Source and Response Soft Charges for Maxwell Theory on AdS$_d$

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arXiv:1908.10385 **[hep-th]**

De Sitter slicing of AdS

$$-X_{-1}^2 + X_a X^a = -\ell^2, \qquad a = 0, \cdots, d-1$$

$$X^a X_a \equiv -(X^0)^2 + X^i X_i = \rho^2 > 0, \qquad X_{-1}^2 = \ell^2 + \rho^2 .$$

$$X^0 = \rho \tan\tau, \qquad \sqrt{X^i X_i} = \frac{\rho}{\cos\tau} .$$

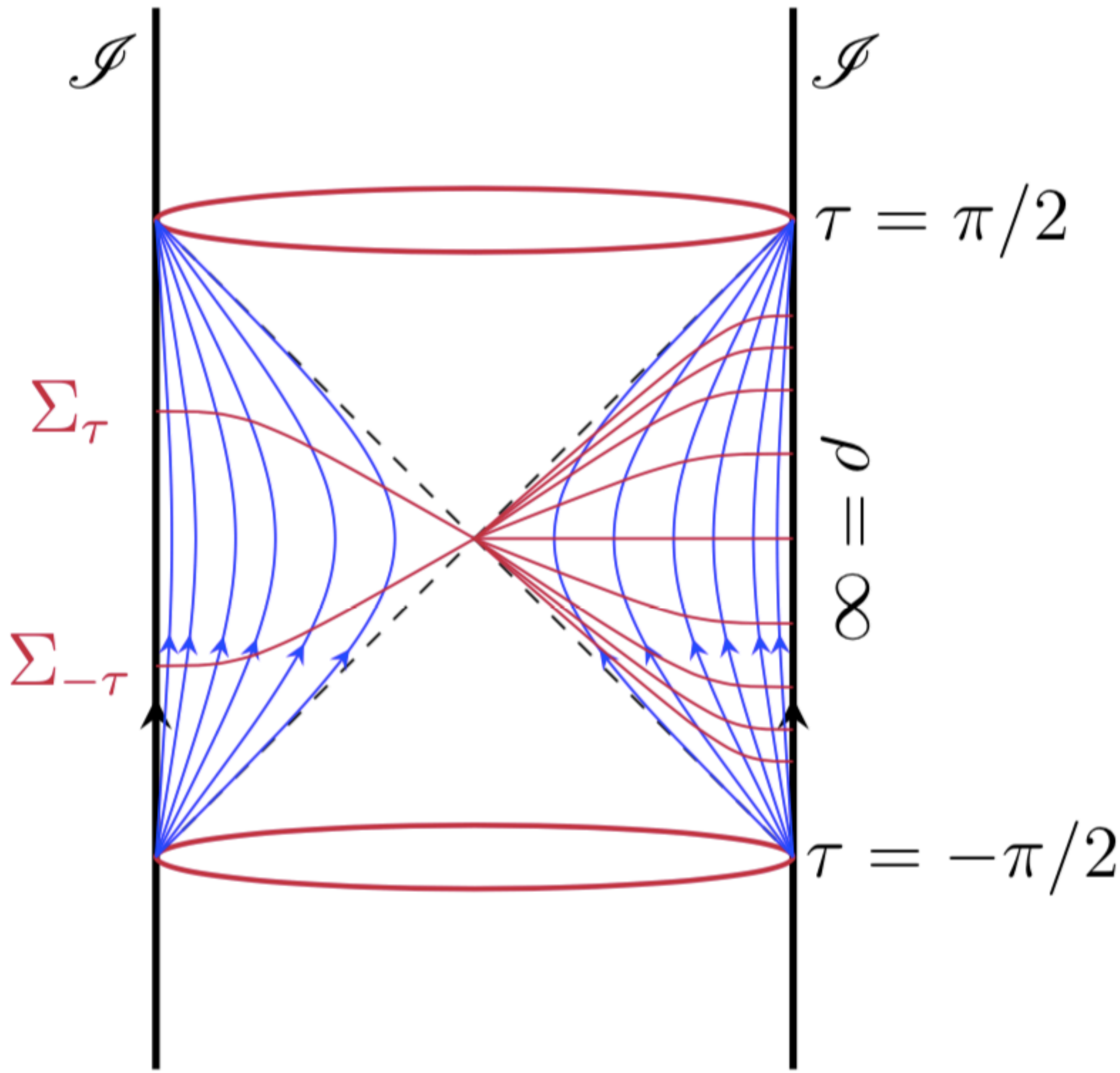
$$ds^2 = \frac{\ell^2 d\rho^2}{\rho^2 + \ell^2} + \rho^2 h_{\mu\nu} dx^\mu dx^\nu, \qquad \mu, \nu = 0, 1, \cdots, d-2 ,$$

$$h_{\mu\nu} dx^\mu dx^\nu = \frac{1}{\cos^2\tau}\Big(- d\tau^2 + d\Omega_{d-2}^2 \Big),$$

Isometries

Lorentz: ξ_L $\qquad L^{ab} = 2\bar{X}^{[a} \bar{M}_\mu^{b]} h^{\mu\nu} \partial_\nu$

AdS-**Translation:** ξ_T $\qquad L^{-1a} = -X_{-1} \bar{X}^a \, \partial_\rho - \frac{X_{-1}}{\rho} \bar{M}_\mu^a h^{\mu\nu} \partial_\nu$



$\mathscr{I}$
$\mathscr{I}$
$\tau = \pi/2$
Σ_τ
$\rho = \infty$
$\Sigma_{-\tau}$
$\tau = -\pi/2$

Action

$$S = -\frac{1}{4}\int d^d x \sqrt{-g}\mathcal{F}_{ab}\mathcal{F}^{ab},$$

Symplectic from

$$\Omega_\tau^{\text{bulk}} = \int_{\Sigma_\tau} \sqrt{g}\tau_a \delta\mathcal{F}^{ab}\delta\mathcal{A}_b$$

Dynamical equations

$$\begin{aligned} \text{Constraints:} \quad & D_\mu \mathcal{F}^{\mu\rho} = 0 \\ \text{Radial evolution equations:} \quad & (\rho^2+\ell^2)^{1/2}\rho^{1-d}\partial_\rho\left(\rho^{d-1}(\rho^2+\ell^2)^{-1/2}\mathcal{F}^{\rho\mu}\right) + D_\nu \mathcal{F}^{\nu\mu} = 0 \end{aligned}$$

Boosted electric charge. The simplest family of solutions is found by assuming $\mathcal{A}_\mu = \partial_\mu \mathcal{C}$ for which $\mathcal{F}_{\mu\nu} = 0$. By closedness of $\mathcal{F}_{ab}$ we have

$$\partial_\mu \mathcal{F}_{\nu\rho} - \partial_\nu \mathcal{F}_{\mu\rho} = 0. \tag{3.4}$$

Then, $\mathcal{F}_{\mu\rho} = \partial_\mu \psi$ for a gauge-invariant scalar ψ,

$$\psi = \mathcal{A}_\rho - \partial_\rho \mathcal{C}\,. \tag{3.5}$$

Equation (3.3) reduces to

$$\frac{\sqrt{\rho^2+\ell^2}}{\rho^{d-1}}\partial_\rho \left(\rho^{d-1}(\rho^2+\ell^2)^{-1/2}\mathcal{F}^{\rho\mu}\right) = 0. \tag{3.6}$$

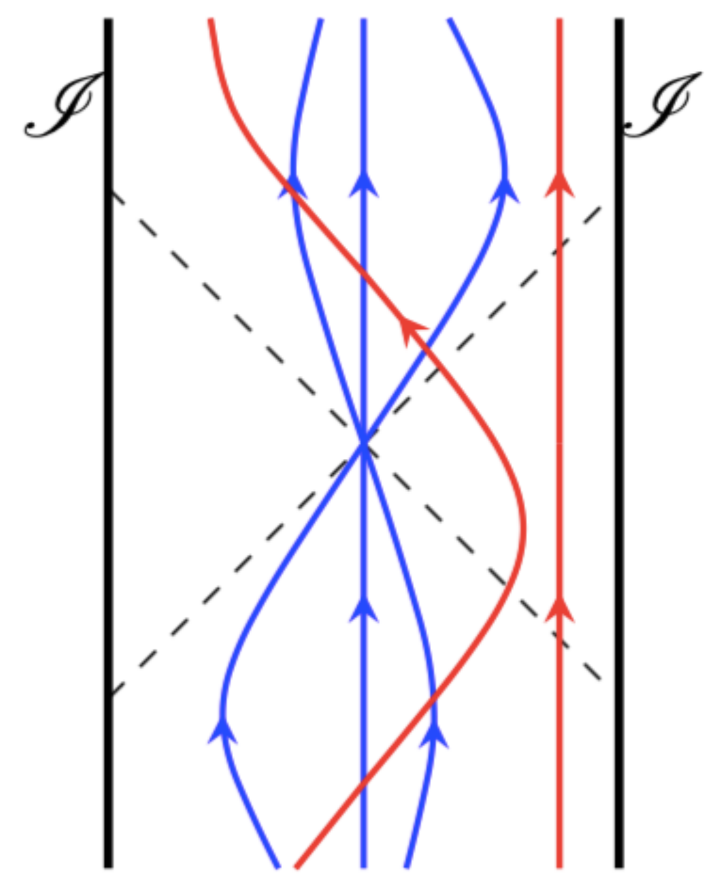
and the solution is (see appendix A)

$$\psi = \rho^{3-d}(\ell^2+\rho^2)^{-1/2}\bar{\psi}(x^\nu), \qquad D^\mu D_\mu \bar{\psi}(x^\nu) = 0 \tag{3.7}$$

This is the exact solution for an electric charge with an arbitrary boost, which crosses the event at the origin of AdS (no dipole moment). To see this, consider a static electric charge at the origin, which in hyperbolic coordinates is given by

$$\mathcal{F}_{\tau\rho} = \frac{\rho^{3-d}\cos^{d-3}\tau}{\sqrt{\rho^2+\ell^2}} = \partial_\tau \psi(\rho,\tau). \tag{3.8}$$

ψ is a Lorentz scalar and under boosts, it acquires angle-dependence so that other components of $\mathcal{F}_{\mu\rho}$ are also turned on. The phase space of all superpositions of boosted electric charges consists of all functions $\psi(\rho, x^\mu)$ satisfying conditions (3.7). Finally, note that near the origin, $\rho \ll \ell$, the behavior is $\psi \propto \rho^{3-d}$ which matches flat space solutions [22, 26]. Near the boundary, however, the behavior is weaker $\psi \propto \rho^{2-d}$.



Displaced electric charge. Let us now relax the condition $\mathcal{A}_\mu = \partial_\mu \mathcal{C}$. Fixing the $\mathcal{A}_\rho = 0$ gauge, the constraint equation (3.2) implies that $\mathcal{A}_\mu$ is a divergenceless de Sitter vector $D^\mu \mathcal{A}_\mu = 0$. Spectrum of the Laplace operator on $\mathcal{A}_\mu$ is given by representation theory of $SO(d-1,1)$ (see appendix A for a derivation using differential equations):

$$D^\nu \mathcal{F}^{[\beta]}_{\nu\mu} = -\beta(\beta + d - 4)\mathcal{A}^{[\beta]}_\mu \,. \tag{3.9}$$

Eq. (3.3) then becomes an equation for the eigen-vector $\mathcal{A}^{[\beta]}_\mu$:

$$\frac{\sqrt{\rho^2+\ell^2}}{\ell^2 \rho^{d-5}} \partial_\rho \left(\rho^{d-3}(\rho^2+\ell^2)^{1/2} \partial_\rho \mathcal{A}^{[\beta]}_\mu\right) - \beta(\beta+d-4)\mathcal{A}^{[\beta]}_\mu = 0. \tag{3.10}$$

Exact expressions can be expressed in terms of hypergeometric functions. We are only interested in two asymptotic limits $\rho \ll \ell$ and $\rho \gg \ell$.

$$\mathcal{A}_t = \frac{\mathcal{Y}_{\beta,m_i}}{r^{\beta+d-3}},$$

where $\mathcal{Y}_{\beta,m_i}$ are harmonics on S^{d-2}, *cf.* Appendix A for conventions. In hyperbolic coordinates (which is related to global coordinates by (2.7)), and in radial gauge $\mathcal{A}_\rho = 0$ this potential is given by

$$\mathcal{A}_\tau = \frac{3-\beta-d}{4-\beta-d}\cos^{d+\beta-3}\tau\, \rho^{4-d-\beta}\mathcal{Y}_{\beta,m_i} \qquad \mathcal{A}_i = -\frac{\tan\tau\cos^{d+\beta-3}\tau}{4-\beta-d}\rho^{4-\beta-d}\partial_i \mathcal{Y}_{\beta,m_i}.$$

Explicitly, looking for solutions to $\mathcal{L}_t \mathcal{A} = 0$ in the $\mathcal{A}_\rho = 0$ gauge, we find the above expressions for the fields in which β is quantized. Straightforward algebra shows that

$$D^i \mathcal{F}_{i\tau} = -\beta(\beta+d-4)\mathcal{A}_\tau \,.$$

This equality holds also in a boosted frame by Lorentz covariance. Thus, $\mathcal{A}^{[\beta]}_\mu$ eigen-vectors with integral values for β correspond to electric 2^β-poles.

Flat region $\rho \ll \ell$. The two solutions for eigen-vector fields (3.9) are

$$\mathcal{A}_\mu^{[\beta]} \propto \rho^\beta, \qquad \mathcal{A}_\mu^{[\beta]} \propto \rho^{4-d-\beta}. \tag{3.11}$$

For integer β, the solutions with $\rho^{4-d-\beta}$ falloff are of course recognized as the multipole moments of the electromagnetic field [26, 29, 30]. Note that (3.9) is Lorentz covariant. Therefore, the eigen-vector field $\mathcal{A}_\mu^{[\beta]}$ describes a system of 2^β-poles moving freely in flat space.[3]

Boundary region $\rho \gg \ell$. For each eigen-vector field $\mathcal{A}_\mu^{[\beta]}$ one employs a near boundary expansion

$$\mathcal{A}_\mu^{[\beta]} = \sum \rho^s A_\mu^{[\beta](s)}. \tag{3.12}$$

Equation (3.10) leads to a recursive relation

$$s(d+s-3)A_\mu^{[\beta](s)} = \beta(\beta+d-4)A_\mu^{[\beta](s+2)} \tag{3.13}$$

Although the recursive relation depends on the moment β (and the exact solution is a hypergeometric function depending on β and d), the leading term is universal in β

$$\mathcal{A}_\mu^{[\beta]} \sim \mathcal{O}(1), \qquad \mathcal{A}_\mu^{[\beta]} \sim \mathcal{O}(\rho^{3-d}) \qquad \forall\beta \tag{3.14}$$

Falloff/Boundary conditions

$$\mathcal{A}_\rho = \frac{A_\rho}{\rho^{d-2}} + \frac{A_\rho^{(3-d)}}{\rho^{d-3}} + \cdots$$

$$\mathcal{A}_\mu = \partial_\mu \Phi + \frac{A_\mu}{\rho^{d-3}} + \frac{A_\mu^{(4-d)}}{\rho^{d-4}} + \cdots$$

$$\lambda = \lambda_S + \frac{\hat{\lambda}}{\rho^{d-3}} + \cdots$$

As in the usual AdS/CFT [18], we fix the radial gauge by setting $\mathcal{A}_\rho = 0$. This gauge is accessible by setting $A_\rho = (d-3)\hat{\lambda}$ and correspondingly for subleading orders, while the leading gauge parameter λ_S is unconstrained. As we see this gauge fixing is not possible in 3d case, as in this case the electric charge yields a logarithmic function of ρ; the 3d should be studied separately. Here we focus on $d > 3$ case.

$$A_\mu = \ell(\partial_\mu \Psi + \hat{A}_\mu), \qquad D^\mu \hat{A}_\mu = 0.$$

E, o, M $\longrightarrow$ $D_\mu D^\mu \Psi = 0.$

$$\delta_\lambda \Phi = \lambda_S, \qquad \delta_\lambda \Psi = \lambda_R, \qquad \delta_\lambda \hat{A}_\mu = -\partial_\mu \lambda_R, \qquad D_\mu D^\mu \lambda_R = 0,$$

Action Principle

$$\delta S = \int_{AdS} \sqrt{g} \nabla_a \mathcal{F}^{ab} \delta \mathcal{A}_b + \int_{AdS} \partial_a \theta^a,$$

$$\delta S \approx - \int_B \sqrt{g} \mathcal{F}^{\rho\mu} \delta \mathcal{A}_\mu = (d-3) \int_B \sqrt{h} (D_\mu \Psi D^\mu \delta \Phi + \hat{A}_\mu D^\mu \delta \Phi)$$
$$= (d-3) \int_B \sqrt{h} D^\mu \Big(\Psi D_\mu \delta \Phi + \hat{A}_\mu \delta \Phi \Big) - (d-3) \int_B \sqrt{h} \Big(\Psi D^\mu D_\mu \delta \Phi + D^\mu \hat{A}_\mu \delta \Phi \Big)$$

$$D^\mu D_\mu \Phi = 0,$$

Conserved Symplectic form

$$\Omega_\tau^{\text{bulk}} = \int_{\Sigma_\tau} \sqrt{g}\tau_a \delta\mathcal{F}^{ab}\delta\mathcal{A}_b$$

$$\Omega_{\tau_2}^{\text{bulk}} - \Omega_{\tau_1}^{\text{bulk}} = \int_{B_{12}} \omega^\rho = \int_{B_{12}} \sqrt{h} D_\mu \delta\kappa^\mu + \int_{B_{12}} \omega^{\text{flux}}$$

$$\Omega \equiv \int_{\Sigma_\tau} \omega + \oint_{\partial\Sigma_\tau} \omega_{\text{b'dry}} = \int_{\Sigma_\tau} \sqrt{g}\tau_a \delta\mathcal{F}^{ab}\delta\mathcal{A}_b + (3-d)\oint_{\partial\Sigma_\tau} \sqrt{h}\tau_\mu \Big(\delta\Psi D^\mu \delta\Phi + \delta\hat{A}^\mu \delta\Phi\Big)$$

Conserved Symplectic form, On-shell

$$\mathcal{A}_a = \partial_a \Phi + \bar{\mathcal{A}}_a$$

$$\Omega = \int_{\Sigma_\tau} \sqrt{g}\tau_a \delta \mathcal{F}^{ab} \delta \bar{\mathcal{A}}_b + (3-d) \oint_{\partial \Sigma_\tau} \sqrt{h}\, \tau_\mu \Big(\delta \Psi D^\mu \delta \Phi + \delta \Phi D^\mu \delta \Psi \Big).$$

$$S_{\text{b'dry}} = (3-d) \int_B \sqrt{h}\, \partial_\mu \Psi \partial^\mu \Phi,$$

$$T^{\text{b'dry}}_{\mu\nu} = \frac{-2}{\sqrt{h}} \frac{\delta S^{\text{b'dry}}}{\delta h^{\mu\nu}} = (d-3) \Big[2\partial_{(\mu} \Psi \partial_{\nu)} \Phi - h_{\mu\nu} \partial \Psi \cdot \partial \Phi \Big]$$

Physical gauge transformations

$$\delta Q_\lambda = \Omega(\cdot, \delta_\lambda).$$

$$Q^S_\lambda[\Psi] = \oint_{\partial\Sigma_\tau} \sqrt{h}\tau_\mu \Big(\lambda_S D^\mu \Psi - \Psi D^\mu \lambda_S\Big),$$
$$Q^R_\lambda[\Phi] = \oint_{\partial\Sigma_\tau} \sqrt{h}\tau_\mu \Big(\lambda_R D^\mu \Phi - \Phi D^\mu \lambda_R\Big),$$

$$\{Q^S_\lambda, Q^S_\chi\} = 0, \qquad \{Q^R_\lambda, Q^R_\chi\} = 0,$$
$$\{Q^S_\lambda, Q^R_\chi\} = -\Omega(\delta_{\lambda_S}, \delta_{\chi_R}) = \oint_{\partial\Sigma_\tau} \sqrt{h}\; \tau_\mu(\lambda D^\mu \chi - \chi D^\mu \lambda).$$

$$\{Q^{S,\sigma}_{l,m_i}, Q^{R,\sigma'}_{l',m'_i}\} = 2i\; \delta(\sigma\sigma' + 1)\; \delta_{l,l'}\delta_{m_i,m'_i},$$

Scaling symmetry

$$\Phi \to M^{-1}\Phi, \qquad \Psi \to M\Psi.$$

$$\mathbf{\Delta} \equiv \oint \sqrt{h}\tau_\mu(\Psi D^\mu \Phi - \Phi D^\mu \Psi),$$

$$\{\mathbf{\Delta}, Q^S_\lambda\} = -Q^S_\lambda, \qquad \{\mathbf{\Delta}, Q^R_\lambda\} = Q^R_\lambda.$$

Boundary gauge transformations and integrability of AdS isometry charges

$$\delta_{\xi_L}(D_\nu D^\nu \Phi) = \mathcal{L}_{\xi_L}(D_\nu D^\nu \Phi)$$
$$\delta_{\xi_L}(D_\nu \hat{A}^\nu) = \mathcal{L}_{\xi_L}(D_\nu \hat{A}^\nu),$$

$$\delta_{\xi_T}(D_\nu D^\nu \Phi) = D_\nu[\bar{\xi}_T^\nu D_\mu D^\mu \Phi + (d-3)\bar{\xi}_T^\rho D^\nu \Phi]$$
$$\delta_{\xi_T}(D_\nu \hat{A}^\nu) = D_\nu[\bar{\xi}_T^\nu D_\mu \hat{A}^\mu + (d-3)\Psi D^\nu \bar{\xi}_T^\rho].$$

$$(\delta_\alpha + \delta_{\xi_T})\Phi = 0, \qquad (\delta_\beta + \delta_{\xi_T})\Psi = 0.$$

$$\hat{\delta}_{\xi_T} D^\nu D_\nu \Phi = \delta_{\xi_T} D^\nu D_\nu \Phi - D^2 \alpha = 0,$$
$$\hat{\delta}_{\xi_T} D^\nu \hat{A}_\nu = \delta_{\xi_T} D^\nu \hat{A}_\nu + D^2 \beta = D_\nu \left[\bar{\xi}_T^\nu (D^\mu \hat{A}_\mu + D^\mu D_\mu \Psi)\right] \approx 0,$$

Lorentz charges are integrable. The action of Lorentz subgroup of isometries defined by Lie derivative enjoys integrable charges. One can see that by computing phase space Lie derivative of the symplectic form under Lorentz transformations,

$$\mathbb{L}_{\xi_L}\Omega = \oint \sqrt{h}\tau_\nu \xi^\nu_L \omega^{\text{flux}}, \tag{6.23}$$

where ω^{flux} is defined in (4.6) and is vanishing on our phase space. Therefore, δ_{ξ_L} has an integrable charge on the constructed phase space and is computed to be (see (B.7)),

$$I_{\xi_L} = \int_{\Sigma_\tau} \sqrt{g}\tau_a \xi^L_b T^{ab} + (3-d)\oint_{\partial\Sigma_\tau} \sqrt{h}\tau_\nu \left[\Psi D^\nu(\bar{\xi}^\mu_L D_\mu \Phi) - D^\nu \Psi \bar{\xi}^\mu_L D_\mu \Phi\right]. \tag{6.24}$$

The notable point here is the boundary term containing the boundary data Ψ and Φ. Using our gauge conditions and Killing equations, one can easily show that the boundary term is nothing but the boundary energy momentum tensor introduced in (4.10) contracted by ξ_L. So, the Lorentz charges can be recast as,

$$I_{\xi_L} = \int_{\Sigma_\tau} \sqrt{g}\tau_a \xi^L_b T^{ab} + \oint_{\partial\Sigma_\tau} \sqrt{h}\tau_\nu \bar{\xi}^L_\mu T^{\mu\nu}_{\text{b'dry}}. \tag{6.25}$$

AdS-Translation charges are not integrable. Under AdS-translations the symplectic form transforms as (see appendix B for details),

$$\mathbb{L}_{\xi_T}\Omega = \oint \sqrt{h}\ \tau_\nu \xi_T^\nu \omega^{\text{flux}} - (d-3)^2 \oint \sqrt{h}\ \tau_\nu \xi_T^\nu\ \delta\Psi\delta\Phi. \tag{6.26}$$

This proves non-integrability of AdS-translations, even if $\omega^{\text{flux}} = 0$.

Improved AdS-translations are integrable. In section 6.2.1 we replaced δ_{ξ_T} by $\hat{\delta}_{\xi_T} = \delta_{\xi_T} + \delta_{\alpha\beta}$ which leaves the boundary gauge conditions invariant. Detailed computations in appendix B show that the transformation $\hat{\delta}_{\xi_T}$ defined above leaves the boundary data Ψ and Φ invariant and importantly, has integrable well-defined canonical charge. In other words, using the equations of motion we can show that $\hat{\delta}_{\xi_T}$ leaves our phase space invariant and is also a canonical transformation on it,

$$\mathbb{L}_{\hat{\xi}}\Omega = (\mathbb{L}_{\xi_T} + \mathbb{L}_{\alpha,\beta})\Omega \approx 0. \tag{6.27}$$

This leads us to,

$$\Omega(\cdot, \hat{\delta}_{\xi_T}) = \delta I_{\hat{\xi}_T} \tag{6.28}$$

where the charge $I_{\hat{\xi}_T}$ turns out to be (see (B.11)),

$$I_{\hat{\xi}_T} = \int_{\Sigma_\tau} \sqrt{g} \tau_a \xi_b^T T^{ab}. \tag{6.29}$$

This is a plausible conclusion since the boundary data are invariant under the improved AdS-translations.

Algebra

$$\{I_{\hat{\xi}_T}, I_{\hat{\zeta}_T}\} = I_{\widehat{[\xi_T,\zeta_T]}}, \qquad \{I_{\zeta_L}, I_{\xi_L}\} = I_{[\xi_L,\zeta_L]}, \qquad \{I_{\xi_L}, I_{\hat{\zeta}_T}\} = I_{\widehat{[\xi_L,\zeta_T]}}$$

$$\{I_{\hat{\xi}_T}, Q^S_\lambda\} = 0, \qquad \{I_{\hat{\xi}_T}, Q^R_\lambda\} = 0$$

$$\{I_{\xi_L}, Q^S_\lambda\} = Q^S_{\mathcal{L}_{\xi_L}\lambda}, \qquad \{I_{\xi_L}, Q^R_\lambda\} = Q^R_{\mathcal{L}_{\xi_L}\lambda}.$$

$$\{\mathbf{\Delta}, I_{\xi_L}\} = 0, \qquad \{\mathbf{\Delta}, I_{\hat{\xi}_T}\} = 0.$$

Thank you