^\tau \qquad \{p_\mu, p_\nu\} = 0$$

Passing from space-time to twistor geometry we make basic replacement

$$\text{D=4 Lorentz } o(3,1) \rightsquigarrow \text{D= 4 conformal su(2,2)}$$

$$\text{D=4 Poincaré } i\,o(3,1) \rightsquigarrow i\,su(2,2), \bar{\imath}\,su(2,2)$$

One can look for twistorial counterparts of various space-time quantum deformations / NC space-time models

- $\theta_{\mu\nu}$ – **deformed Poincaré-Hopf algebra** $\rightsquigarrow$ θ_{AB} **-deformed inhomogeneous** $su(2,2)$ **Hopf algebra (Palatial twistors)**
- **Lorentzian Snyder model** $\rightsquigarrow$ **conformal - spinorial Snyder model**
- κ – **deformed NC space time** $\rightsquigarrow$ κ **-deformed NC twistors**

Two types of noncommutativity of twistors $Z_a = (t_A, \bar{t}_A)$**:**

$$[Z_a^{QM}, Z_b^{QM}] \sim \hbar \qquad [Z_a^{QG}, Z_b^{QG}] \sim (\lambda_p)^k \quad \text{k= 1,2 \ldots} \qquad \lambda_p = \frac{\hbar}{m_p\, c}$$

2. TWIST-DEFORMED QM TWISTORS AND INHOMOGENEOUS CONFORMAL SYMMETRIES: TOWARDS PALATIAL TWISTORS

Penrose (2015) proposed concrete choice of holomorphic complex symplectic structure as describing noncommutativity of T^4 (θ_{AB} complex, constant)

$$\omega_2 = \theta_{AB} dt^A \wedge dt^B \quad t^A = (\pi_\alpha, \omega^{\dot\alpha}), \quad t_A = (\omega^\alpha, \pi_{\dot\alpha}) = \eta_{AB} \bar{t}^B \quad \bar\theta_{AB} = \theta^{AB}$$

and have chosen θ_{AB} as the infinity twistor for D=4 AdS space-time written down in Lorentz-covariant basis ($[\Lambda] = L^{-2}$- cosmological constant, $[\pi_\alpha] = L^{-1/2}$, $[\omega^{\dot\alpha}] = L^{1/2}$)

$$\theta_{AB} = \lambda_p \begin{pmatrix} \frac{\Lambda}{6}\epsilon_{\alpha\beta} & 0 \\ 0 & \epsilon^{\dot\alpha\dot\beta} \end{pmatrix} \qquad \theta^{AB} = \lambda_p \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \frac{\Lambda}{6}\epsilon_{\dot\alpha\dot\beta} \end{pmatrix} \qquad \theta_{AB}\theta^{BC} = -\frac{\Lambda\,\lambda_p^2}{6}\delta_A^C$$

Such symplectic structure related with the quantization of QM twistors by means of canonical conformal twist $F = F_{(1)} \otimes F_{(2)} \in U(\bar\imath\, su(2,2)) \otimes U(\bar\imath\, su(2,2))$ (Brain, Majid 2007). Analogy:

space-time: Poincaré algebra with 2-cocycle twist $F \in U(\mathcal{P}^{3;1}) \otimes U(\mathcal{P}^{3;1})$	$\Longleftrightarrow$	twistor space: inhomogeneous conformal algebra with 2-cocycle twist $F \in U(\bar\imath\, su(2,2)) \otimes U(\bar\imath\, su(2,2))$

Two possible choices of inhomogeneous $su(2,2)$:

α) $\imath\, su(2,2) \equiv su(2,2) \rtimes T^4$ – $t^A \in T^4$ as "twistor coordinates" ($\bar{T}^4$ as module)

β) $\bar{\imath}\, su(2,2) \equiv su(2,2) \rtimes \bar{T}^4$ – $t_A \in \bar{T}^4$ as "twistor momenta" (T^4 as module)

One uses two following twists F and $\bar{F}$ related by the map $t^A \to t_A = \eta_{AB}\bar{t}^B$ (twistor analogue of Born map $\hat{x}^\mu \to \hat{p}_\mu, \lambda \to \lambda^{-1}; [\lambda] = L$)

α) $F = \exp i\lambda_p \theta_{AB} t^A \wedge t^B$ $\quad (\leftarrow \exp i\frac{\theta_{\mu\nu}}{\lambda^2} x^\mu \wedge x^\nu)$

β) $\bar{F} = \exp i\lambda_p \theta^{AB} t_A \wedge t_B$ $\quad (\leftarrow \exp i\,\lambda^2 \theta^{\mu\nu} p_\mu \wedge p_\nu \;\Leftarrow$ standard Moyal twist)

where $t^A \to t_A$ means $\pi_\alpha \to \omega^\alpha = (\omega^{\dot{\alpha}})^\star$ and $\omega^{\dot{\alpha}} \to \pi_{\dot{\alpha}} = (\pi_\alpha)^\star$.

Twist quantizations by F and $\bar{F}$ provides two different twist deformations, each applied to two different inhomogeneous conformal Hopf algebras α) and β) with algebra unchanged, coalgebra modified by similarity transformation

$$\Delta_{(F)}(\hat{g}) = F^{-1} \circ \Delta_{(0)}(\hat{g}) \circ F \qquad \hat{g} \in SU(2,2)$$

(for $\Delta_{(\bar{F})}$ analogously) and twisted antipodes changed also by similarity map.

Further one considers deformation of t^A ($\bar{t}_A$ in case β)) as twist-deformed $\imath\, su(2,2)$-module ($\bar{\imath}\, su(2,2)$ - module in case β), which is given according to the formula

$$\hat{t}^A_{(F)} = (F^{-1}_{(1)} \triangleright t^A) F^{-1}_{(2)} \qquad\qquad (F^{-1} = F^{-1}_{(1)} \otimes F^{-1}_{(2)})$$

in case α) and analogously for $\bar{F}$ in case β).

One can check that the classical commutators $[t^A, \hat{g}]$ are also F-deformed, but $[t_A, \hat{g}]$ remains classical (for $\bar{F}$-twisting - opposite alternative)

Explicite formulae for Palatial choice of twists F, $\bar{F}$:

a) Twisted coproducts of su(2,2) generators $\hat{g}^{\ A}_{B} = t^A t_B - \frac{1}{4}(t,t)\delta^A_B$: $\quad ((t,t) = t^A t_A)$

$$\Delta_{(F)}(\hat{g}^{\ A}_{B}) = F \circ \Delta(\hat{g}^{\ A}_{B}) \circ F^{-1} = \Delta_0(\hat{g}^{\ A}_{B}) + \frac{1}{4}\theta_{CD}[(\eta^C_B t^A \otimes t^D + \eta^C_B t^D \otimes t^A) - (C \leftrightarrow D)]$$

$$\Delta_{(\bar{F})}(\hat{g}^{\ B}_{A}) = \bar{F} \circ \Delta_0(\hat{g}^{\ B}_{A}) \circ \bar{F}^{-1} = \Delta_0(\hat{g}^{\ B}_{A}) + \frac{1}{4}\theta^{CD}[(\eta^B_C t_A \otimes t_D + \eta^B_D t_C \otimes t_A) - (C \leftrightarrow D)]$$

Twisted antipodes are equal to the classical ones.

Further remains to consider for F-twist (case α)) and $\bar{F}$-twist (case β)) the twist deformation of module algebras.

b) Twisted quantum-mechanical twistors obtained as modules of twisted Hopf algebras $U_\theta(\bar{i}\, su(2,2))$ and $U_{\bar\theta}(i\, su(2,2))$:

$$i)\quad U_\theta(\bar{i}\, su(2,2)): \qquad \hat{t}^A_{(F)} = t^A - \theta^{AB} t_B \qquad \hat{\bar{t}}_{(F)\,A} = t_A$$

$$ii)\quad U_{\bar\theta}(i\, su(2,2)): \qquad \hat{t}^A_{(F)} = t^A \qquad \hat{\bar{t}}_{(\bar{F})\,A} = \bar{t}_A - \theta_{AB} t^B$$

We obtain pair of algebras which are Born-dual but not Hermitean-dual

c) Quantum conformal covariance of deformed twistors

From Hopf-algebraic action formula

$$\hat{g} \triangleright \hat{a} \cdot \hat{b} = (\hat{g}_{(1)} \triangleright \hat{a})(\hat{g}_{(2)} \triangleright \hat{b}) \qquad \hat{g} \in U_\theta(\bar{i}\, su(2,2)); \quad \hat{a}, \hat{b} \in T^4$$

by using deformed coproducts of $su(2,2)$ generators one gets

$$\hat{g} \triangleright \{[\hat{t}^A_{(F)}, \hat{t}^B_{(F)}] - \theta^{AB}\} = 0 \qquad \text{for } \hat{g} \in U_\theta(\bar{i}\, su(2,2))$$

as well as

$$\hat{g} \triangleright \{[t_{(\hat{F})A}, t_{(\bar{F})B}] - \theta_{AB}\} = 0 \qquad \text{for } \hat{g} \in U_{\bar\theta}(i\, su(2,2))$$

The twist modification of classical coproducts of conformal algebra is exactly the one which is needed for quantum conformal covariance

d) Quantum D=4 Heisenberg-conformal algebra (Hopf algebroid)

We introduce twistorial canonical phase space $H^{4;4} = T^4 \underset{\hbar}{\ltimes} \bar{T}^4$ as inhomogeneous algebraic sector added to su(2,2):

$$\mathcal{H} \equiv su(2,2) \underset{\theta,\bar{\theta}}{\ltimes} H^{4;4} \equiv su(2,2) \underset{\theta,\bar{\theta}}{\ltimes} (T^4 \underset{\hbar}{\ltimes} \bar{T}^4)$$

where $\hbar$ indicates that $H^{4;4}$ is twistorial QM phase space

Properties:

1) One can not introduce Hopf-algebraic twist which will deform simultaneously T^4 and $\bar{T}^4$ in $su(2,2) \ltimes H^{4,4}$ in "Palatial way" - due to QM CCR ($\hbar \neq 0$) in $H^{4;4}$ the algebra $\mathcal{H}$ is not a Hopf algebra, but Hopf algebroid – $F, \bar{F}$ become algebroid twists.

2) In order to get the Hermitean-dual Palatial algebraic relations in $\mathcal{H}$ with θ^{AB}–deformed T^4 and θ_{AB}-deformed $\bar{T}^4$ one can use the relations obtained from F and $\bar{F}$ twisting procedure in the form of so-called Bopp shifts

$$\hat{t}^A = t^A - \theta^{AB} t_B \qquad\qquad \hat{t}_A = t_A - \theta_{AB} t^B$$

as the quantization maps (without coalgebra structure), what leads to the following relation (in general case θ^{AB} and θ_{AB} can be independent, for Palatial choice are c.c.)

$$[\hat{t}^A, \hat{t}_B] = \delta^A_B + \theta^{AC} \bar{\theta}_{BC} = (\hbar - \frac{\Lambda \lambda_p^2}{6}) \delta^A{}_B \quad \leftarrow \text{ Palatial case}$$

3. QUANTUM DEFORMATIONS OF SU(2,2), CONFORMAL SYMMETRY BREAKING AND NC TWISTORS

Quantum deformations are described infinitesimally by classical r-matrices satisfying Yang-Baxter (YB) equations

i) homogeneous YB equation ($r_{12} = r_{(1)} \otimes r_{(2)} \otimes 1, r_{13} = r_{(1)} \otimes 1 \otimes r_{(2)}$ etc.) Ho

$$\ll r, r \gg \equiv [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \qquad r = r_{(1)} \otimes r_{(2)}$$

Such deformation, called triangular, provide explicite formulas for the twist quantization of symmetry algebras and their modules (NC representations). They provide explicitly the quantization maps $t_A \to \hat{t}_A$ of twistors as modules of quantum conformal Hopf algebras, providing the contribution to t_A^{QG}

$$\hat{t}^A = (F_{(1)} \triangleright t^A) F_{(2)} \qquad F = F_{(1)} \otimes F_{(2)}$$

Most important triangular deformations are described by Jordanian (generalized Jordanian) nonstandard r-matrices, with the carrier algebra belonging to Borel subalgebras ($B_-(\hat{g})$) spanned by Cartan generators h_i and positive (e_{+A}) (negative (e_{-A})) root generators; further the reality conditions should map $B_\pm(\hat{g}) \to B_\pm(\hat{g})$.

ii) inhomogeneous YB equation, defining quasitriangular r-matrices

$$\ll r, r \gg = \pm\Omega \qquad \begin{array}{ll} \text{sign} + & - \text{ split r-matrices} \\ \text{sign} - & - \text{ non-split r-matrices} \end{array}$$

where $\Omega \in \hat{g} \wedge \hat{g} \wedge \hat{g}$ is $\hat{g}$-invariant $([\Delta^{(3)}(\hat{g}), \Omega] = 0)$
For $M_{KL} = -M_{LK} \in O(n; C)$ $(K, L, M = 1 \ldots n)$ one gets

$$\Omega = M_{KL} \wedge M_{LN} \wedge M_{NK}$$

n=6 (O(6;C)) provides complexified D=4 conformal algebra.

The reality conditions for quasi-triangular r-matrices necessarily involve maps $e_{+A} \to C_A^{\ B} e_{-B}$ from $B_\pm$ to $b_\mp$; the most popular "canonical" standard Drinfeld-Jumbo r-matrix is invariant under the reality condition $(e_{+A})^\dagger = e_{-A}$ $(C_A^{\ B} = \delta_A^{\ B})$.

Because generators P_μ is standard physical basis are described by e_{+A} $([P_\mu] = M)$ and conformal generators K_μ by e_{-A} $([K_\mu] = M^{-1})$, the DJ classical r-matrix

$$r_{DJ} = i\, q \sum_A e_{+A} \wedge e_{-A}$$

has dimensionless deformation parameter $q \Rightarrow$ DJ quantization does not generate the mass-like conformal symmetry breaking terms.

More about $o(6;c) \simeq sl(4;C)$ (complexified D=4 conformal):

Cartan-Weyl basis of generators:

$$(h_1, h_2, h_3; \underbrace{e_{+1}, e_{+2}, e_{+3}}_{\text{simple positive roots}}, \underbrace{e_{+4}, e_{+5}, e_{+6}}_{\text{composite positive roots}}; e_{-1} \underbrace{\ldots e_{-3}, e_{-4} \ldots e_{-6}}_{\text{negative roots}})$$

h_1, h_2, h_3: Cartan generators

The algebra can be written shortly as (A,B = 1 … 6)

$$[e_A, e_{-A}] = \delta_{AB}\, h_B \qquad h_4, h_5, h_6 \text{ - extended Cartan generators}$$

$$[h_A, e_{+B}] = A_{AB}\, e_{+B}$$

$$A_{AB} \text{ – extended Cartan matrix}$$

$$[h_A, e_{-B}] = -A_{AB}\, e_{-B}$$

Real forms of sl(4;C) giving su(2,2) ≃ o(4,2) are well-known.
If one uses reality conditions which maps $B_\pm(o(4,2)) \to B_\pm(o(4,2))$ one can have at most eight-dimensional carrier of triangular r-matrices, spanned by a pair of Cartan generators and six generators e_{+A} (A= 1, … 6)

Some references on SU(2,2) quantum deformations:

1. V.K. Dobrev, "Canonical q-deformations of noncompact Lie (super-) algebras, J. Phys. A26, 1317 (1993)

2. J. Lukierski, A. Nowicki, J. Sobczyk, "All real forms of $U_q(sl(4;C))$ and D=4 conformal quantum algebras", J. Phys. A26, 4047 (1993)

3. J. Lukierski, P. Minnaert, M. Mozrzymas, "Quantum deformations of conformal algebras introducing fundamental mass parameters", Phys. Lett. B371, 215(1996)

4. N. Aizawa, F. Herranz, J. Negro, M.A. del Olmo, "Twisted conformal algebra so(4,2)", J. Phys. A35, 8179 (2002)

5. J. Lukierski, V. Lyakhovsky, M. Mozrzymas, "κ-deformation of D=4 Weyl and conformal symmetries", Phys. Lett. B538, 375 (2002)

– recently (2014 - ....): papers by F. Delduc, A. Magro, B. Vicedo, K. Yoshida, S. van Tongeren, B. Hoare, R. Borsato, L. Wulff etc. with calculations of YB deformations of D=10 superstring model ($AdS_5 \simeq so(4,2)$)

The class of triangular su(2,2) r-matrices with carrier $\in B_+(o(4,2)) \equiv B_+(su(2,2))$:

$$r = H_2 \wedge e_2 + H_6 \wedge e_6 + c\, e_2 \wedge e_6 + e_1 \wedge e_5 - e_3 \wedge e_4 \quad \leftarrow \quad \text{for sl(4;C)}$$

$$H_k = \alpha_k^{(i)} h_i \qquad k = 2,6 \quad i = 1,2,3 \quad \alpha_2^{(i)}, \alpha_6^{(i)}, c \quad - 7 \text{ complex parameters}$$

After imposing su(2,2) reality conditions 7 complex parameters $\rightarrow$ 4 real parameters (ref. [5])

Some properties of such triangular deformation which can be applied to twistors:

i) As subcase one obtains the light-cone κ-deformation of Poincaré algebra, with r-matrix, satisfying homogeneous YB equation and describing twist quantization

$$r^{LC}_{Poinc} = \frac{1}{\kappa}[L_3 \wedge P_+ - (J_1 - L_2) \wedge P_2 + (J_2 - L_1) \wedge P_1]$$

J_i - o(3) space rotations

L_i - boosts

ii) By replacement in r^{LC}_{Poinc}: $L_3 \rightarrow L_3 - D$ (D - scale generator) one obtains κ-deformation of Weyl-Poincaré algebra $(P_\mu, M_{\mu\nu}, D)$.

iii) In considered generalized Jordanian r-matrix there are not present conformal generators K_μ because they belong to Borel subalgebra $B_-(o(4,2))$

4. QUANTUM DEFORMATIONS OF CM(4) IN SIX-DIMENSIONAL APPROACH AND GRASSMANIAN FRAMEWORK

Compactified D=4 Minkowski space describes all light-cone directions in $R^{4,2}$ (O(4,2) light rays modulo scaling):

$$ds^2 = dx_k\, dx^k = dx_\mu\, dx^\mu - dx^+\, dx^- \qquad x^\pm = x_5 \pm x_6$$

$$k = 1, \ldots 6 \qquad \text{metric } (-,+,+,+,+,-) \qquad k = [AB] \quad A, B = 1,2,3,4$$

One can choose the scaling gauge ($x_k\, x^k = \tilde{x}_k\, \tilde{x}^k = 0$)

$$x_k = (x_\mu, x_5, x_6) \to \tilde{x}_k = (\tilde{x}_\mu = \frac{x_\mu}{x_5}, 1, \tilde{x}_\mu^2)$$

Introducing the light-cone condition in the form

$$\epsilon^{ABCD} x_{[AB]} x_{[CD]} = 0 \quad \xrightarrow{\text{follows}} \quad x_{[AB]} \sim t^1_{[A}\, t^2_{B]}$$

One gets D=6 incidence relation ($A = \alpha, \dot{\alpha},\ \beta = \beta, \dot{\beta}$) for null 3-planes

$$x_{[AB]}\, t^B = 0 \qquad x_{[AB]} = \begin{pmatrix} x_+ \epsilon_{\dot{\alpha}\dot{\beta}} & x_{\dot{\alpha}}^{\ \beta} \\ -x^{\alpha}_{\ \dot{\beta}} & -x_- \epsilon^{\alpha\beta} \end{pmatrix}, \qquad t^B = \begin{pmatrix} \omega^{\dot{\beta}} \\ \pi_\beta \end{pmatrix}$$

↑ D=6 generalized α-plane

In D=4 notation D=6 incidence relation takes the form

(1) $x_+ \epsilon_{\dot{\alpha}\dot{\beta}} \omega^{\dot{\beta}} + x^{\beta}_{\dot{\alpha}} \pi_\alpha = 0 \quad \xrightarrow{x_+ = 1}$ D=4 incidence relation

(2) $-x^{\alpha}_{\dot{\beta}} \omega^{\dot{\beta}} - x_- \epsilon^{\alpha\beta} = 0 \quad \xrightarrow{\text{inserting (1)}} \quad x^- = x_\mu x^\mu \quad \rightarrow$ D=6 light cone

Three observations:

α) One can introduce D=6 dual incidence relations

$X^{[AB]} \tilde{t}_B = 0 \quad \longleftarrow$ D=6 generalized β-planes (null 3-planes)

In order to get nonzero intersection of D=6 generalized α-planes and β-planes it is necessary to impose $t^A \tilde{t}_A = 0$ i.e.

complex compactified $CM(4) \simeq CP(5) \Leftrightarrow Q_5 = \{CP(3) \otimes CP(3); t^A \tilde{t}_A = 0\}$

β) one can replace light cone by AdS_5 with radius R

$x_k x^k = 0 \quad \longrightarrow \quad x_k x^k = -R^2 \quad \Longrightarrow$ AdS_5 space-time (complexified)

The incidence relations for complexified AdS_5 take the following modified form using a pair of orthogonal twistors ($t^A \tilde{t}_A = 0$)

$$X_{AB} t^B = \frac{1}{2} R \tilde{t}_A \qquad X^{AB} \tilde{t}_B = \frac{1}{2} R t^A \qquad \Rightarrow \qquad X_{AB}\, X^{BC} = \frac{1}{4} R^2 \delta_A^{\ C}$$

γ) Infinity twistor breaks conformal covariance - to Poincaré group in flat case and to AdS if R $\neq$0 (Palatial case).

Complex Minkowski space from Grassmanian $G(4;2)$:

$$CP(3) \simeq \frac{SU(2,2)}{U(2,1)} \longrightarrow G(4,2) \simeq \frac{SU(2,2)}{S(U(2)\otimes U(2))} \in CM^4$$

projective twistors $t^A \simeq \lambda t^A$ composite CM^4 coordinates X^{AB}

Quantum deformation of twistor theory ⇝ Quantum deformations of conformal group SU(2,2) (Brain Majid 2007)

Twist quantization of U(su(2,2)) enveloping algebra ⇝ Cotwist quantization of conformal group SU(2,2)

twist quantization

– multiplication in Hopf algebra **not changed**
– coproducts **changed**

$\xleftrightarrow[\text{duality}]{\text{Hopf}}$

cotwist quantization

– multiplication in Hopf algebra **changed**
– coproducts **unchanged** (remains classical)

Modified multiplication of quantum SU(2,2) group elements g, h

$$g \bullet h = \mathfrak{F}_\theta(g_{(1)}, h_{(1)})\, g_{(2)} h_{(2)} \qquad \mathfrak{F}_\theta(g,h) = <\bar{F}_\theta | g \otimes h>$$

The cotwisting by $\bar{F}$ results in **RTT relations**

$$g^A{}_B \in S_\theta U(2,2): \qquad R^{AC}_{BD}\, g^B_E\, g^D_F = g^C_D\, g^A_B\, R^{BD}_{EF}$$

$$\mathfrak{F}_\theta(g^A{}_B, g^C{}_D) \equiv \mathfrak{F}^{AC}_{\theta\ BD} \qquad \Rightarrow\ R^{AC}_{BD} = (\mathfrak{F}^T_\theta\, \mathfrak{F}^{-1}_\theta)^{AC}_{BD} \quad (\theta^{AB} - \text{dependent})$$

calculated from cotwist multiplication formula

One can also get **the deformed commutators of $x^{[AB]} \in CM(4)$**:

⟶ **in general case** one obtains the **quadratic algebra**(Brain+Majid)

⟶ **for special choices** of θ^{AB} one gets some coordinates $x^{[AB]}$ **central and some satisfying c-number noncommutativity** (Hannabus)

5. FINAL REMARKS

i) Twistor space is a phase space $\longrightarrow$ its quantum deformations are described in the formalism of NC geometry by Hopf (Courant) algebroids

ii) Analogy of geometric methods in QG and twistor theory:

	PHYSICS:		GEOMETRY:
SPACE-TIME:	Classical gravity	$\longleftrightarrow$	(pseudo) Riemannian geometry
	$\downarrow \hbar \neq 0$		$\downarrow \hbar \neq 0$
	Quantum gravity	$\longleftrightarrow$	NC (pseudo) Riemannian geometry $\nwarrow$ (S.Beggs, S.Majid book just appeared in Springer)
	$\updownarrow \updownarrow$		$\updownarrow \updownarrow$
	Classical twistor theory	$\longleftrightarrow$	spinorial (curved) conformal geometry
TWISTORS:	$\downarrow \hbar \neq 0$		$\downarrow \hbar \neq 0$
	Quantum twistor theory	$\longleftrightarrow$	NC spinorial-conformal curved geometry

iii) Twistorial curved geometry and “bosonic SUSY”

Conformal algebra $\hat{g} = su(2,2)$

$\xrightarrow{\text{SUSY}}$ $su(2,2;1): \quad \{\hat{Q},\hat{Q}\} \in \hat{g} \quad Q \in \frac{su(2,2;1)}{u(2,2)}$

$\xrightarrow[\text{SUSY}]{\text{bosonic}}$ $su(2,3): \quad \{\hat{T},\hat{T}\} \in \hat{g} \quad \hat{T} \in \frac{su(2,3)}{u(2,2)}$

Twistors $\hat{T}$ are spinorial (as $\hat{Q}$) but bosonic! If “fermionic twistors” are introduced $\Longrightarrow$ one gets superconformal algebra.

Analogy with space-time approach:

Lorentz symmetry $o(3,1) \longleftrightarrow su(2,2)$ symmetry

AdS symmetry $o(3,2) \longleftrightarrow su(2,3)$ symmetry

Such “bosonic SUSY” geometries did appear already in literature, e.g.

S. Fedoruk, E. Ivanow, J. Lukierski, “Massless higher spin D=4 particle with both N=1 SUSY and its bosonic counterpart, PLB641, 226 (2008).
(by Taylor expansion in “bosonic” spinor variables of generating fields one gets infinite-component HS multiplets).

THANK YOU!