3d conformal geometry and 4d prepotentials for fermionic higher-spin fields

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3d conformal fermionic higher-spin fields: symmetric tensor-spinors with gauge symmetries

$$\delta\psi_{i_1 i_2 \cdots i_s} = \partial_{(i_1}\xi_{i_2\cdots i_s)} + \gamma_{(i_1}\lambda_{i_2\cdots i_s)} \quad (i = 1, 2, 3)$$

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2. Application: prepotentials and twisted self-duality for 4d (non-conformal) fermionic higher spins

→ Uses 1. to solve the Hamiltonian constraint

→ New "prepotential" action gives twisted self-duality directly; not manifestly Lorentz-invariant

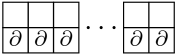
→ cf. [Deser, Teitelboim '76] for Maxwell; works in many other cases [Bunster/Teitelboim, Henneaux, Hillmann, Hörtner, Julia, VL, Leonard, Matulich, Prohazka, ...]

1. Conformal Geometry

Goal: get a complete set of invariants under
$\delta\psi_{i_1 i_2 \cdots i_s} = \partial_{(i_1}\xi_{i_2\cdots i_s)} + \gamma_{(i_1}\lambda_{i_2\cdots i_s)}$.

Invariant for ξ transformations: "Riemann" tensor $R = d^s_{(s)}\psi$,

$$R_{i_1 j_1 \cdots i_s j_s} = 2^s\,\partial_{j_1}\cdots\partial_{j_s}\psi_{i_1\cdots i_s}\,, \quad \text{with } [i_k j_k]$$



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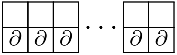

Two theorems, based on the cohomology of $d_{(s)}$ with $d^{s+1}_{(s)} = 0$:

1. Completeness: $R = 0 \Leftrightarrow \psi = d_{(s)}\xi$
2. Poincaré lemma: $d_{(s)} T = 0\,,\ T \sim (s, s) \Leftrightarrow T = R[\psi]$

[Olver '82; Dubois-Violette, Henneaux '99, '02; Bekaert, Boulanger '02, '04]

1. Conformal Geometry

Invariant under λ transformations? 3D: no Weyl tensor !

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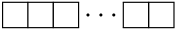
1. Einstein tensor $G = \star^s R$, 
2. Schouten tensor made of G and its (gamma-)traces
   $\rightarrow$ simple transformation $\delta S = d_{(s)}\nu$
3. $d^s_{(s)} S$ is invariant since $d^{s+1}_{(s)} = 0$
   $\rightarrow$ Cotton $\boxed{D = \star^s d^s_{(s)} S}$

D is symmetric, divergenceless and gamma-traceless.

1. Conformal Geometry

Example: spin 5/2 ($s = 2$) $\delta\psi_{ij} = \partial_{(i}\xi_{j)} + \gamma_{(i}\lambda_{j)}$

1. Einstein tensor:

$$G_{ij} = \varepsilon_{ikm}\varepsilon_{jln}\partial^k\partial^l\psi^{mn}$$

2. Schouten tensor:

$$S_{ij} = G_{ij} + b_1\gamma_{(i}\not{G}_{j)} + a_0\delta_{ij}G_k{}^k$$

Requiring $\delta S_{ij} = \partial_{(i}\nu_{j)}$ for some ν_j fixes $b_1 = -\frac{1}{2}$, $a_0 = -\frac{1}{4}$

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3. Cotton tensor: $4 = 2s$ derivatives

$$D_{ij} = \varepsilon_{ikm}\varepsilon_{jln}\partial^k\partial^l S^{mn}$$

$$\Bigg[D_{ij} = \Delta^2\left(\psi_{ij} - \frac{1}{2}\gamma_{(i}\not{\psi}_{j)} - \frac{1}{4}\delta_{ij}\bar{\psi}\right)$$
$$+ \frac{\Delta}{4}\left(\partial_i\partial_j\bar{\psi} + 2\not{\partial}\partial_{(i}\not{\psi}_{j)} + \partial^k(\delta_{ij}\partial^l\psi_{lk} - 10\partial_{(i}\psi_{j)k} + 2\gamma_{(i}\not{\partial}\psi_{j)k} + 2\partial_{(i}\gamma_{j)}\not{\psi}_k)\right)$$
$$+ \frac{1}{4}\partial_i\partial_j\left(5\partial^k\partial^l\psi_{kl} - 2\not{\partial}\partial^k\not{\psi}_k\right) - \frac{1}{2}\partial_{(i}\gamma_{j)}\partial^k\partial^l\not{\partial}\psi_{kl}\Bigg]$$

Two important theorems:

1. Completeness:

$$D[\psi] = 0 \iff \psi = d_{(s)}\xi + \gamma\lambda$$

2. Conformal Poincaré lemma:

$$T_{i_1 i_2 \dots i_s} = T_{(i_1 i_2 \dots i_s)},\ \partial \cdot T = 0,\ \gamma \cdot T = 0 \iff T = D[\psi]$$

2. Twisted self-duality

Maxwell's equations (spin 1) in 4D can be rewritten as

$$\begin{pmatrix} \star F \\ \star G \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} , \quad F = dA \, , \; G = d\tilde{A}$$

which makes the $SO(2)$ duality manifest.

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Generalizes to

- Extended (ungauged) supergravity $\star\mathcal{G} = \Omega\mathcal{M}(\phi)\,\mathcal{G}$
  $\to$ duality group $G \subset Sp(2n_v, \mathbb{R})$
- Linearized gravity, $\begin{pmatrix} \star R \\ \star \tilde{R} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} R \\ \tilde{R} \end{pmatrix}$
- Bosonic higher-spin fields
- ...

Genuine *off-shell* symmetries ! This can be made manifest by going to the Hamiltonian and solving the constraints.

2. Twisted self-duality

Free (non-conformal) fermionic higher-spin field in four dimensions:

1. Fang-Fronsdal equation

$$\mathcal{F}_{\mu_1 \cdots \mu_s} \equiv \not{\partial} \psi_{\mu_1 \cdots \mu_s} - s \partial_{(\mu_1} \not{\psi}_{\mu_2 \cdots \mu_s)} = 0$$

Gauge variation $\delta \mathcal{F} = d^2_{(s)} \not{\xi}$ under $\delta \psi_{\mu_1 \cdots \mu_s} = \partial_{(\mu_1} \xi_{\mu_2 \ldots \mu_s)}$
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2. Drop $\not{\xi} = 0$ condition: higher-derivative geometric equation

$$\not{R} = 0$$

Equivalent because $\not{R} = d^{s-1}_{(s)}\mathcal{F}$.
[Bekaert, Boulanger '03; Bandos, Bekaert, de Azcarraga, Sorokin, Tsulaia '05]

2. Twisted self-duality

Also equivalent to twisted self-duality

$$R + \gamma_5 \star R = 0$$

with spatial constraint

$$\gamma^{kl} R_{kl\, i_2 j_2 \cdots\, i_s j_s} = 0$$

- $\rightarrow$ Duality rotations $R \leftrightarrow \star R$ act as chirality transformations $\psi \leftrightarrow \gamma_5 \psi$
- $\rightarrow$ Equivalent to the constraint associated to the Lagrange multiplier $\psi_{0 i_2 \dots i_s}$ in the Hamiltonian formalism

[Deser, Kay, Stelle '77] for spin 3/2;
see also [Deser, Seminara '04; Henneaux, Teitelboim '13]

2. Twisted self-duality

Constraint is $\not{G}[\psi_{\text{spatial}}] = 0$. Also symmetric and divergenceless...
$\Rightarrow$ use conformal Poincaré lemma !

$$\exists \chi_{i_1 \dots i_s} \text{ such that } G[\psi] = D[\chi] \quad (\Leftrightarrow \psi = S[\chi])$$

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$\rightarrow$ Equations of motion:

$$\dot{D}^{i_1 \dots i_s}[\chi] + \gamma_5 \varepsilon^{i_1 jk} \partial_j D_k{}^{i_2 \dots i_s}[\chi] = 0$$

$\rightarrow$ Action:

$$S[\chi] = -i \int dt\, d^3x\, \chi^\dagger_{i_1 \dots i_s} \left(\dot{D}^{i_1 \dots i_s}[\chi] + \gamma_5 \varepsilon^{i_1 jk} \partial_j D_k{}^{i_2 \dots i_s}[\chi] \right)$$

with $SO(2)$ invariance $\chi \rightarrow e^{\alpha \gamma_5} \chi$

4. Conclusions

1. Conformal geometry of 3D fermionic HS fields

$\rightarrow$ Cotton tensor

$\rightarrow$ Proofs of completeness + conformal Poincaré lemma

2. Twisted self-duality for 4D fermionic HS fields

$\rightarrow$ Use 1. to solve Hamiltonian constraint

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Some open questions:

- Similar form for all bosonic and fermionic fields
  $\rightarrow$ Symmetries? Hypersymmetry, Sp(8), ...
- Extension to (A)dS? to massive/partially massless fields?
- Conformal self-dual rectangles in six dimensions?
- Interactions?

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Thank you for your attention !