Covariant Actions for Chiral $p-$forms

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Standard approach to free theories

The Lorentz-covariant field variable is taken in the same representation as that of the little group carried by the corresponding particle

Examples

- Trivial representation of the little group corresponds to the spin-zero particle. Lorentz covariant variable – scalar field.
- Vector representation of the little group corresponds to the spin-one particle and is described by a Lorentz vector field (Maxwell potential).
- Symmetric tensor of the little group corresponds to the spin-two particle and is described by the linearised Einstein equations (Fierz-Pauli) and is described by symmetric Lorentz tensor.

Particles and fields

Wigner classification of particles $\leftrightarrow$ field equations (unique?)

For massless spin-zero particle the simplest option is the Klein-Gordon equation

$$\Box\,\phi = 0$$

The scalar here is a single field that carries one degree of freedom: trivial representation of the massless little group. The Lagrangian is

$$\mathcal{L} \sim \frac{1}{2}\phi\,\Box\,\phi$$

Alternative

An alternative formulation of the scalar field is given by so-called Notoph Lagrangian by Ogievetsky and Polubarinov:

$$\mathcal{L} \sim \partial^{\mu} B_{\mu\nu}\partial_{\lambda} B^{\lambda\nu}$$

Duality

A $p-$form and its dual

The Lagrangian is given in the form of ("Maxwell Lagrangian")

$$\mathcal{L} \sim F \wedge \star F \,, \quad F = dA \,.$$

Massless $p-$form and a $(d-2-p)$-form fields describe correspondingly partilces of $p-$form and a $(d-2-p)$-form representations of the massless little group $ISO(d-2)$, which are dual to each other.

Attention!

Dual formulations do not admit the same interacting deformations!

Self-dual (Chiral) fields

There are special representations of the Poincaré algebra which are described by self-dual forms. The covariant equations describing such representations are given as:

$$F = \pm \star F\,, \quad F = dA$$

which implies the regular "Maxwell equations" $d \star F = 0$.

Lagrangian?

Lagrangian formulation of the (free) chiral fields has a long history. Siegel '84, Kavalov-Mkrtchyan '87, Florianini-Jackiw '87, Henneaux-Teitelboim '88, Harada '90, Tseytlin '90, McClain-Yu-Wu '90, Wotzasek '91, ..., Pasti-Sorokin-Tonin '95,...

Duality-symmetric fields

Maxwell action for $p-$forms and $(d-2-p)-$forms describes the same particle content.
When $d = 2p+2$, the dual variables are of the same type and the Maxwell action itself takes the same form in both variables.

Twisted self-duality equations

The Maxwell equations are equivalent to first-order equations involving both dual potentials:

$$F = \pm \star \tilde{F}\,, \quad F = dA\,, \quad \tilde{F} = d\tilde{A}$$

Duality-symmetric formulations

Zwanziger '70,..., Tseytlin'90, Schwarz-Sen '93, Gibbons-Rasheed '95, Pasti-Sorokin-Tonin '96, Cederwall-Westerberg '97, Ivanov-Zupnik '02, ..., Kuzenko-Theisen '00,...

Chiral $p-$forms in $d = 4k + 2$ Minkowski space

Minkowski vs Euclidean

Since $\star^2 = (-1)^{\sigma+p+1}$ where σ is the number of time directions, only even-forms can be self-dual (chiral) in Minkowski space.

$p = 2k$ forms in $d = 4k + 2$ dimensions

For even $p-$form potentials in special dimensions the corresponding particles are not irreducible but contain two irreps — chiral and anti-chiral halves.

New action for Chiral fields

The new Lagrangian

$$\mathcal{L} = -\frac{1}{2(p+1)}\, F_{\mu_1\dots\mu_{p+1}}\, F^{\mu_1\dots\mu_{p+1}} + G^{\mu\nu}\partial_{[\mu}c_{\nu]}$$
$$-\frac{1}{2(p+1)}\,(\mathcal{F}_{\mu_1\dots\mu_{p+1}} - (p+1)\, c_{[\mu_1}\, R_{\mu_2\dots\mu_{p+1}]})\times$$
$$\times\, (\mathcal{F}^{\mu_1\dots\mu_{p+1}} - (p+1)\, c^{[\mu_1}\, R^{\mu_2\dots\mu_{p+1}]})$$

where

$$F = d\varphi\,, \quad \mathcal{F} = F + \star F\,,$$

or, equivalently,

$$\mathcal{L} \sim -F\wedge\star F + \star G\wedge d\,c - (\mathcal{F} - c\wedge R)\wedge\star(\mathcal{F} - c\wedge R)$$

Equivalence to PST

Integrating out the auxiliary field

We solve the algebraic equation of motion for the field $R_{\mu_1\ldots\mu_p}$,

$$\mathcal{F}_{\mu_1\ldots\mu_{p+1}}\,c^{\mu_1} + (-1)^{p+1}\,p\,c_{[\mu_2}\,R_{\mu_3\ldots\mu_{p+1}]\mu_1}\,c^{\mu_1} - c^2\,R_{\mu_2\ldots\mu_{p+1}} = 0$$

as

$$R_{\mu_1\ldots\mu_p} = \frac{1}{c^2}\mathcal{F}_{\nu\mu_1\ldots\mu_p}\,c^\nu + c_{[\mu_1}\lambda_{\mu_2\ldots\mu_p]}$$

and plug back into the action to get:

$$\mathcal{L} = -\frac{1}{2(p+1)}\,F_{\mu_1\ldots\mu_{p+1}}\,F^{\mu_1\ldots\mu_{p+1}} + G^{\mu\nu}\partial_{[\mu}c_{\nu]} + \frac{1}{2\,c^2}\,\mathcal{F}_{\mu_1\ldots\mu_p\nu}\,c^\nu\,\mathcal{F}^{\mu_1\ldots\mu_p\rho}\,c_\rho$$

classically equivalent to the celebrated PST action.

Classically equivalent form

Integrating out the field G, we get a classically equivalent Lagrangian

$$\mathcal{L} = -\frac{1}{2(p+1)}\, F_{\mu_1\ldots\mu_{p+1}}\, F^{\mu_1\ldots\mu_{p+1}}$$
$$-\, \frac{1}{2(p+1)}\, (\mathcal{F}_{\mu_1\ldots\mu_{p+1}} - (p+1)\,\partial_{[\mu_1} a\, R_{\mu_2\ldots\mu_{p+1}]})\times$$
$$\times\, (\mathcal{F}^{\mu_1\ldots\mu_{p+1}} - (p+1)\,\partial^{[\mu_1} a\, R^{\mu_2\ldots\mu_{p+1}]})\,,$$

or

$$\mathcal{L} \sim -\,F \wedge \star F - (\mathcal{F} - d\,a \wedge\, R) \wedge \star(\mathcal{F} - d\,a \wedge\, R)\,,$$

Manifest double gauge symmetries

Rearranging the Lagrangian

After a field redefinition $\varphi_{\mu_1\ldots\mu_p} \to \varphi_{\mu_1\ldots\mu_p} + a\, R_{\mu_1\ldots\mu_p}$ can be rewritten in the form:

$$\mathcal{L} = -\frac{1}{2(p+1)}\,(F_{\mu_1\ldots\mu_{p+1}} + a\,Q_{\mu_1\ldots\mu_{p+1}})(F^{\mu_1\ldots\mu_{p+1}} + a\,Q^{\mu_1\ldots\mu_{p+1}})$$

$$-\frac{1}{(p+1)\,(p+1)!}\epsilon_{\mu_1\ldots\mu_{p+1}\nu_1\ldots\nu_{p+1}}\,a\,F^{\nu_1\ldots\nu_{p+1}}\,Q^{\mu_1\ldots\mu_{p+1}}\,,$$

where $Q_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1} R_{\mu_2\ldots\mu_{p+1}]}.$, or

$$\mathcal{L} \sim -(F + a\,Q) \wedge \star(F + a\,Q) - a\,F \wedge Q$$

where $F = d\,\varphi\,, \quad Q = d\,R\,.$

Yet another form of the action

Finally, one can rewrite the action in the form:

$$\mathcal{L} = -\frac{1}{2(p+1)}\, F_{\mu_1\ldots\mu_{p+1}}\, F^{\mu_1\ldots\mu_{p+1}} - \frac{1}{2(p+1)}\, (\mathcal{F}_{\mu_1\ldots\mu_{p+1}} + a\, Q_{\mu_1\ldots\mu_{p+1}})^2$$

We will use the notation:

$$Q^{\pm}_{\mu_1\ldots\mu_{p+1}} = Q_{\mu_1\ldots\mu_{p+1}} \pm \frac{1}{(p+1)!}\epsilon_{\mu_1\ldots\mu_{p+1}\nu_1\ldots\nu_{p+1}} Q^{\nu_1\ldots\nu_{p+1}}$$

for (anti)self-dual part of the $(p+1)-$form $Q_{\mu_1\ldots\mu_{p+1}}$.

Equations and PST symmetry

Combining the equations of motion E^{φ} , E^{R} for the fields $\varphi_{\mu_1 \dots \mu_p}$ and $R_{\mu_1 \dots \mu_p}$ one gets

$$\begin{aligned} E^{R}_{\mu_2 \dots \mu_{p+1}} + a\, E^{\varphi}_{\mu_2 \dots \mu_{p+1}} = \partial^{\mu_1} a\, P_{\mu_1 \dots \mu_{p+1}} = 0\,, \\ P_{\mu_1 \dots \mu_{p+1}} \equiv \mathcal{F}_{\mu_1 \dots \mu_{p+1}} + a\, Q^{+}_{\mu_1 \dots \mu_{p+1}}\,, \end{aligned}$$

which implies

$$P_{\mu_1 \dots \mu_{p+1}} = 0\,,$$

automatically satisfying the equation of motion E^{a} for the a field,

$$E^{a} = Q_{\mu_1 \dots \mu_{p+1}}\, P^{\mu_1 \dots \mu_{p+1}} = 0\,.$$

This indicates the existence of a PST like symmetry (shift for a).

A Generalisation

An interesting generalisation of the Lagrangian is:

$$\mathcal{L} = -\frac{1}{2}\, f(a)\, (\sqrt{a}\, F + \frac{1}{\sqrt{a}}\, Q)^2 + \, f(a)\, F \wedge \, Q\,.$$

For $f(a) \sim 1/a$, this Lagrangian is equivalent to the chiral one written earlier and describes a single chiral p-form carried in field φ. For $f(a) \sim a$, it describes an anti-chiral $p-$form field carried by R. The exchange $\varphi \leftrightarrow R\,, a \to -\frac{1}{a}\,, f(a) \to -f(a)$ is a symmetry of the Lagrangian.

$d = 2$

One can integrate out the c_μ field in the original action:

$$c_\mu = \frac{1}{R}\mathcal{F}_\mu + \frac{1}{R^2}\epsilon_{\mu\nu}\partial^\nu \tilde{r}\,,$$

and plugging back into action (renaming $\frac{1}{R} \to r$) to get:

$$S = \int \Big(-\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\,r^2\,\partial_\mu\tilde{r}\,\partial^\mu\tilde{r} - r\,\mathcal{F}^\mu\,\partial_\mu\tilde{r}\Big)\,d^2x\,.$$

Another parametrisation gives:

$$\mathcal{L}_\pm = -\frac{1}{8}\left[(r+1)\partial_\mu\varphi \pm (r-1)\partial_\mu\tilde{\varphi}\right]^2 + \frac{1}{4}\,\epsilon^{\mu\nu}\,r\,\partial_\mu\varphi\,\partial_\nu\tilde{\varphi}\,,.$$

where different signs correspond to different chiralities. Here $\varphi = \varphi_+ + \varphi_-$ and $\tilde{\varphi} = \varphi_+ - \varphi_-$. The two actions transform into each other under $\varphi \leftrightarrow \tilde{\varphi}\,,\ r \to -r$.

A compact form

The Lagrangian in a simpler form

$$\mathcal{L} \sim -\mathcal{M}_{IJ}\, F^I \wedge \star F^J - \mathcal{K}_{IJ}\, F^I \wedge F^J\,,$$

with

$$\mathcal{M}_{IJ} = \begin{bmatrix} 1 & a \\ a & a^2 \end{bmatrix}, \quad \mathcal{K}_{IJ} = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}, \quad F^I = \begin{bmatrix} F \\ Q \end{bmatrix},$$

where F^I is a two-vector with $p+1$-form components. $\mathcal{M}$ is of rank one. The "background matrix" $\mathcal{E} = \mathcal{M} + \mathcal{K}$ is invertible.

An observation

The same action with the inverted background matrix $\mathcal{E}^{-1}$ describes the same degrees of freedom, exchanging the roles of φ and R.

Duality-symmetric Electromagnetism

The Lagrangian for a single massless spin-one field

$$\mathcal{L} = -\frac{1}{8}\, F^a_{\mu\nu}\, F^{a\,\mu\nu} + G^{\mu\nu}\, \partial_{[\mu} c_{\nu]}$$
$$-\frac{1}{8}\, (\mathcal{F}^a_{\mu\nu} - 2\, c_{[\mu}\, R^a_{\nu]})(\mathcal{F}^{a\,\mu\nu} - 2\, c^{[\mu}\, R^{a\,\nu]})\,,$$

where $a, b = 1, 2$, and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu\,, \quad \mathcal{F}^a_{\mu\nu} = F^a_{\mu\nu} - \frac{1}{2}\epsilon_{ab}\, \varepsilon_{\mu\nu\lambda\rho} F^{b\,\lambda\rho}\,,$$
$$\epsilon_{ab} = -\epsilon_{ba}\,, \quad \epsilon_{12} = 1 = \epsilon^{12}\,, \quad \varepsilon_{0123} = 1 = -\varepsilon^{0123}\,.$$

The following identities hold (Einstein summation rule is assumed for both types of indices):

$$\mathcal{F}^a_{\mu\nu}\, \mathcal{F}^{a\,\mu\nu} = 0\,, \quad \varepsilon_{\mu\nu\lambda\rho} \mathcal{F}^{a\,\lambda\rho} = 2\, \epsilon^{ab}\, \mathcal{F}^b_{\mu\nu}\,,$$

Other form of the action

$$\mathcal{L} = -\frac{1}{8}\, F^a_{\mu\nu}\, F^{a\,\mu\nu} - \frac{1}{8}\, (\mathcal{F}^a_{\mu\nu} + a\, Q^a_{\mu\nu})\, (\mathcal{F}^{a\,\mu\nu} + a\, Q^{a\,\mu\nu})$$

where $Q^a_{\mu\nu} = \partial_\mu\, R^a_\nu - \partial_\nu\, R^a_\mu$. This Lagrangian describes a single Maxwell field, using four vectors and a scalar. It can be written as:

$$\mathcal{L} = -\frac{1}{8}\, \mathcal{M}_{IJ}\, F^I_{\mu\nu}\, F^{J\,\mu\nu} - \frac{1}{16}\, \mathcal{K}_{IJ} \epsilon^{\mu\nu\alpha\beta}\, F^I_{\mu\nu}\, F^J_{\alpha\beta}\,,$$

where

$$\mathcal{M}_{IJ} = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & 0 & a \\ a & 0 & a^2 & 0 \\ 0 & a & 0 & a^2 \end{bmatrix}, \quad \mathcal{K}_{IJ} = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \\ 0 & -a & 0 & 0 \\ a & 0 & 0 & 0 \end{bmatrix}, \quad F^I = \begin{bmatrix} F^1 \\ F^2 \\ Q^1 \\ Q^2 \end{bmatrix}$$

Set of long-standing problems.

A list of related problems

- Duality-symmetric formulation for non-abelian gauge theory.
- Interacting theory of non-abelian Chiral (self-dual) $p-$forms.

Thank you for your attention!