Breaking of arbitrary amount of $d = 3$ supersymmetry

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Spontaneous breaking of supersymmetry

It is known that while single branes preserve one half the original supersymmetry, some multiple brane configurations may exhibit different patterns of supersymmetry breaking, such as $\frac{1}{4}$, $\frac{1}{8}$, $\frac{3}{16}$ and more. They are typically interpreted as systems of branes, intersecting at different angles. Not a great number of supersymmetric actions with such spontaneously broken and unbroken theories are known, and they are primarily concentrated in the field of mechanics, (multiple)$D0$ and $M0$ branes.

It would be, therefore, of interest to try to employ the formalism of nonlinear realizations to find the way to construct the actions of such kind. For test purposes, we discuss relatively simple case of spontaneous breaking of some $N=1$, $d=3$ supersymmetries to the single one. As treatment of unspecified number of supersymmetries will be difficult within the usual superfield formalism, we prefer to use the formalism of nonlinear realizations to construct the on-shell component action.

We find that it is possible to construct the action of arbitrary number N_0 of $N=1$, $d=3$ Goldstone scalar multiplets. The constructed action exhibits $SO(N_0)$ symmetry, and its structure is universal for more than four multiplets. No such action is found for vector multiplets.

Basic objects

The algebra we consider is just a direct sum of N_0+1 $N=1$, $d=3$ superalgebras,

$$\left\{Q_\alpha, Q_\beta\right\} = 2\left(\sigma^A\right)_{\alpha\beta} P_A,\;\; \left\{S^i_\alpha, S^j_\beta\right\} = 2\delta^{ij}\left(\sigma^A\right)_{\alpha\beta} P_A,\;\; \left\{Q_\alpha, S^i_\beta\right\} = 2\epsilon_{\alpha\beta} Z^i,$$

as well as Lorentz $SO(1,2)$ and automorphism $SO(N_0+1)$ generators. Here, $\alpha,\beta = 1,2$, $A,B = 0,1,2$ and $i,j = 1,\ldots,N_0$.
We associate the generators Q_α, P_A with the coordinates of $N=1$, $d=3$ superspace and Z^i, S^i_α with the Goldstone bosons and fermions. The corresponding coset element reads

$$g = e^{\mathrm{i}x^A P_A} e^{\theta^\alpha Q_\alpha} e^{\psi^{i\alpha} S^i_\alpha} e^{\mathrm{i}\mathbf{q}^i Z^i}.$$

The transformation in this coset space are induced by left multiplication, $g_0 g = g' h$,

$$\begin{aligned} g_Q &= e^{\epsilon^\alpha Q_\alpha} \;\Rightarrow\; \delta_Q\theta^\alpha = \epsilon^\alpha,\; \delta_Q x^A = \mathrm{i}\epsilon^\alpha\theta^\beta\left(\sigma^A\right)_{\alpha\beta},\; \delta_Q\psi^{\alpha i} = 0,\; \delta_Q\mathbf{q}^i = 0,\\ g_S &= e^{\varepsilon^{i\alpha} S^i_\alpha} \;\Rightarrow\; \delta_S\psi^{\alpha i} = \varepsilon^{\alpha i},\; \delta_S x^A = \mathrm{i}\varepsilon^{\alpha i}\psi^{\beta i}\left(\sigma^A\right)_{\alpha\beta},\; \delta_Q\theta^\alpha = 0,\; \delta_Q\mathbf{q}^i = \mathrm{i}\varepsilon^i_\alpha\theta^\alpha. \end{aligned}$$

The following Maurer-Cartan forms are invariant with respect to the supersymmetry transformations:

$$g^{-1}dg = \mathrm{i}\triangle x^A P_A + d\theta^\alpha Q_\alpha + d\psi^{i\alpha} S^i_\alpha + \mathrm{i}\triangle\mathbf{q}^i Z^i,$$

$$\text{where}\;\; \triangle x^A = dx^A + \mathrm{i}\left(d\theta^\alpha\,\theta^\beta + d\psi^{i\alpha}\,\psi^{i\beta}\right)\left(\sigma^A\right)_{\alpha\beta},\;\; \triangle\mathbf{q}^i = d\mathbf{q}^i + 2\mathrm{i}d\theta^\alpha\,\psi^i_\alpha.$$

Covariant derivatives

Using the invariant forms $\triangle x^A$ and $d\theta^\alpha$, one can establish a set of derivatives, which are covariant with respect to $N_0 + 1$ supersymmetries:

$$\nabla_A = \left(E^{-1}\right)_A{}^B \partial_B, \quad E_A{}^B = \delta_A^B + \mathrm{i}\partial_A \psi^{i\mu}\, \psi^{i\nu} \left(\sigma^B\right)_{\mu\nu},$$

$$\nabla_\alpha = D_\alpha - \mathrm{i}\nabla_\alpha \psi^{i\mu}\, \psi^{i\nu} \left(\sigma^B\right)_{\mu\nu} \partial_B, \quad D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \mathrm{i}\theta^\beta \left(\sigma^C\right)_{\alpha\beta} \partial_C.$$

They satisfy the following commutation relations:

$$\begin{aligned}
\left\{\nabla_\alpha, \nabla_\beta\right\} &= -2\mathrm{i}\left(\sigma^A\right)_{\alpha\beta} \nabla_A - 2\mathrm{i}\nabla_\alpha \psi^{j\mu}\, \nabla_\beta \psi^{j\nu} \left(\sigma^C\right)_{\mu\nu} \nabla_C, \\
\left\{\nabla_A, \nabla_\alpha\right\} &= -2\mathrm{i}\nabla_\alpha \psi^{j\mu}\, \nabla_A \psi^{j\nu} \left(\sigma^C\right)_{\mu\nu} \nabla_C, \\
\left[\nabla_A, \nabla_B\right] &= 2\mathrm{i}\nabla_A \psi^{k\mu} \nabla_B \psi^{k\nu} \left(\sigma^C\right)_{\mu\nu} \nabla_C.
\end{aligned}$$

Relating Goldstone fields

As typically happen in the systems with spontaneously broken spacetime symmetries, not all the Goldstone fields are independent. In particular, as $\triangle \mathbf{q}^i$ and $d\theta^\alpha$ forms are invariant, one can covariantly put the $d\theta^\alpha$ projection of $\triangle \mathbf{q}^i$ to zero and obtain

$$d\theta^\alpha \left(\nabla_\alpha \mathbf{q}^i + 2\mathrm{i}\boldsymbol{\psi}^i_\alpha \right) = 0 \;\Rightarrow\; \boldsymbol{\psi}^i_\alpha = \frac{\mathrm{i}}{2} \nabla_\alpha \mathbf{q}^i .$$

This is an example of the so called Inverse Higgs Effect. Note that from the relation above one can obtain a new relation

$$\nabla_\alpha \boldsymbol{\psi}^i_\beta + \nabla_\beta \boldsymbol{\psi}^i_\alpha = \frac{\mathrm{i}}{2} \{ \nabla_\alpha, \nabla_\beta \} \mathbf{q}^i = \left(\sigma^C\right)_{\alpha\beta} \nabla_C \mathbf{q}^i + \nabla_\alpha \boldsymbol{\psi}^{j\mu} \nabla_\beta \boldsymbol{\psi}^{j\nu} \left(\sigma^C\right)_{\mu\nu} \nabla_C \mathbf{q}^i .$$

It expresses the symmetric part of $\nabla_\alpha \boldsymbol{\psi}_\beta$ in terms of $\nabla_A \mathbf{q}^i$, while the antisymmetric part remains undetermined. If we define the components as

$$\mathbf{q}^i|_{\theta\to 0} = q^i, \;\; \boldsymbol{\psi}^i_\alpha|_{\theta\to 0} = \psi^i_\alpha, \;\; \nabla_A \mathbf{q}^i|_{\theta\to 0} = \mathcal{D}_A q^i, \;\; \nabla_\alpha \boldsymbol{\psi}^i_\beta|_{\theta\to 0} = \epsilon_{\alpha\beta} A^i + \frac{1}{2} \left(\sigma^C\right)_{\alpha\beta} J^i_C ,$$

the component limit of the superfield relation above implies

$$J^i_A = \left(1 + A^k A^k - \frac{1}{4} J^k_C J^{Ck} \right) \mathcal{D}_A q^i + \frac{1}{2} J^k_A J^{Bk} \mathcal{D}_B q^i - \epsilon_A{}^{BC} \mathcal{D}_B q^i J^m_C A^m .$$

Thus the system remains off-shell for any number of multiplets!

Equation of motion

To make the construction of the sample action of many scalar multiplets simpler, one can assume that it retains the explicit $SO(N_0)$ invariance. In this case, the only possible equation of motion for general N_0 is $A^i = 0$. Indeed, if we sum up all indices according to the standard rule, the only terms in the Lagrangian, which contain A^i field, are quadratic in it: $A^i A^i$, $\mathcal{D}_A q^i A^i \mathcal{D}^A q^j A^j$, et.c. Then the algebraic equation of motion of A^i would read $\Lambda^{ij} A^j = 0$. If the action contains standard A^2 term, the matrix $\Lambda^{ij} A^j = 0$ begins with δ^{ij} and is not degenerate. Thus it is safe to assume that $A^i = 0$ from the beginning.
Also let us note the situation is different in the particular case $N_0 = 4$, one can construct the term

$$A^i \epsilon^{ijkl} \epsilon^{ABC} \mathcal{D}_A q^j \mathcal{D}_B q^k \mathcal{D}_C q^l,$$

which is linear in A^i and would lead to the nontrivial equation of motion. As combination $\mathcal{D}_K q^i \epsilon^{ijkl} \epsilon^{ABC} \mathcal{D}_A q^j \mathcal{D}_B q^k \mathcal{D}_C q^l = 0$, being antisymmetric in A, B, C, K, the possible equation of motion in this case is

$$A^i = \Phi(\mathcal{D}q) \epsilon^{ijkl} \epsilon^{ABC} \mathcal{D}_A q^j \mathcal{D}_B q^k \mathcal{D}_C q^l.$$

The broken supersymmetry

The components transform with respect to the broken supersymmetry as

$$\delta_S^\star q^i = -U^C\partial_C q^i,\ \ \delta_S^\star \psi^{i\alpha} = \varepsilon^{i\alpha} - U^C\partial_C\psi^{i\alpha},\ \ \delta_S^\star A^i = -U^C\partial_C A^i,$$
$$\delta_S^\star J_A^i = -U^C\partial_C J_A^i,\ \ U^A = \mathrm{i}\varepsilon^{k\mu}\psi^{k\nu}\left(\sigma^A\right)_{\mu\nu}.$$

The previously defined broken supersymmetry covariant derivative reads

$$\mathcal{D}_A = \left(\mathcal{E}^{-1}\right)_A{}^B\partial_B,\ \ \mathcal{E}_A{}^B = \delta_A^B - \mathrm{i}\psi^{k\mu}\partial_A\psi^{k\nu}\left(\sigma^B\right)_{\mu\nu},$$
$$\delta_S^\star\mathcal{D}_A q^i = -U^C\partial_C\mathcal{D}_A q^i,\ \ \delta_S^\star \det\mathcal{E} = -\partial_C\left(U^C \det\mathcal{E}\right)$$

Therefore, the main part of the action, with nontrivial bosonic limit, can be constructed in standard way

$$S_0 = -\int d^3x \det\mathcal{E}\, F\left(\mathcal{D}_A q^i\right) = -\int d^3x \det\mathcal{E}\,\widetilde{F}\left(J_A^i\right).$$

The $SO(N_0)$ invariant action should depend on Lorentz and $SO(N_0)$ invariants, such as $\mathrm{tr}\, d^n$, $n=1,2,3$, if $d_{AB} = \mathcal{D}_A q^i \mathcal{D}_B q^i$, or $\mathrm{tr} X^n$, $n=1,2,3$, $X_{AB} = J_A^i J_B^i$.

The Wess-Zumino terms

For arbitrary N_0, one can construct one Wess-Zumino term

$$\begin{aligned}
\mathcal{L}_{WZ1} &= \mathrm{i}\det\mathcal{E}\,\epsilon^{ABC}\mathcal{D}_A q^i \mathcal{D}_B q^j \left(\psi^{i\alpha}\mathcal{D}_C\psi^j_\alpha - \psi^{j\alpha}\mathcal{D}_C\psi^i_\alpha\right) \;\Rightarrow \\
\delta^\star_S \mathcal{L}_{WZ1} &= -\partial_K\left(U^K\mathcal{L}_{WZ}\right) + \mathrm{i}\det\mathcal{E}\,\epsilon^{ABC}\mathcal{D}_A q^i \mathcal{D}_B q^j \left(\varepsilon^{i\alpha}\mathcal{D}_C\psi^j_\alpha - \varepsilon^{j\alpha}\mathcal{D}_C\psi^i_\alpha\right) = \\
&= -\partial_K\left(U^K\mathcal{L}_{WZ}\right) + \mathrm{i}\partial_C\Big(\epsilon^{ABC}\partial_A q^i \partial_B q^j \left(\varepsilon^{i\alpha}\psi^j_\alpha - \varepsilon^{j\alpha}\psi^i_\alpha\right)\Big).
\end{aligned}$$

In the case $N_0 = 4$, the second Wess-Zumino term can be added

$$\mathcal{L}_{WZ2} = \mathrm{i}\det\mathcal{E}\,\epsilon^{ABC}\epsilon^{ijkl}\mathcal{D}_A q^i \mathcal{D}_B q^j \psi^{k\alpha}\mathcal{D}_C\psi^l_\alpha.$$

It is invariant for the same reasons.

The unbroken supersymmetry

The unbroken supersymmetry transformations of the components can be derived with the help of the formula $\delta^\star_Q f = \epsilon^\gamma D_\gamma \mathbf{f}|_{\theta\to 0}$ or

$$\delta^\star_Q f = \epsilon^\gamma \nabla_\gamma \mathbf{f}|_{\theta\to 0} + H^C\partial_C f,\quad H^A = -\mathrm{i}\epsilon^\mu \psi^{k\nu} A^k \left(\sigma^A\right)_{\mu\nu} + \frac{\mathrm{i}}{2}\epsilon^\gamma \psi^i_\gamma J^{Ai} + \frac{\mathrm{i}}{2}\epsilon^{ABC}\epsilon^\mu \psi^{i\nu} J^i_B \left(\sigma_C\right)_{\mu\nu}.$$

The case of general N_0 turns out to be simpler, as $A^i = 0$. Then the variations of the basic components read

$$\delta^\star_Q q^i = -2\mathrm{i}\epsilon^\gamma \psi^i_\gamma + H^C\partial_C q^i,\quad \delta^\star_Q \psi^{i\alpha} = \frac{1}{2}\epsilon^\gamma \left(\sigma^C\right)_\gamma{}^\alpha J^i_C + H^C \partial_C \psi^{i\alpha}.$$

Note that the variation of ψ^i_α, and, as a consequence, of $\mathcal{E}_A{}^B$, contains J^i_A, which should be expressed in terms of $\mathcal{D}_A q^i$ with use of the equation

$$J^i_A = \left(1 - \frac{1}{4}J^k_C J^{Ck}\right)\mathcal{D}_A q^i + \frac{1}{2}J^k_A J^{Bk}\mathcal{D}_B q^i.$$

This relatively simple equation, however, appears to be very hard to solve in terms of J^i_A. Use of the J^i_A as a basic variable also has drawbacks: to derive its correct transformation law, one should find $\nabla^2\psi^i_\alpha|_{\theta\to 0}$ if $A^i \neq 0$ (otherwise the transformation laws will contain ψ on-shell). This could be done, but it leads to rather complicated expression for $\nabla^2\psi^i_\alpha|_{\theta\to 0}$.

The unbroken supersymmetry

The most practical approach to this problem is to find the variation of the d_{AB}

$$\delta^{\star}_{Q} d_{AB} = -2\mathrm{i}\epsilon^{\gamma}\mathcal{D}_A\psi^i_{\gamma}\mathcal{D}_B q^i + \mathrm{i}\epsilon^{\gamma}\mathcal{D}_A\psi^k_{\gamma}J^{kC}d_{CB} + \\ +\mathrm{i}\epsilon^{\mu}\mathcal{D}_A\psi^{\nu k}\left(\sigma_D\right)_{\mu\nu}J^k_C\epsilon^{CDE}d_{BE} + (A \leftrightarrow B)$$

and of traces of its powers and express all the $\mathcal{D}_A q^i$ in terms of J^i_A using the formula

$$\begin{aligned} J^i_A &= M_A{}^B\mathcal{D}_B q^i, \quad M_A{}^B = \left(1 - \frac{1}{4}\mathrm{tr}X\right)\delta^B_A + \frac{1}{2}X_A{}^B \quad \Rightarrow \mathcal{D}_A q^i = \left(M^{-1}\right)_A{}^B J^i_B, \\ \left(M^{-1}\right)_A{}^B &= \frac{1}{\det M}\left[\left(1 + \frac{1}{16}\left(\mathrm{tr}X\right)^2 - \frac{1}{8}\mathrm{tr}X^2\right)\delta^B_A - \frac{1}{2}\left(1 + \frac{1}{4}\mathrm{tr}X\right)X_A{}^B + \frac{1}{4}\left(X^2\right)_A{}^B\right], \\ \det M &= 1 - \frac{1}{4}\mathrm{tr}X + \frac{1}{16}\left(\mathrm{tr}X\right)^2 + \frac{1}{192}\left(\mathrm{tr}X\right)^3 - \frac{1}{8}\mathrm{tr}X^2 - \frac{1}{32}\mathrm{tr}X\mathrm{tr}X^2 + \frac{1}{24}\mathrm{tr}X^3. \end{aligned}$$

The unbroken supersymmetry

Afterwards, one can find the variations of $\mathrm{tr}\left(X^n\right)$ by solving the linear system of equations

$$\delta \mathrm{tr} d = \frac{\partial \mathrm{tr} d}{\partial \mathrm{tr} X}\delta \mathrm{tr} X + \frac{\partial \mathrm{tr} d}{\partial \mathrm{tr} X^2}\delta \mathrm{tr} X^2 + \frac{\partial \mathrm{tr} d}{\partial \mathrm{tr} X^3}\delta \mathrm{tr} X^3, \ldots$$

While it is not practical to write down $\delta\mathrm{tr}\left(X^n\right)$ explicitly, it is worth to note that they, and consequently the variation of the main part of the Lagrangian, consist of terms

$$\epsilon^\alpha \mathcal{D}_A \psi^i_\alpha J^{iA}, \;\; \epsilon^\alpha \mathcal{D}_A \psi^i_\alpha X^{AB} J^i_B, \;\; \epsilon^\alpha \mathcal{D}_A \psi^i_\alpha \left(X^2\right)^{AB} J^i_B,$$
$$\epsilon^\alpha \mathcal{D}_A \psi^{i\beta}\left(\sigma_D\right)_{\alpha\beta} J^i_C \epsilon^{CDA}, \;\; \epsilon^\alpha \mathcal{D}_B \psi^{i\beta} X^B_A \left(\sigma_D\right)_{\alpha\beta} J^i_C \epsilon^{CDA}, \;\; \epsilon^\alpha \mathcal{D}_B \psi^{i\beta}\left(X^2\right)^B_A \left(\sigma_D\right)_{\alpha\beta} J^i_C \epsilon^{CDA}$$

with some scalar functions. All the terms which do not contain the ϵ^{ABC}-symbol can not combine into the full divergence even if J^i_A were expressed in terms of $\mathcal{D}_A q^i$. Therefore, coefficients of them in the variation of the Lagrangian should vanish. This condition is strong enough to fix the main part of the Lagrangian as

$$\mathcal{L}_0 = -\det\mathcal{E}\,\frac{2\left(1-\frac{1}{4}\mathrm{tr}X\right)^2}{\det M}, \quad \delta^\star_Q \mathcal{L}_0 = 2\mathrm{i}\,\frac{\det\mathcal{E}}{\det M}\left(1-\frac{1}{4}\mathrm{tr}X\right)\epsilon^\alpha \mathcal{D}_A \psi^{i\beta} M_B{}^A \left(\sigma_D\right)_{\alpha\beta} J^i_C \epsilon^{BCD}.$$

The variation of WZ term

The variation of the Wess-Zumino term after some simplifications and neglecting full divergencies can be written as

$$\delta_Q^\star \mathcal{L}_{WZ} = -2\mathrm{i}\epsilon^{ABC}\partial_A q^i\, \partial_B q^j\, \epsilon^\alpha \left(\sigma^D\right)_{\alpha\beta}\partial_C\psi^{j\beta} J^i_D + 8\epsilon^{ABC}\epsilon^\beta \psi^i_\beta\, \partial_A q^j\, \partial_B \psi^{i\alpha}\partial_C\psi^j_\alpha .$$

Substituting here $\mathcal{D}_A q^i = \left(M^{-1}\right)_A{}^B J^i_B$, one can rewrite the first term as

$$\mathrm{i}\det\mathcal{E}\epsilon^{ABC}\mathcal{D}_A q^i \mathcal{D}_B q^j \epsilon^\alpha \left(\sigma^D\right)_{\alpha\beta}\mathcal{D}_C\psi^{j\beta} J^i_D = -2\mathrm{i}\frac{\det\mathcal{E}}{\det M}\epsilon^{ABC} J^i_A J^i_D J^j_B \epsilon^\alpha \left(\sigma^D\right)_{\alpha\beta} M_C{}^K \mathcal{D}_K\psi^{j\beta} =$$

$$= -4\mathrm{i}\det\mathcal{E}\epsilon^{ABC}\mathcal{D}_B q^j \epsilon^\alpha \left(\sigma_A\right)_{\alpha\beta}\mathcal{D}_C\psi^{j\beta} - 4\mathrm{i}\frac{\det\mathcal{E}}{\det M}\left(1-\frac{1}{4}\mathrm{tr}X\right)\epsilon^{ABC}\epsilon^\alpha J^i_A \left(\sigma_B\right)_{\alpha\beta} M_C{}^K \mathcal{D}_K\psi^{j\beta}.$$

Therefore, the complete variation reduces to

$$\delta_Q^\star \mathcal{L}_{WZ} = -4\mathrm{i}\epsilon^{ABC}\left(\sigma_A\right)_{\alpha\beta}\epsilon^\alpha\partial_C\psi^{j\beta}\,\partial_B q^j - 2\epsilon^{ABC}\partial_B q^j \epsilon^\alpha \partial_A\left(\psi^i_\lambda\psi^{i\lambda}\right)\partial_C\psi^{j\beta} +$$
$$-4\mathrm{i}\frac{\det\mathcal{E}}{\det M}\left(1-\frac{1}{4}\mathrm{tr}X\right)\epsilon^{ABC}\epsilon^\alpha J^i_A \left(\sigma_B\right)_{\alpha\beta} M_C{}^K \mathcal{D}_K\psi^{j\beta}.$$

Therefore, $\mathcal{L}_0 + 1/2\,\mathcal{L}_{WZ}$ is invariant with respect to the unbroken supersymmetry.

The complete action

In the particular cases of $N_0=1$, $N_0=2$ the obtained Lagrangian reduces to the Lagrangians of the membranes in $D=4$ and $D=5$, and in the latter case one may expect an additional hidden unbroken $N=1$, $d=3$ supersymmetry. It does not have Nambu-Goto form for higher number of fields, however. In particular, one can formally solve the equation that relates J^i_A and $\mathcal{D}_A q^i$ to find that in the bosonic limit

$$J^i_A = \left(1-\frac{1}{4}\mathrm{tr}X\right)\partial_A q^i + \frac{1}{2}J^k_A J^{Bk}\partial_B q^i \;\Rightarrow\; \left(1-\frac{1}{4}\mathrm{tr}X\right)=\Lambda,\; d_{AB}=\partial_A q^i \partial_B q^i,$$

$$J^i_A = 2\Lambda\left(\frac{1}{1+\sqrt{1-2\Lambda d}}\right)_A{}^B\partial_B q^i,\;\; 2+\Lambda = 2\Lambda\mathrm{tr}\left(\frac{1}{1+\sqrt{1-2\Lambda d}}\right),$$

$$\mathcal{L}_{bos} = -\frac{1}{4\Lambda}\det\left(1+\sqrt{1-2\Lambda d}\right).$$

Expanded as a power series, the Lagrangian reads

$$\begin{aligned} L_{bos} \;=\; & -2+\left[\frac{1}{2}\mathrm{tr}d\right]+\left[\frac{1}{4}\mathrm{tr}d^2-\frac{1}{8}\left(\mathrm{tr}d\right)^2\right]+\left[\frac{1}{24}\left(\mathrm{tr}d\right)^3-\frac{3}{16}\mathrm{tr}d\,\mathrm{tr}d^2+\frac{5}{24}\mathrm{tr}d^3\right]+ \\ & +\left[\frac{7}{384}\left(\mathrm{tr}d\right)^4-\frac{3}{32}\left(\mathrm{tr}d\right)^2\mathrm{tr}d^2+\frac{1}{32}\left(\mathrm{tr}d^2\right)^2+\frac{1}{12}\mathrm{tr}d\,\mathrm{tr}d^3\right]+\ldots \end{aligned}$$

The $N_0 = 4$ case

The special case $N_0 = 4$ can be treated by the means similar to the described above. If the equation of motion for A^i is assumed to be

$$A^i = \Phi(J)\epsilon^{ABC}\epsilon^{ijkl}J^j_A J^k_B J^l_C,$$

more terms in the variations appear, which contain

$$\epsilon^{ABC}\epsilon^{ijkl}J^j_A J^k_B J^l_C \epsilon^{\mu}\left(\sigma^K\right)_{\mu\nu}\mathcal{D}_K\psi^{k\nu},\quad \epsilon^{ABC}\epsilon^{ijkl}J^j_A J^k_B J^l_C \epsilon^{\mu}\left(\sigma^K\right)_{\mu\nu}\mathcal{D}_L\psi^{k\nu}X^L_K,$$
$$\epsilon^{ABC}\epsilon^{ijkl}J^j_A J^k_B J^l_C \epsilon^{\mu}\left(\sigma^K\right)_{\mu\nu}\mathcal{D}_L\psi^{k\nu}\left(X^2\right)^L_K.$$

The ansatz for the Lagrangian now contains two Wess-Zumino terms,

$$\mathcal{L}_{N0=4} = -\det\mathcal{E}F(J) + C_1 \mathrm{i}\det\mathcal{E}\,\epsilon^{ABC}\mathcal{D}_A q^i \mathcal{D}_B q^j\left(\psi^{i\alpha}\mathcal{D}_C\psi^j_\alpha - \psi^{j\alpha}\mathcal{D}_C\psi^i_\alpha\right) +$$
$$+C_2 \mathrm{i}\det\mathcal{E}\,\epsilon^{ABC}\epsilon^{ijkl}\mathcal{D}_A q^i \mathcal{D}_B q^j \psi^{k\alpha}\mathcal{D}_C\psi^l_\alpha.$$

The unbroken supersymmetry transformations do not fix the constants completely. The Φ function has to satisfy the cubic equation

$$-C_2 + C_1\Phi\left(24 - 6\mathrm{tr}X + 864\det X\Phi^2\right) = 0.$$

The main function can be determined as ($C_1 = 1/2$, $C_2 = \pm 1/2$- membrane in $D=7$)

$$F = \frac{1}{\det M}\left(4C_1\left(1-\frac{1}{4}\mathrm{tr}X\right)^2 + 18\det X\Phi\left(C_2+8C_1\right)\left(1-\frac{1}{4}\mathrm{tr}X\right)\right).$$

The vector multiplets

While one can dualize the scalar fields in the actions above into the electromagnetic fields, one can try to construct the action of many vector multiplets from the first principles. Contrary to the expectation, it turns out that the analog of system with arbitrary number of scalars does not exist.
The $SO(N_0)$ invariant system of N_0 vector multiplets has to be described by the irreducibility conditions $\nabla_\alpha \psi^{\alpha i} = 0$, as no $SO(N_0)$ covariant terms can be added to right-hand side. The terms like $\nabla_\alpha \psi^{\beta k} \nabla_\beta \psi^{\gamma k} \nabla_\gamma \psi^{\alpha i}$ contain $\nabla_\alpha \psi^{\alpha i}$, and, therefore, lead to the equivalent condition $\Lambda^{ij} \nabla_\alpha \psi^{\alpha j} = 0$.
Unlike the known case $N_0 = 1$, the condition for N_0 multiplets is not acceptable. Indeed, one should be able to use the irreducibility conditions to derive the Bianchi identity for the field strength, which should be equivalent to $\partial_A F^{Ai} = 0$. In the present case it reads

$$\nabla^\beta \nabla_\beta \nabla_\alpha \psi^{\alpha i} = 0 \;\Rightarrow\; \left\{\nabla_\alpha, \nabla_\beta\right\} \nabla^\alpha \psi^{i\beta} + \text{fermions} = 0.$$

One can define $\nabla_\alpha \psi^i_\beta|_{\theta\to 0} = \frac{1}{2}\left(\sigma^A\right)_{\alpha\beta} V^i_A$ and obtain in the bosonic limit.

$$\partial_A V^{Ai}\big(1 - \frac{1}{4} V^j_B V^{Bj}\big) + \frac{1}{2} V^{Aj} V^{Bj} \partial_A V^i_B = 0.$$

The vector multiplets

Considering the identity

$$\partial_A V^{Ai}\left(1-\frac{1}{4}V_B^j V^{Bj}\right)+\frac{1}{2}V^{Aj}V^{Bj}\partial_A V_B^i=0.$$

in the cubic approximation in the fields, one can rewrite the last term as

$$\partial_A\left(V^{Aj}V^{Bj}V_B^i\right)-\partial_A V^{Aj}\,V^{Bj}\partial_A V_B^i-V^{Aj}V^{Bi}\partial_A V_B^j$$

The first term is suitable, the second term $\partial_A V^{Aj}\,V^{Bj}\partial_A V_B^i$ can be canceled by multiplying the identity by the suitable matrix, but the last one can not unless the equation of motion of V^{Ai} $(\partial_A V_B^i-\partial_B V_A^i+\ldots=0)$ is taken into account.
Now one can think that the system is inherently on-shell and should be formulated in terms of the vector potential. Let us assume that the system is described by the Goldstone fermion and the vector potential with the following broken supersymmetry transformation laws

$$\delta_S^\star\psi_\alpha^i=\varepsilon_\alpha^i-U^C\partial_C\psi_\alpha^i,\;\;\delta_S^\star\mathcal{A}_A^i=\epsilon_{ABC}U^B\mathcal{F}^{Ci},\;\;U^A=\mathrm{i}\varepsilon^{j\mu}\psi^{j\nu}\left(\sigma^A\right)_{\mu\nu},\;\;\mathcal{F}^{Ai}=\epsilon^{ABC}\partial_B\mathcal{A}_C^i.$$

All the terms in the transformation law of the vector potential are assumed to be gauge invariant. These transformations close off-shell properly,

$$\left[\delta_{S1}^\star,\delta_{S2}^\star\right]\psi_\alpha^i=-2\mathrm{i}\varepsilon_2^{\mu k}\varepsilon_1^{\nu k}\left(\sigma^A\right)_{\mu\nu}\partial_A\psi_\alpha^i \text{ and } \left[\delta_{S1}^\star,\delta_{S2}^\star\right]\mathcal{A}_A^i=2\mathrm{i}\varepsilon_2^{\mu k}\varepsilon_1^{\nu k}\left(\sigma^B\right)_{\mu\nu}\left(\partial_A\mathcal{A}_B^i-\partial_B\mathcal{A}_A^i\right).$$

The vector multiplets

The unbroken supersymmetry transformation law of ψ^i_α, which commutes with broken supersymmetry off-shell, necessary has structure

$$\delta^\star_Q \psi^i_\alpha = \hat{\delta}_Q \psi^i_\alpha - H^K \partial_K \psi^i_\alpha, \quad H^K = \mathrm{i}\psi^{j\mu} \hat{\delta}_Q \psi^{j\nu} \left(\sigma^K\right)_{\mu\nu}, \quad \delta^\star_S \hat{\delta}_Q \psi^i_\alpha = -U^K \partial_K \hat{\delta}_Q \psi^i_\alpha .$$

If we assume that the unbroken supersymmetry transformation law of the potential is $\delta^\star_Q \mathcal{A}^i_A = \hat{\delta}_Q A^i_A + \epsilon_{ABC} H^B \mathcal{F}^{Ci}$, then

$$\left[\delta^\star_Q, \delta^\star_S\right] \mathcal{A}^i_A = -\hat{\delta}_S \left(\hat{\delta}_Q \mathcal{A}^i_A\right) + U^B \partial_A \hat{\delta}_Q \mathcal{A}^i_B + \partial_A \left(\epsilon_{BCD} U^B H^C \mathcal{F}^{Di}\right).$$

If the transformation law of the potential $\hat{\delta}_Q \mathcal{A}^i_A = \mathrm{i}\epsilon^\alpha \psi^{\beta i} \left(\sigma_A\right)_{\alpha\beta} + \ldots$ begins just like the law of the free system, the expression $-\hat{\delta}_S \left(\hat{\delta}_Q \mathcal{A}^i_A\right) + U^B \partial_A \hat{\delta}_Q \mathcal{A}^i_B$ can not be reduced to gauge transformation. Indeed, in the lowest approximation in the fermions

$$U^B \partial_A \hat{\delta}_Q \mathcal{A}^i_B \approx \epsilon^\alpha \varepsilon^j_\alpha \psi^{\beta j} \partial_A \psi^i_\beta + \epsilon^\alpha \varepsilon^{j\beta} \psi^j_\alpha \partial_A \psi^i_\beta .$$

The most general, up to gauge transformation, cubic contribution to $\hat{\delta}_Q \mathcal{A}^i_A$ is

$$\hat{\delta}_Q \mathcal{A}^i_A = \mathrm{i}\epsilon^\alpha \psi^{\beta i} \left(\sigma_A\right)_{\alpha\beta} + a\epsilon^\alpha \partial_A \psi^i_\alpha \, \psi^{j\beta} \psi^j_\beta + b\epsilon^\alpha \psi^{i\beta} \psi^j_\alpha \partial_A \psi^j_\beta ,$$

$$-\hat{\delta}_S \left(\hat{\delta}_Q \mathcal{A}^i_A\right) = -\mathrm{i}\epsilon^\alpha \varepsilon^{\beta i} \left(\sigma_A\right)_{\alpha\beta} - 2a\epsilon^\alpha \varepsilon^{j\beta} \psi^j_\beta \partial_A \psi^i_\alpha - b\epsilon^\alpha \varepsilon^{i\beta} \psi^j_\alpha \partial_A \psi^j_\beta + b\epsilon^\alpha \varepsilon^j_\alpha \psi^{i\beta} \partial_A \psi^j_\beta .$$

Therefore, $-\hat{\delta}_S \left(\hat{\delta}_Q \mathcal{A}^i_A\right) + U^B \partial_A \hat{\delta}_Q \mathcal{A}^i_B$ is not a gauge transformation even if $\partial_{\alpha\beta} \psi^{i\beta} = 0$.

Conclusion

In this talk, we considered the systems with arbitrary number of Goldstone $N=1$, $d=3$ multiplets. It was found that in the case of scalar multiplets, it is possible to construct the action, invariant with respect N_0 broken supersymmetries and one unbroken supersymmetry (with the simplifying assumption of $SO(N_0)$ invariance). This action exists for arbitrary N_0 and, aside of some low values of N_0, has universal structure. Unlike the scalar case, it was found to be impossible to construct the action of arbitrary large number of vector multiplets, at least while preserving the $SO(N_0)$ symmetry.

It would be interesting to learn whether analogs of these systems exist in higher dimensions and whether systems, that combine the scalar and vector multiplets, exist. As theories with arbitrarily large number of supersymmeties do not have clear string theory interpretation, it remains a question how to properly position these theories in the broader context. The idea of $SO(N_0)$ invariance may also be too restrictive.

Let us also note that construction of the mechanics with $\frac{1}{4}$ breaking of global supersymmetry revealed that the underlying algebra can be interpreted as a reduction of some higher dimensional super Poincare algebra, extended by tensorial central charges. It would be interesting to identify the appropriate higher dimensional algebras that correspond to the considered systems.