Thick morphisms and spinors

Hovhannes Khudaverdian

University of Manchester, Manchester, UK

SQS XIX
26 August—31 August, Yerevan, Armenia

The talk is based on the work with Ted Voronov

Contents

Papers that talk is based on are

[1]. H.M.Khudaverdian, Th.Voronov "Thick morphisms, higher Koszul brackets, and L_∞-algebroids", math-arXiv:180810049

[2] Th. Voronov, *Nonlinear pullback on functions and a formal category extending the category of supermanifolds]*, arXiv: 1409.6475

[3] Th. Voronov, *Microformal geometry*, arXiv: 1411.6720

Abstract...

For an arbitrary morphism $\varphi\colon M \to N$ of (super)manifolds, the pull-back $\phi^*C^\infty(N) \to C^\infty(M)$ is a linear map of space of functions. Moreover it is homomorphism of algebra $C^\infty(N)$ into algebra $C^\infty(M)$.

In 2014 Voronov have introduced *thick morphisms* of (super)manifolds which define generally non-linear pull-back of functions. This construction was introduced as an adequate tool to describe L_∞ morphisms of algebras of functions provided with the structure of homotopy Poisson algebra.

Thick morphism $\Phi = \Phi_S$: $M \Rightarrow N$ can be defined by the "action" $S(x, q)$, where x are local coordinates on M and q are coordinates of momenta in T^*N. The pull-back of thick morphism Φ_S^*: $C^\infty(N) \to C^\infty(M)$, is non-linear map in the case if the action $S(x, q)$ is not linear over q.
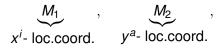
In this approach we come to fundamental concepts of Quantum Mechanics.

In particular thick morphisms with quadratic action give naturally the spinor representation.

Let M_1, M_2 be two (super)manifolds.

$$\underbrace{M_1}_{x^i\text{- loc.coord.}}, \qquad \underbrace{M_2}_{y^a\text{- loc.coord.}},$$

Consider also cotangent bundles T^*M_1 and T^*M_2.

$$\underbrace{T^*M_1}_{x^i, p_j\text{- loc.coord.}}, \qquad \underbrace{T^*M_2}_{y^a, q_b\text{- loc.coord.}},$$

p_i are components of momenta which are conjugate to x^i, respectively q_a are components of momenta which are conjugate to y^a.

Remark We consider even and odd coordinates, i.e. M_1, M_2 are supermanifolds; parity of any coordinate coincide with the parity of corresponding component of momenta:

$$p(p_i) = p(x^i), \quad p(q_b) = p(y^b).$$

Definition of thick morphism. (T.Voronov)

M_1, M_2–two (super)manifolds
Consider symplectic manifold $T^*M_1 \times (-T^*M_2)$
equipped with canonical symplectic structure

$$\omega = \omega_1 - \omega_2 = \underbrace{dp_i \wedge dx^i}_{\text{coord. on } T^*M_1} - \underbrace{dq_a \wedge dy^a}_{\text{coord. on } T^*M_2}$$

Function $S = S(x, q)$—action
It defines Lagrangian surface $\Lambda_S \subset T^*M_1 \times (-T^*M_2)$:

$$\Lambda_S = \left\{ (x, p, y, q): \quad p_i = \frac{\partial S(x, q)}{\partial x^i}, y^b = \frac{\partial S(x, q)}{\partial q_b} \right\}$$

Lagrangian surface—canonical relation—thick morphism

Lagr. surf. Λ_S is canon. relation Φ_s in $T^*M_1 \times (-T^*M_2)$

$$(x^i, p_j) \sim_S (y^a, q_b) \leftrightarrow (x^i, p_j, y^a, q_b) \in \Lambda_S, (\Phi_S = \sim_S).$$

$$\Phi = \Phi_s \text{ is a thick morphism } M_1 \Rrightarrow M_2$$

It defines pull-back Φ_S^* of functions

$$\Phi_S^* \colon \mathfrak{M}_2 = C(M_2) \to \mathfrak{M}_1 = C(M_1),$$

such that for every function $g = g(y) \in \mathfrak{M}_2$,

$$f = f(x) = (\Phi_S^* g)(x) \colon \Lambda_f = \Phi_S \circ \Lambda_g,$$

where Λ_f, Λ_g are Lagrangian surfaces, graphs of df, dg in T^*M_1, T^*M_2.

Explicit expression

Thick morphism $M \Rightarrow N$ defines the pull-back
$\Phi_g^* : C^\infty(N) \to C^\infty(M)$, such that

$$\Phi_S^* g = f(x) = g(y) + S(x,q) - y^a q_a,$$

where y^a and q_a are defined from the equations

$$y^a = \frac{\partial S(x,q)}{\partial q_a}, \quad q_a = \frac{\partial g(y)}{\partial y^a}$$

We see that $\Lambda_f = \Phi_S \circ \Lambda_g$ since

$$p_i = \frac{\partial f}{\partial x^i} = \frac{\partial}{\partial x^i}\left(g(y) + S(x,q) - y^a q_a\right) = \frac{\partial S(x,q)}{\partial x^i}.$$

Thick morphism is usual map if $S(x, q) = S^a(x)q_q$

Example

Generating function $S = S^a(x)q_a$

$$(\Phi_S^* g)(x) = g(y) + S(x, q) - y^a q_a = g(y) + \underbrace{(S^a(x) - y^a)}_{\text{vanishes}} q_a = g(S^a(x))$$

Thick morphism $M_1 \overset{\Phi_s}{\Rightarrow} M_2$ is usual morphism $M_1 \overset{y^a = S^a(x)}{\rightarrow} M_2$.

Thick morphism in general case

In general case the pull-back is non-linear:

$$f(x) = (\Phi_S^* g)(x) = \Big(S(x,q) + g(y) - y^i q_i \Big)\Big|_{y = \frac{\partial S(x,q)}{\partial q}, q = \frac{\partial g}{\partial y}},$$

Example

$S(x,q) = xq + \frac{1}{2}aq^2$, $g(y) = \frac{1}{2}ky^2$ then $q = ky$,
$y = y(x)$ is defined by relation

$$y = \frac{\partial S(x,q)}{\partial q} = x + aq = x + ky \Rightarrow y = \frac{x}{1-k},$$

and

$$f(x) = \Phi_S^*(g)(x) = S(x,q) + g(y) - yq = \frac{kx^2}{2(1-k)}.$$

Application of thick morphisms: L_∞ morphisms

Consider two homotopy Poisson algebras defined on space of functions $C^\infty(M_i)$ by Hamiltonian Q_i ($i = 1, 2$)

We say that Hamiltonians Q_1, Q_2 are connected by the action $S(x, q)$ if

$$Q_1\left(x^i, p_j = \frac{\partial S(x, q)}{\partial x^j}\right) \equiv Q_2\left(y^a = \frac{\partial S(x, q)}{\partial q_a}, q_b\right)$$

(x^i-coordinates on M_1 and q_a momenta on fibers of T^*M_2)

Theorem

The pull-back Φ_S^* of the thick morphism Φ_S is L_∞ morphism of homotopy Poisson algebra $(C^\infty(M_2), Q_2)$ on homotopy Poisson algebra $(C^\infty(M_1), Q_1)$. (Th.Voronov, 2014)

What is it 'Homotopy Poisson algebras'. Recalling

Let M be a supermanifold, and let $Q = Q(x, p)$ be an odd Hamiltonian, odd function defined on cotangent bundle T^*M
This Hamiltonian defines hmotopy Poisson bracket on algebra of functions $C^\infty(M)$. The chain of brackets ican be defined by

$$\{f_1\} = (Q, f_1)\big|_M, \quad \{f_1, f_2\} = ((Q, f_1), f_2)\big|_M,$$

$$\{f_1, f_2, f_3\} = (((Q, f_1), f_2) f_3)\big|_M, \quad \text{and so on:}$$

$$\{f_1, \ldots, f_n\} = \underbrace{(\ldots(}_{n-\text{times}} Q, f_1), \ldots, f_n)\big|_M,$$

$(\, , \,)$— canonical even Poisson bracket on T^*M
Q obeys condition $(Q, Q) = 0$—Jacobi identity.
The chain of brackets $\{f_1, \ldots, f_n\}$ becomes an usual odd
Poisson) bracket if Hamiltonian Q is quadratic on momenta.

The function $S = S(x, q)$ which defines thick morphism $\Phi_S \colon M \Rightarrow N$ we call in this paper 'action'[1]. Why?

Let $H = H(x, p)$ be Hamiltonian defined in cotangent bundle T^*M and let $\mathfrak{S} = \mathfrak{S}(t, x, y)$ be the action of classical mechanics for the path $x(\tau)$, $0 \leq \tau \leq t$ which obeys equations of motion, starts at the point x at $\tau = 0$, and ends at the point y at time $\tau = t$.

---

[1] In the pioneer works of T.Voronov, where thick morphism was suggested, this function was called just "generating function"

Its Legendre transform

$$S(t,x,q) = yq - \mathfrak{S}(t,x,y), \text{where } y = \frac{\partial \mathfrak{S}}{\partial q}.$$

It obeys Hamilton Jacobi equation

$$S(x,q,t)\colon \begin{cases} \frac{\partial S}{\partial t} = H\left(\frac{\partial S}{\partial q}, q\right) \\ S(x,q)\big|_{t=0} = xq \end{cases}$$

Denote $S(t,x,q) = \exp tH$.

Example
free particle

$$H_{\text{free}} = \frac{p^2}{2m}$$

$$\exp tH_{\text{free}}:$$

$$\mathfrak{S}(t,x,y) = \frac{m(y-x)^2}{2t}, \quad S(t,x,q) = xq + \frac{q^2 t}{2m},$$

Example

harmonic oscillator

$$H = \frac{p^2 + x^2}{2},$$

$$\exp tH_{\text{oscillator}} :$$

$$\mathfrak{S}(t,x,y) = \frac{x^2 + y^2}{2}\cotan t - \frac{yx}{\sin t}, \; S(t,x,q) = xq\cos t + \frac{x^2 + q^2}{2}\tan t,$$

Theorem

Let action $S(t, x, q)$ is an exponent of Hamiltonian H:

$$S(t, x, q) = \exp tH$$

Consider the one-parametric group of thick morphism $\Phi_t \colon M \Rightarrow M$ generated by $S(t, x, q)$. For an arbitrary function $g = g(x)$ consider

$$f_t(x) = \Phi_t^*(g)$$

The function $f_t(x)$ obeys the Hamilton-Jacobi equation:

$$\frac{\partial f_t(x)}{\partial t} = H\left(x, \frac{\partial f}{\partial x}\right), \quad f_t(x)\big|_{t=0} = g(x).$$

Quantum thick morphisms

$S_\hbar(x, q)$-quantum action, power series in q and $\hbar$

The corresponding `quantum thick morphism` performs the pull-back:

$$\Phi_{S_\hbar}^{*\text{quant.}}(w)(x) = \int_{T^*N} e^{\frac{i}{\hbar}\left(S_\hbar(x,q) - y^i q_i\right)} w(y) Dq Dy.$$

$DqDp$ is invariant Lioville measure on T^*M

Quantum thick morphisms $\to$ classical thick morphisms

One can see this using stationary phase method:

For $w_\hbar = e^{\frac{i}{\hbar}g(y)}$

$$\lim_{\hbar \to 0} \left[\frac{\hbar}{i} \left(\log \left(\Phi_{S_\hbar}^{*\text{quant.}} (w_\hbar) \right) \right) \right] = \lim_{\hbar \to 0} \left[\frac{\hbar}{i} \left(\log \left(e^{\frac{i}{\hbar} \left(g(y) + S_\hbar(x,q) - y^i q_i \right)} \right) \right) \right] =$$

$$g(y_0) + S(x, q^0) - y_0^i q_i^0,$$

where $y_0 = y_0(x)$ and $q^0 = q^0(x)$ are defined (depending on x) by the sationary point condition: $y_0^i = \frac{\partial S(x,q)}{\partial q_i} |q_i = q_i^0$ and $q_i^0 = \frac{\partial g(y)}{\partial y^i} \big|_{y^i = y_0^i}$, and $S(x,q) = \lim_{\hbar \to 0} S_\hbar(x,q)$.

We come to the classical thick morphism:

$$\lim_{\hbar \to 0} \left[\frac{\hbar}{i} \left(\log \left(\Phi_{S_\hbar}^{*\text{quant.}} \left(e^{\frac{i}{\hbar g(y)}} \right) \right) \right) \right] = \Phi_{S_0}^{\text{class}}(g)(x).$$

Legendre transform $\rightarrow$ Fourier transform

Legendre transform is quasiclassics of Fourier transform:
Legendre: $g(p) = G(x) = px$ such that $G'(x) = p$

$$e^{\frac{i}{\hbar}(G(x)-px)} dx \approx e^{\frac{i}{\hbar}g(p)}.$$

Classical thick morphisms — Hamilton Jacobi equation
Quantum thick morphisms— Shrodinger equation:

What is a spinor

Thick morphism acts on functions on n variables. On the other hand it is defined by an action $S(x, q)$ which depends on $2n$ variables.

This strongly resembles spinor representation if one recalls that the spinor representation (in the orthogonal or symplectic settings) can be seen as action of transformations of a large space on objects such as functions or half-forms that live on a (half-dimensional) maximally isotropic subspace.

Symplectic (orthogonal) spinor is a function on space of half-dimensions, which transforms under the action of spinor group

V-vector space, $\dim V = N$, $X = V \oplus V^*$, $\dim X = 2N$.

$$X \ni \mathbf{A} = \begin{pmatrix} a^i \\ \alpha_j \end{pmatrix} \longrightarrow h_\mathbf{A} = a^i \hat{p}_i + \alpha_i \hat{q}^i = \frac{\hbar}{i} a^i \frac{\partial}{\partial x^i} + \alpha_j x^j,$$

$$X \ni \mathbf{B} = \begin{pmatrix} b^i \\ \beta_j \end{pmatrix} \longrightarrow h_\mathbf{B} = b^i \hat{p}_i + \beta_j \hat{q}^j = \frac{\hbar}{i} b^i \frac{\partial}{\partial x^i} + \beta_j x^j,$$

$\mathbf{A}, \mathbf{B}$-vectors in $2N$-dimensional space $\rightarrow h_\mathbf{A}, h_\mathbf{B}$ operators on space of functions on N variables.

Symplectic scalar product $\rightarrow$ commutators

$$\langle \mathbf{A}, \mathbf{B} \rangle = a^i \beta_i - \alpha_j b^j = \frac{i}{\hbar} [h_\mathbf{A}, h_\mathbf{B}]$$

$$\left([\hat{p}_i, \hat{p}_j] = [\hat{q}^i, \hat{q}^j] = 0, [\hat{p}_i, \hat{q}^j] = \frac{\hbar}{i} \right)$$

$$h_{\mathbf{A}} \to \mathfrak{O}^{-1} h_{\mathbf{A}} \mathfrak{O} = h_{\mathbf{A}'}, \; h_{\mathbf{B}} \to \mathfrak{O}^{-1} h_{\mathbf{B}} \mathfrak{O} = h_{\mathbf{B}'},$$

$$\mathbf{A} \longrightarrow \mathbf{A}' = g(\mathbf{A}), \, \mathbf{B} \longrightarrow \mathbf{B}' = g(\mathbf{B}),$$

$$\langle \mathbf{A}, \mathbf{B} \rangle = \frac{i}{\hbar}[h_{\mathbf{A}}, h_{\mathbf{B}}] = \frac{i}{\hbar} \left[\mathfrak{O}^{-1} h_{\mathbf{A}} \mathfrak{O}, \mathfrak{O}^{-1} h_{\mathbf{B}} \mathfrak{O} \right] = \frac{i}{\hbar}[h_{\mathbf{A}'}, h_{\mathbf{B}'}] = \langle \mathbf{A}', \mathbf{B}' \rangle.$$

This transformation preserves symplectic scalar product.

spinor group $\ni \mathfrak{O} \to g \colon \mathbf{A} \to \mathbf{A}'$, $g \in$ symplectic group $Sp(n)$

Where are usual spinors?

$$SO(N) = Sp(-N)$$

We come to usual (orthogonal spinors) changing a parity

symplectic group $Sp(n)$	$-----$	Orthogonal group $O(n)$
$X = V \oplus V^*$	$---$	$\Pi X = \Pi V \oplus \Pi V^*$
symplectic space	$---$	Euclidean space
linear operator $h_\mathbf{A}$	$---$	linear operator $\gamma_\mathbf{A}$
symplectic group $Sp(N)$	$---$	Orthogonal group $O(N)$
acting on space of functions	$---$	acting on space of functions
of commuting coordinates	$---$	of anticommuting coordinates
$x_1, \ldots, x_N$	$---$	$\xi_1, \ldots, \xi_N$
Spinor representation	$---$	Spinor representation
is infinite dimensional	$---$	is finite dimensional

In the symplectic case spinors called metaplectic spinors.

Spinor group and thick morphisms

Return to thick morphisms

Spinor group $\{\mathfrak{O}\}$ can be defined as subgroup of qiantum thick morphisms corresponding to quadratic Hamiltonians.

Let M, N be two (super)manifolds. Recall that the classical action $S = S(x, q)$ connects Hamiltonian H_M on T^*M with Hamiltonian H_N on T^*N if

$$H_M \left(x, \frac{\partial S(x, q)}{\partial x} \right) \equiv H_N \left(\frac{\partial S(x, q)}{\partial q}, q \right).$$

Let $\Delta_M = H_M(\hat{x}, \hat{p})$ be a linear operator on $M =$ the quantum Hamiltonian (operator depending on $\hat{x} = x$ and $\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$, and respectively let $\Delta_N = H_N(\hat{y}, \hat{y})$ be a linear operator on $N =$ the quantum Hamiltonian (operator depending on $\hat{y} = y$ and $\hat{q} = \frac{\hbar}{i} \frac{\partial}{\partial y}$,

Definition

We say that the quantum thick morphism $\Phi_{S_h} \, M \Rrightarrow N$ connects operators Δ_M and Δ_N if the pull-back Φ_{S_h} of quantum thick morphism commutes with these operators. i.e.

$$\Delta_M \circ \Phi_{S_h}^* = \Phi_{S_h}^* \circ \Delta_N, \quad \left(\Delta_N = \left(\Phi_{S_h}^* \right)^{-1} \circ \Delta_M \Phi_{S_h}^* \right)$$

Quantum morpshims $\rightarrow$ classical morphisms

Theorem

Let $S_\hbar(x, q)$ be a quantum action such that quantum thick morphism $\Phi_{S_\hbar}$ connects quantum Hamiltonians Δ_M and Δ_N. then

- ▶ *classical thick morphism Φ_{S_0} defined by classical action $S_0(x, q) = \lim_{\hbar \to 0} S_\hbar$ connects classical Hamiltonians H_M and H_N (symbols of operators Δ_M and Δ_N).*

- ▶ *If Δ_M and Δ_N are operators, such that Hamiltonians (their symbols) H_M, H_N are linear then the condition that quantum thick morphism $\Phi_{S_\hbar}$ connects quantum Hamiltonians $\hat{H}_M$ and $\hat{H}_N$ does not depend on $\hbar$; in particular the condition that classical action connects two linear classical Hamiltonians is equivalent to the condition quantum version.*

Definition of spinor group in terms of thick morphisms

To define spinor group we have to consider thick morphisms corresponding to quadratic Hamiltonians/

Definition
Spinor group is the group of thick diffeomorphisms Φ_S corresponding to quadratic Hamiltonians.