Cyclic elements in semisimple Lie algebras

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Nilpotent elements of nilpotent type

A nilpotent element e has nilpotent type iff depth of the grading is odd.

Then, a generic cyclic element $e + F$ is nilpotent.

For every nilpotent element of nilpotent type, $\mathfrak{g}_{-d}$ carries a nondegenerate skew-symmetric bilinear form,
and the centralizer of the $\mathfrak{sl}(2)$-triple acts on it through the standard representation of the symplectic algebra for this form.

For classical algebras: nilpotent elements e in $\mathfrak{g} = \mathfrak{so}(n)$ with partitions $(n_1 \geqslant n_2 \geqslant \cdots \geqslant n_k)$, $n_1 + n_2 + ... + n_k = n$,
such that n_1 is odd and $n_2 = n_1 - 1$.

Depth $d = 2n_1 - 3$, $\dim(\mathfrak{g}_{-d}) =$ multiplicity of n_2.

Generic cyclic element has partition with the largest part $3n_1 - 2$, multiplicity of n_2 two less than for e, and all other parts unchanged.

Nilpotent elements of nilpotent type

For exceptional algebras:

$\mathfrak{g}$	WDD	label	depth $= d$	$\dim(\mathfrak{g}_{-d})$	generic $e + F$
E_6	0 − 0 − 1 − 0 − 0 0	$3A_1$	3	2	D_4
E_6	1 − 0 − 1 − 0 − 1 0	$2A_2 + A_1$	5	2	E_6
E_7	0 − 0 − 0 − 0 − 1 − 0 0	$[3A_1]'$	3	2	D_4
E_7	1 − 0 − 0 − 0 − 0 − 0 1	$4A_1$	3	6	$D_4 + A_1$
E_7	0 − 1 − 0 − 0 − 1 − 0 0	$2A_2 + A_1$	5	2	E_6
E_8	0 − 1 − 0 − 0 − 0 − 0 − 0 0	$3A_1$	3	2	D_4
E_8	0 − 0 − 0 − 0 − 0 − 0 − 0 1	$4A_1$	3	8	$D_4 + A_1$
E_8	0 − 1 − 0 − 0 − 0 − 0 − 1 0	$2A_2 + A_1$	5	2	E_6
E_8	0 − 0 − 0 − 1 − 0 − 0 − 0 0	$2A_2 + 2A_1$	5	4	$E_6 + E_1$
E_8	0 − 0 − 0 − 1 − 0 − 0 − 1 0	$2A_3$	7	4	D_7
E_8	0 − 1 − 0 − 0 − 1 − 0 − 0 0	$A_4 + A_3$	9	2	E_8
E_8	0 − 1 − 1 − 0 − 1 − 0 − 1 0	A_7	15	2	E_8
F_4	0 — 1 ⇐ 0 — 0	$A_1 + \tilde{A}_1$	3	2	C_3
F_4	0 — 1 ⇐ 0 — 1	$\tilde{A}_2 + A_1$	5	2	F_4
G_2	0 ⇛ 1	$\tilde{A}_1$	3	2	G_2

Irreducible nilpotent elements of semisimple type

For all irreducible nilpotent elements, $(\mathfrak{g}_{-d}, *)$ is a commutative algebra $\mathscr{C}_\lambda(n)$ generated by p_1, ..., p_n with defining relations $p_i^2 = p_i$, $i = 1, \dots, n$, and

$$p_i p_j = \lambda(p_i + p_j), \qquad i \neq j.$$

The singular set of $\mathfrak{g}_{-d}$ coincides with the union of all proper subalgebras of this algebra.

For classical algebras:

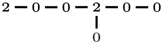
$\mathfrak{g}$	partition	depth	rank	$Z(\mathfrak{s})\vert\mathfrak{g}_{-d}$	$(\mathfrak{g}_{-d}, *)$
$\mathfrak{sl}(2k+1)$	$[2k+1]$	$4k$	1	1	1
$\mathfrak{sp}(2k)$	$[2k]$	$4k-2$	1	1	1
$\mathfrak{so}(2k+1)$	$[2k+1]$	$4k-2$	1	1	1
$\mathfrak{so}(4k+4)$	$[2k+3, 2k+1]$	$4k+2$	2	1	$\mathscr{C}_{-k}(2)$

Irreducible nilpotent elements of semisimple type

For exceptional algebras:

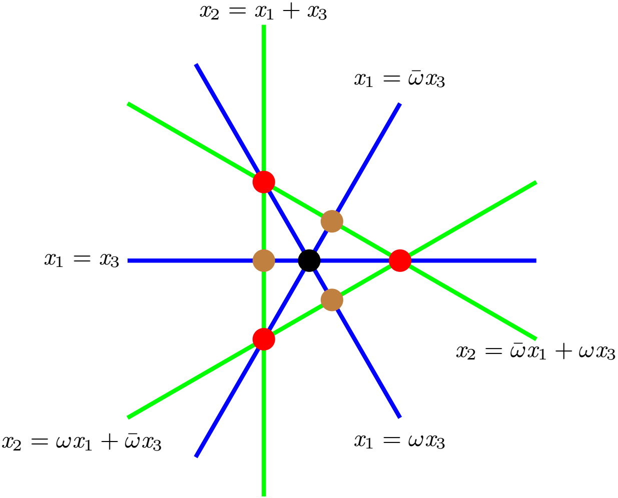
$\mathfrak{g}$	WDD	label	depth	rank	$Z(\mathfrak{s})\vert\mathfrak{g}_{-d}$	$(\mathfrak{g}_{-d}, *)$
E_6	2 — 2 — 0 — 2 — 2 2	$E_6(a_1)$	16	1	1	1
E_7	2 — 2 — 2 — 2 — 2 — 2 2	E_7	34	1	1	1
E_7	2 — 2 — 2 — 0 — 2 — 2 2	$E_7(a_1)$	26	1	1	1
E_7	2 — 0 — 0 — 2 — 0 — 0 0	$E_7(a_5)$	10	3	$\sigma_3 \oplus \mathbf{1}$	$\mathscr{C}_{-\frac{1}{3}}(3)$
E_8	2 — 2 — 2 — 2 — 2 — 2 — 2 2	E_8	58	1	1	1
E_8	2 — 2 — 2 — 2 — 0 — 2 — 2 2	$E_8(a_1)$	46	1	1	1
E_8	2 — 2 — 0 — 2 — 0 — 2 — 2 2	$E_8(a_2)$	38	1	1	1
E_8	2 — 0 — 2 — 0 — 2 — 0 — 2 0	$E_8(a_4)$	28	1	1	1
E_8	0 — 2 — 0 — 0 — 2 — 0 — 2 0	$E_8(a_5)$	22	2	$\sigma_2 \oplus \mathbf{1}$	$\mathscr{C}_{-\frac{2}{7}}(2)$
E_8	0 — 2 — 0 — 0 — 2 — 0 — 0 0	$E_8(a_6)$	18	2	σ_3	$\mathscr{C}_{-1}(2)$
E_8	0 — 0 — 0 — 2 — 0 — 0 — 0 0	$E_8(a_7)$	10	4	σ_5	$\mathscr{C}_{-\frac{1}{3}}(4)$
F_4	2 — 2 ⇐ 2 — 2	F_4	22	1	1	1
F_4	0 — 2 ⇐ 0 — 2	$F_4(a_2)$	10	2	$\sigma_2 \oplus \mathbf{1}$	$\mathscr{C}_{-\frac{1}{3}}(2)$
G_2	2 ⇛ 2	G_2	10	1	1	1

Irreducible nilpotent elements of semisimple type – examples

E_7, diagram  ,

$\dim(\mathfrak{g}_{-d}) = 3$, algebra $\mathscr{C}_{-\frac{1}{3}}(3)$ has six 2-dimensional subalgebras.

Image of the singular set in the projective plane: union of six lines.

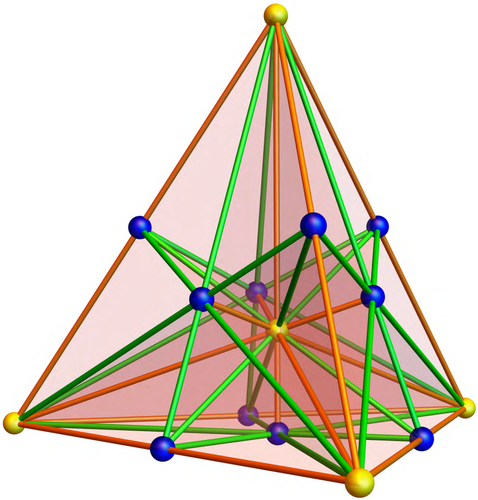


Irreducible nilpotent elements of semisimple type – examples

E_8, diagram
$$\begin{tikzcd}0 \arrow[r, dash] & 0 \arrow[r, dash] & 0 \arrow[r, dash] & 2 \arrow[r, dash] & 0 \arrow[r, dash] \arrow[d, dash] & 0 \arrow[r, dash] & 0 \\ & & & & 0 & & \end{tikzcd}$$
,

$\dim(\mathfrak{g}_{-d}) = 4$, algebra $\mathscr{C}_{-\frac{1}{3}}(4)$ has ten 3-dimensional subalgebras.

Image of the singular set in the projective 3-space: union of ten planes.



Thank you for your attention!