Quantum matrix algebras: a review

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Plan

Let V be a vector space over the field $\mathbb{C}$ and P be the usual flip acting in $V^{\otimes 2}$ or its matrix.
Also, let $M = (m_i^j)$ be a numerical $N \times N$ matrix. Consider the system

$$P\, M_1\, M_2 - M_1\, M_2\, P = 0, \quad M_1 = M \otimes I, \quad M_2 = I \otimes M.$$

Note that $M_2 = P\, M_1\, P$ and consequently, this system can be cast under the form

$$P\, M_1\, P\, M_1 - M_1\, P\, M_1\, P = 0.$$

This system written via the entries reads

$$m_i^j\, m_k^l = m_k^l\, m_i^j, \quad \forall\, i, j, k, l,$$

i.e. the entries commute with each other.

Example $N = 2$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

$$M_1 = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}, \quad M_2 = \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}.$$

The corresponding system reads

$$ab = ba, \; ac = ca, \ldots$$

Let us introduce some symmetric polynomials of M (namely, elementary ones and power sums)

$$\det(M - t\,I) = \sum_0^N (-t)^{N-k} e_k(M), \;\; p_k(M) = Tr\, M^k.$$

If M is a triangular matrix these elements are respectively elementary symmetric polynomials and power sums in the eigenvalues μ_i of M. Namely, we have

$$e_k = \sum_{i_1 < \ldots < i_k} \mu_{i_1} \ldots \mu_{i_k}, \;\; p_k(M) = \sum \mu_i^k.$$

Also, note that these symmetric polynomials of M are related by the Newton identities

$$k\, e_k - p_1\, e_{k-1} + p_2\, e_{k-2} + \cdots + (-1)^k p_k\, e_0 = 0.$$

Together with the initial system $P\,M_1\,P\,M_1 - M_1\,P\,M_1\,P = 0$ consider its inhomogeneous analog

$$P\,M_1\,P\,M_1 - M_1\,P\,M_1\,P = P\,M_1 - M_1\,P.$$

In terms of the entries we have the relations

$$m_i^j\,m_k^l - m_k^l\,m_i^j = m_i^l\,\delta_k^j - m_k^j\,\delta_i^l,$$

which define the enveloping algebra $U(gl(N))$.

Note that if in the homogeneous (inhomogeneous) system we replace P by the super-flip $P_{m|n}$, we get the defining relations of the super-commutative algebra $Sym(gl(m|n))$ (resp., the enveloping algebra $U(gl(m|n))$).

Now, deform $P \to R$ in the corresponding systems–homogeneous and not. And do the same with the super-flip $P_{m|n}$. Namely, take R as follows (here $N = 2, \quad m = n = 1$)

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q - q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -q^{-1} \end{pmatrix}.$$

Note that for $q \to 1$ we respectively recover the flip P and the super-flip $P_{1|1}$.

If we deform the system $P\,M_1\,P\,M_1 - M_1\,P\,M_1\,P = 0$ and its inhomogeneous analog, we get

$$R\,M_1\,R\,M_1 - M_1\,R\,M_1\,R = 0.$$

$$R\,M_1\,R\,M_1 - M_1\,R\,M_1\,R = R\,M_1 - M_1\,R.$$

The first one will be called Reflection Equation (RE) algebra. The second one–modified RE algebra.

If we deform P in the system $P\,M_1\,M_2 - M_1\,M_2\,P = 0$, we get

$$R\,M_1\,M_2 - M_1\,M_2\,R = 0 \;\Leftrightarrow\; R\,M_1\,P\,M_1\,P - M_1\,P\,M_1\,P\,R = 0.$$

This algebra will be called RTT algebra.

Note that all these algebras make sense for some other braidings R. Question: for what R deforming P these algebras are deformations of commutative ones?

We call an invertible linear operator $R : V^{\otimes 2} \to V^{\otimes 2}$ *braiding* if it satisfies the so-called *braid relation*

$$R_{12}\, R_{23}\, R_{12} = R_{23}\, R_{12}\, R_{23}, \quad R_{12} = R \otimes I, \;\; R_{23} = I \otimes R.$$

Then the operator $\mathcal{R} = R\,P$ where P is the usual flip is subject to the QYBE

$$\mathcal{R}_{12}\, \mathcal{R}_{13}\, \mathcal{R}_{23} = \mathcal{R}_{23}\, \mathcal{R}_{13}\, \mathcal{R}_{12}.$$

A braiding R is called *involutive symmetry* if $R^2 = I$.
A braiding is called *Hecke symmetry* if it is subject to the Hecke condition

$$(q\,I - R)(q^{-1}\,I + R) = 0,\ q \in \mathbb{C},\ q \neq 0,\ q \neq \pm 1.$$

In particular, such a symmetry comes from the QG $U_q(sl(N))$. For $N = 2$ it is just the example above.

We assume q to be generic. This means that $k_q \neq 0$ for any integer k.

As for the braidings coming from the QG of other series B_n, C_n, D_n, each of them has 3 eigenvalues and it is called BMW symmetry.

In order to classify Hecke symmetries, consider "R-symmetric" and "R-skew-symmetric" algebras

$$Sym_R(V) = T(V)/\langle Im(qI - R)\rangle, \ \bigwedge\nolimits_R(V) = T(V)/\langle Im(q^{-1}I + R)\rangle,$$

where $T(V)$ is the free tensor algebra. Also, consider the corresponding Poincaré-Hilbert series

$$P_+(t) = \sum_k dim\, Sym_R^{(k)}(V) t^k, \ P_-(t) = \sum_k dim \bigwedge\nolimits_R^{(k)}(V) t^k,$$

where the upper index (k) labels homogenous components of these quadratic algebras.
If R is involutive, we put $q = 1$ in these formulae.

Example

Let us compare two symmetries. The first one is Hecke coming from $U_q(sl(2))$, the second one is involutive:

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For the first (resp., second) symmetry we have

$$Sym_R = T(V)/ < xy - qyx >, \bigwedge\nolimits_R = T(V)/ < x^2, y^2, qxy + yx > .$$

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Observe that the algebras $Sym_R(V)$ are similar, but $\bigwedge_R(V)$ are not.

One example more. Consider an involutive symmetry

$$\begin{pmatrix} 1 & a & -a & ab \\ 0 & 0 & 1 & -b \\ 0 & 1 & 0 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$Sym_R(V) = T(V)/ < xy - yx + b\,y^2 >,$$

$$\bigwedge_R(V) = T(V)/ < x^2 + \frac{a}{2}(xy - yx), xy + yx, y^2 > .$$

If $b = 0$, $a \neq 0$, the algebra $Sym_R(V)$ is usual but $\bigwedge_R(V)$ is not.

The following holds $P_-(-t)P_+(t) = 1$.

Proposition. (Phung Ho Hai)

The HP series $P_-(t)$ (and hence $P_+(t)$) is a rational function:

$$P_-(t) = \frac{N(t)}{D(t)} = \frac{1 + a_1\, t + \ldots + a_r\, t^r}{1 - b_1\, t + \ldots + (-1)^s\, b_s\, t^s} = \frac{\prod_{i=1}^r (1 + x_i t)}{\prod_{j=1}^s (1 - y_j t)},$$

where a_i and b_i are positive integers, the polynomials $N(t)$ and $D(t)$ are coprime, and all the numbers x_i and y_i are real positive.

We call the couple $(r|s)$ bi-rank. In this sense all involutive and Hecke symmetries are similar to super-flips, for which the role of the bi-rank is played by the super-dimension $(m|n)$.

Examples. If R comes from the QG $U_q(sl(m))$, then

$$P_-(t) = (1+t)^m.$$

If R is a deformation of the super-flip $P_{m|n}$, then

$$P_-(t) = \frac{(1+t)^m}{(1-t)^n}.$$

Also, there exist "exotic" examples: for any $N \geq 2$ there exits a Hecke symmetry such that

$$P_-(t) = 1 + Nt + t^2.$$

Here $dimV = N$, the bi-rank is $(2|0)$.
If $P_-(t)$ is a polynomial, i.e. the bi-rank of R is $(m|0)$, R is called *even*.

Given an even Hecke symmetry R, how to construct a category, similar to that $Rep - U_q(sl(m))$?
Observe that in general we have no object of QG $U_q(gl(N))$ type.
First, let us extend R up to a braiding

$$R = R^{V\oplus V^*} : (V \oplus V^*)^{\otimes 2} \to (V \oplus V^*)^{\otimes 2},$$

where V^* is the dual space with the paring $\langle\, ,\, \rangle : V \otimes V^* \to \mathbb{C}$.

We fixe a basis $\{x_i\} \in V$. The basis $\{x^j\} \in V^*$ such that $< x_i, x^j >= \delta_i^j$ is called "right dual".
We want to define the extension $R^{V\oplus V^*}$ so that it would be in a sense coordinated with this pairing.

The following method of extending R belongs to V.Lyubashenko. Let us present the symmetry R (involutive or Hecke) in the basis $\{x_i\} \in V$:

$$R(x_i \otimes x_j) = R_{ij}^{kl} x_k \otimes x_l.$$

We say that a braiding R is *skew-invertible* if there exists an operator $\Psi : V^{\otimes 2} \to V^{\otimes 2}$ such that

$$\mathrm{Tr}_2 R_{12} \Psi_{23} = P_{13} \quad \Leftrightarrow \quad R_{ij}^{kl} \Psi_{lp}^{jq} = \delta_i^q \delta_p^k,$$

If it is so, then the mentioned extension is

$$R^{V \oplus V^*}(x^k \otimes x^l) = R_{ji}^{lk} x^i \otimes x^j, \;\; R^{V \oplus V^*}(x_i \otimes x^j) = (R^{-1})_{ik}^{jl} x^k \otimes x_l,$$

$$R^{V \oplus V^*}(x^j \otimes x_i) = \Psi_{li}^{kj} x_k \otimes x^l.$$

Also, introduce two operators

$$B = \mathrm{Tr}_1 \Psi \quad \Leftrightarrow \quad B_i^j = \Psi_{ki}^{kj}, \qquad C = \mathrm{Tr}_2 \Psi \quad \Leftrightarrow \quad C_i^j = \Psi_{ik}^{jk}.$$

Then we define

$$< x^j, x_i >= B_i^j$$

Also, for any $N \times N$ matrix A (may be with NC entries) we put

$$Tr_R A = Tr\, C\, A.$$

This R-trace (or quantum trace) has many remarkable properties.

Below, we use the following notions. Let A be an $N \times N$ matrix. Then we put

$$A_{\overline{1}} = A_1, \;\; A_{\overline{2}} = R\, A_{\overline{1}}\, R^{-1},$$

$$A_{\overline{3}} = R_2\, A_{\overline{2}}\, R_2^{-1} = R_2\, R_1\, A_1\, R_1^{-1}\, R_2^{-1}$$

and so on.

Note that

$$A_2 = I \otimes A = P\, A_1\, P = P(A \otimes I)P.$$

One of the main properties of the quantum trace is (Dubna's group)

$$Tr_{R(2)}\, A_{\overline{2}} = Tr_{R(1)}\, A_{\overline{1}}.$$

Note that in the classical case we have

$$Tr_2\, A_2 = Tr_1\, A_1.$$

It is natural to put $dim_R V = Tr_R I_V$.

Example. If R is an even symmetry of rank $(2, 0)$ (i.e. $P_-(t) = 1 + Nt + t^2$), then

$$dim\, V = N, \quad dim_R\, V = q^{-2}\, 2_q$$

for a Hecke R, and $dim_R\, V = 2$ for an involutive involutive R.

Example. If R is the above Hecke coming from $U_q(sl(2))$, then

$$C = diag(q^{-3}, q^{-1}),\ B = diag(q^{-1}, q^{-3}).$$

Observe that in general $dim_R V = q^{n-m}(m-n)_q$ depends only on the bi-rank $(m|n)$ of the initial symmetry R.

Now, pass to defining quantum determinants in RTT and RE algebras.
Let us assume R to be Hecke. Consider the projectors (idempotents) $V^{\otimes k} \to \bigwedge^{(k)}(V)$, called R-skew-symmetrizers

$$A_R^{(1)} = I, \;\; A_R^{(k)} = \frac{1}{k_q} A_R^{(k-1)} \left(q^{k-1} I - (k-1)_q R_{k-1} \right) A_R^{(k-1)}, \;\; k = 2, 3...$$

For instance,

$$A_R^{(2)} = \frac{qI - R}{2_q},$$

$$A_R^{(3)} = \frac{q^3 I - q^2 R_{12} - q^2 R_{23} + qR_{12}R_{23} + qR_{23}R_{12} - R_{12}R_{23}R_{12}}{2_q \, 3_q}.$$

If R is involutive, we put $q = 1$.

If R has the bi-rank $(m|0)$ the space $Im\, A^{(m)}$ is one-dimensional. Consequently, there exist two tensors

$$u = (u_{i_1 \dots i_m}) \text{ and } v = (v^{j_1 \cdots j_m}),$$

such that

$$A_R^{(m)}(x_{i_1} \otimes \dots \otimes x_{i_m}) = u_{i_1 \dots i_m}\, v^{j_1 \cdots j_m} x_{j_1} \otimes \dots \otimes x_{j_x},$$

$$< v, u >:= v^{i_1 \dots i_m}\, u_{i_1 \dots i_m} = 1.$$

The element $v^{j_1 \cdots j_m} x_{j_1} \otimes \dots \otimes x_{j_m}$ is a generator of $Im\, A_R^{(m)}$.
Note that the tensors u and v are defined up to a renormalization

$$u \to au, \;\; v \to a^{-1}v, \;\; a \in \mathbb{C}, a \neq 0.$$

Again, consider the above symmetries.

For the latter symmetry (involutive) we have

$$u = (u_{11}, u_{12}, u_{21}, u_{22}) = \frac{1}{2}(0, 1, -q^{-1}, 0),$$

$$v = (v^{11}, v^{12}, v^{21}, v^{22}) = (0, 1, -q, 0).$$

For the former one (Hecke, coming from $U_q(sl(2))$) we have

$$u = \frac{1}{2_q}(0, q^{-1}, -1, 0), \quad v = (0, 1, -q, 0).$$

Observe that the tensors v corresponding to these symmetries coincide with each other and, consequently, the algebras

$$Sym_R(V) = T(V)/ < v >$$

are the same. Nevertheless, the tensors u are different. Also, the algebras $\bigwedge_R(V)$ are different as well. We have

$$\bigwedge\nolimits_R(V) = T(V)/ < x^2, y^2, qxy - yx >$$

$$\bigwedge\nolimits_R(V) = T(V)/ < x^2, y^2, xy - qyx >$$

respectively for the Hecke and involutive symmetries.

How to find relations in QMA, if we know the algebra $Sym_R(V)$? Let us assume that the relations in $Sym_R(V)$ are $x_i\,x_j - q\,x_j\,x_i = 0$, $i<j$. Apply the coproduct $x_i \to \sum_k t_i^k \otimes x_k$ to this relation. We have

$$(\sum_k t_i^k \otimes x_k)\,(\sum_l t_j^l \otimes x_l) - q\,(\sum_l t_j^l \otimes x_l)\,(\sum_k t_i^k \otimes x_k) = 0.$$

Now, we have to take away the terms t_j^l from the second factors by transposing them with x_k.

However, the result depends on the way of transposing the factors t_j^l and these x_k. Thus, by imposing different ways we get different algebras (RTT or RE).

Now, introduce the determinants in the QM algebras. Recall that the RTT algebra corresponding to R is defined by the system

$$RT_1T_2 - T_1T_2R, \;\; T = (t_i^j), 1 \le i,j \le m,$$

and the corresponding RE one is defined by that

$$RL_1RL_1 - L_1RL_1R = 0, \;\; L = (l_i^j), \;\; 1 \le i,j \le m.$$

Also, remind the above notation

$$L_{\overline{1}} = L_1, \;\; L_{\overline{2}} = R_{12}\, L_{\overline{1}}\, R_{12}^{-1}, \;\; L_{\overline{3}} = R_{23}\, L_{\overline{2}}\, R_{23}^{-1} = R_{23}\, R_{12}\, L_{\overline{1}}\, R_{12}^{-1}\, R_{23}^{-1}, \ldots$$

In this notation the defining relations of the RE algebra become similar to the RTT ones

$$RL_{\overline{1}}L_{\overline{2}} = L_{\overline{1}}L_{\overline{2}}R.$$

Let R be a symmetry (involutive or Hecke) and F be a skew-invertible. Let us define the quantum determinant in the algebra RTT and RE by assuming R to be of bi-rank $(m|0)$.

Definition

The element

$$det_{\mathcal{L}(R,F)}(L) :=< v|L_{\overline{1}}...L_{\overline{m}}|u >:= v^{i_1...i_m}\,(L_{\overline{1}}...L_{\overline{m}})_{i_1...i_m}^{j_1...j_m}\,u_{j_1...j_m},$$

is called quantum determinant of the generating matrix L in the RE algebra. In RTT one it is necessary only to replace the overlined indexes with usual ones.

Quantum analogs of the elementary symmetric polynomials and power sums in the RTT algebras are respectively defined as follows

$$e_k(L) = Tr_{(12\dots k)} A^{(k)}_R \, L_{\overline{1}} \dots L_{\overline{k}}.$$

$$p_k(L) = Tr_{(12\dots k)} R_{k-1\,k} \dots, R_{23} R_{12} \, L_{\overline{1}} \dots L_{\overline{k}}.$$

In the RE algebra the usual trace $Tr_{(12\dots k)}$ should be replaced by $Tr_{R(12\dots k)}$.

Note that if R is of bi-rank $(m|0)$, the element e_m is a multiple of the quantum determinant.

As shown in [IOP], they are related by the quantum version of the Newton identities

$$p_k - q p_{k-1}\, e_1 + (-q)^2 p_{k-2}\, e_2 + \ldots + (-q)^{k-1} p_1\, e_k + (-1)^k k_q e_k = 0$$

and commute with each other.

The algebra generated by these quantum symmetric polynomials is called *Bethe*.

Note that in the RE algebras the power sums can be reduced to the form similar to the classical one:

$$p_k = Tr_R\, L^k.$$

Moreover, in this case there exists a quantum analog of the Cayley-Hamilton identity similar to the classical one

$$L^m - q\, L^{m-1}\, e_1 + (-q)^2 L^{m-2}\, e_2 + \ldots + (-q)^{m-1} L\, e_{m-1} + (-q)^m\, I\, e_m = 0.$$

In this case we can also define the so-called quantum characteristic polynomial

$$ch(t) = t^m - q\, t^{m-1}\, e_1 + (-q)^2 t^{m-2}\, e_2 + \ldots$$

$$+(-q)^{m-1} t\, e_{m-1} + (-q)^m\, 1\, e_m = 0,$$

such that $ch(L) = 0$.

Observe that the polynomial $det_R(L - tI)$ is well defined but it is not equal to $ch(t)$.

Consider the quantum determinants in the RTT algebras $\mathcal{L}(R, P)$, corresponding to the symmetries R above.
Below, we denote $a = l_1^1$, $b = l_1^2$, $c = l_2^1$, $d = l_2^2$. Then the defining relations in the algebra $\mathcal{L}(R, P)$, corresponding to the involutive symmetry above are

$$ab = q^{-1}ba, \quad ac = qca, \quad ad = da, \quad bc = q^2cb,$$

$$bd = qdb, \quad cd = q^{-1}dc.$$

The quantum determinant in this algebra is

$$\det(L) = \frac{1}{2}(ad - qcb - q^{-1}bc + da) = ad - q^{-1}bc = da - qcb. \quad (1)$$

The defining relations in the algebra corresponding to the Hecke matrix are

$$ab = qba, \quad ac = qca, \quad ad - da = (q - q^{-1})bc, \quad bc = cb,$$

$$bd = qdb, \quad cd = qdc.$$

The corresponding quantum determinant is

$$\det(L) = \frac{1}{2_q}(q^{-1}ad - bc - cb + qda) = ad - qbc = da - q^{-1}cb. \quad (2)$$

Now, introduce the corresponding Generalized Yangians (GY). The famous Yang braiding is $R(u,v) = P - \frac{I}{u-v}$.

Proposition.

1. If R is an involutive symmetry, then

$$R(u,v) = R - \frac{a\,I}{u-v}$$

is an R-matrix, i.e. it meets the quantum Yang-Baxter equation

$$R_{12}(u,v)\,R_{23}(u,w)\,R_{12}(v,w) = R_{23}(v,w)\,R_{12}(u,w)\,R_{23}(u,v).$$

2. If $R = R(q)$ is a Hecke symmetry, then the same is valid for

$$R(u,v) = R(q) - \frac{(q-q^{-1})u\,I}{u-v}.$$

The Drinfeld's Yangian $\mathbf{Y}(gl(N))$ is in fact an RTT algebra defined by

$$R(u,v)\,T_1(u)\,T_2(v) = T_1(v)\,T_2(u)\,R(u,v)$$

with the Yang braiding and under a assumption that $T(u)$ is a series

$$T(u) = \sum_{k\geq 0} T[k]\,u^{-k}$$

and $T[0] = I$.

Introduce two types of GY in a similar manner.

1. Generalized Yangians of RTT type are defined by

$$R(u,v)\,T_1(u)\,T_2(v) = T_1(u)\,T_2(v)\,R(u,v),$$

where $R(u,v)$ is one of the above current braidings.
2. GY of RE type (also called braided Yangians) are defined by

$$R(u,v)L_{\overline{1}}(u)\,L_{\overline{2}}(v) = L_{\overline{1}}(v)\,L_{\overline{2}}(u)\,R(u,v).$$

Here $L_{\overline{2}} = R\,L_{\overline{1}}\,R^{-1}$.
These relations can be also presented as follows

$$R(u,v)L_1(u)RL_1(v) = L_1(v)RL_1(u)R(u,v).$$

If a braiding $R(u, v)$ arises from an involutive symmetry R, the corresponding GY $Y(R, P)$ is called *rational*. If R is Hecke, then $Y(R, R)$ is called *trigonometrical*.

If R is of bi-rank $(m|0)$, we define quantum determinants in the rational (resp., trigonometrical) GY as follows

$$\det_{\mathbf{Y}(R,F)}(L(u)) =< v|L_{\overline{1}}(u)\, L_{\overline{2}}(u-1)\dots L_{\overline{m}}(u-m+1)|u>,$$

$$\det_{\mathbf{Y}(R,F)}(L(u)) =< v|L_{\overline{1}}(u)\, L_{\overline{2}}(q^{-2}u)\dots L_{\overline{m}}(q^{-2(m-1)}u)|u> .$$

Thus, the determinants are defined by formulae similar to those above but with shifts in arguments of the matrices $L(u)$, additive in the rational cases and multiplicative in the trigonometrical ones.

As for quantum elementary symmetric polynomials $e_k(u)$ in $Y(R,R)$, they are in the trigonometrical case

$$e_k(u) = Tr_{R(1\dots k)}\left(A_R^{(k)} L_{\overline{1}}(u) L_{\overline{2}}(q^{-2}u)\dots L_{\overline{k}}(q^{-2(k-1)}u)\right),\ k\geq 1,$$

Quantum powers of the generating matrices in the Generalized Yangians of RE type are defined in the trigonometrical case by

$$L^{[k]}(u) = L(q^{-2(k-1)}u)L(q^{-2(k-2)}u)\dots L(u),\ k\geq 1.$$

The quantum power sums are defined in this case by

$$p_k(u) = Tr_R\, L^{[k]}(u) = Tr_R\, L(q^{-2(k-1)}u)L(q^{-2(k-2)}u)\dots L(u).$$

Here, also the quantum determinant and the highest quantum elementary polynomial differ from each other by a numerical factor.

Let us exhibit the quantum Newton relations and Cayley-Hamilton identities in the Generalized Yangians $Y(R,R)$ of RE type

Proposition.

$$p_k(u) - qp_{k-1}(q^{-2}u)e_1(u) + (-q)^2 p_{k-2}(q^{-4}u)e_2(u) + \dots$$

$$+(-q)^{k-1} p_1(q^{-2(k-1)}u)e_k(u) + (-1)^k k_q e_k(u).$$

Proposition.

$$\sum_{p=0}^{m} (-q)^p L^{[m-p]}(q^{-2p}u)e_p(u) = 0.$$

Observe that in the GY of RE type $\mathbf{Y}(R,R)$ there is an evaluation morphism similar to the that in the Drinfeld's Yangian.

Now, consider the case of general symmetries R (not necessary even) in more detail.

If a given symmetry R is of bi-rank $(m|n)$ $n \neq 0$, the generating matrix L of the RE algebra also meets the Cayley-Hamilton identity

$$a_{m+n}\, L^{m+n} + a_{m+n-1}\, L^{m+n-1} + \cdots + a_0\, I = 0,$$

where all the coefficients a_k belong to the center of the algebra $\mathcal{L}(R,R)$. Note that in this case the leading coefficient a_{m+n} does not equal 1. Upon dividing this relation by $a_0 L$, we can express the matrix L^{-1} as a linear combinations of the matrices L^k, $0 \leq k \leq m+n-1$ with the coefficients $-a_k/a_0$.

Observe that for any Schur diagrams (partitions) $\lambda = (\lambda_1 \geq ... \geq \lambda_k)$ there exists an analog of the Schur functor $V \mapsto V_\lambda$ and the corresponding Schur polynomial p_λ.

Then the quantum determinant and quantum Berezinian are defined by some fractions p_λ / p'_λ. For the quantum determinant we have $p_\lambda = a_0, \; p'_\lambda = a_{m+n}$.

Many thanks