Five-point functions from AdS/CFT integrability

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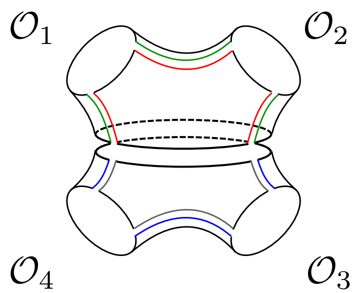
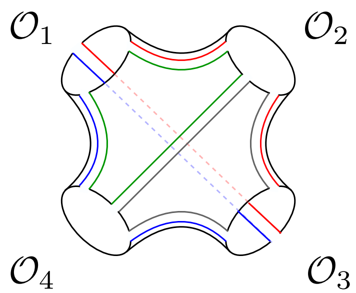
M. de Leeuw, B. Eden, D. le Plat, and T. Meier, work in progress

Introduction

- The **spectrum problem** in AdS/CFT is fairly well-understood (anomalous dimensions of composite operator $\sim$ energy levels of strings).
- **Three-point** functions by **hexagon operators** [Basso, Komatsu, Vieira (2015)]
- **Four-point** functions by **hexagon tilings** [Eden, Sfondrini (2016)], [Fleury, Komatsu (2016)]
- This is a promising **non-OPE** approach: no sum over intermediate physical states.





Recent work: M. de Leeuw, B. Eden, Y. Jiang, D. le Plat, T. Meier, A. Sfondrini (2017—18)

- Tree-level four-point functions with two BMN and two BPS operators point at **colour dressing**.
- Applications to the **torus** contribution to the **two-point function**, **multi-trace operators**, the empirical **selection rules** of [Fleury, Komatsu (2016)].
- The formalism yields a **well-defined weak coupling expansion**.
- **Currently:** integration of the **BPS one-loop five-point process** of [Fleury, Komatsu (2017)]

BMN operators [Berenstein, Maldacena, Nastase (2002)], [Minahan, Zarembo (2002)]

Half-BPS operators:

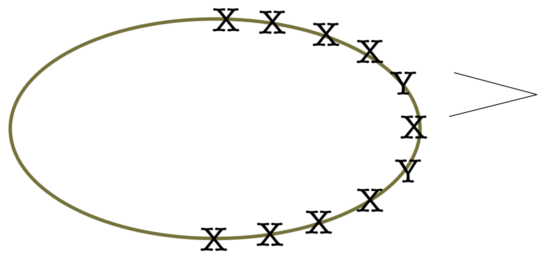
$$\mathcal{O}_L = \frac{1}{\sqrt{LN^L}} \mathrm{Tr}(Z^L)$$

with Z a scalar field. All fields transform in the adjoint representation of the colour group $SU(N)$.

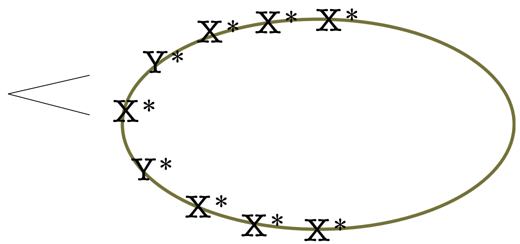
$SU(2)$ sector **BMN-operators** with two scalar **excitations**:

$$\mathcal{B}_L^k = \mathrm{Tr}(Z^{L-k-2} Y Z^k Y)\,, \qquad Y \neq \bar{Z}\,.$$

Two-point function at $\mathbf{g^2}$:



$$g^2 \int Tr([X^*, Y^*]\,[X\,, Y\,])$$



Length 4: basis $\{\mathcal{O}_4^0, \mathcal{O}_4^1\}$. Mixing matrix

$$\Gamma_1/N = \begin{pmatrix} 2 & -4 \\ -2 & 4 \end{pmatrix}, \qquad \gamma_{1,0} = 0,\ \underline{v}_0 = 2\,\mathcal{O}_4^0 + \mathcal{O}_4^1 = \mathrm{Tr}(Y \partial Z)^2 \mathrm{Tr}(Z^4)\,, \qquad \gamma_{1,1} = 6,\ \underline{v}_1 = \mathcal{O}_4^0 - \mathcal{O}_4^1$$

Spin chains and Bethe ansatz

- **Planar single-trace** operators
- **Heisenberg chain:** identify $X = \downarrow$, $Y = \uparrow$, the four-vertex as $\mathbb{I} - \mathbb{P}$.
- **momentum/rapidity** $u_i = \frac{1}{2}\cot\left(\frac{p_i}{2}\right)$ for each excitation

Shift operator and **scattering matrix**

$$\exp(i\,p) = \left(\frac{u + \frac{i}{2}}{u - \frac{i}{2}}\right) , \qquad S(u_2, u_1) = \frac{u_2 - u_1 + i}{u_2 - u_1 - i} .$$

Moving the first magnon once around the chain implies the **Bethe equations**

$$\left(\frac{u_j + \frac{i}{2}}{u_j - \frac{i}{2}}\right)^L \frac{u_k - u_j + i}{u_k - u_j - i} = 1 .$$

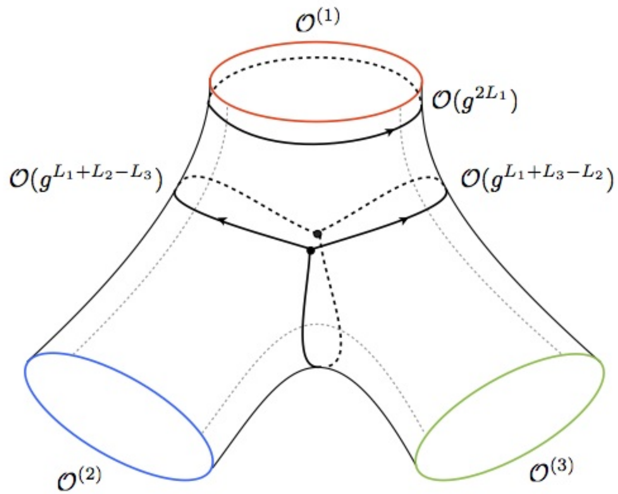
Momentum conservation: $p_1 + p_2 = 0 \Leftrightarrow u_1 + u_2 = 0 \Rightarrow$

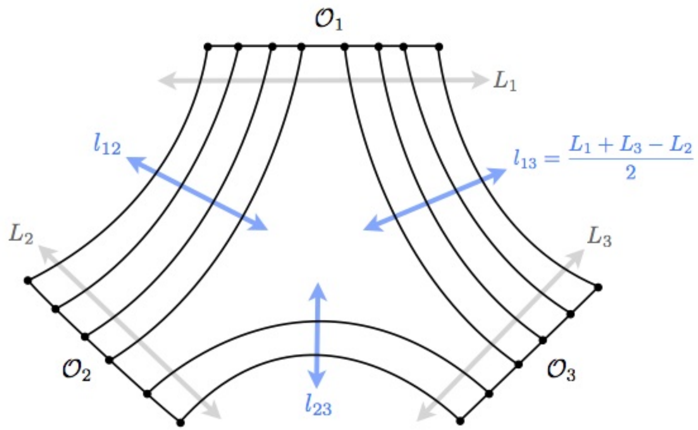
$$\left(\frac{u_1 + \frac{i}{2}}{u_1 - \frac{i}{2}}\right)^{L-1} = 1 , \qquad u_2 = -u_1 .$$

Energy or anomalous dimension

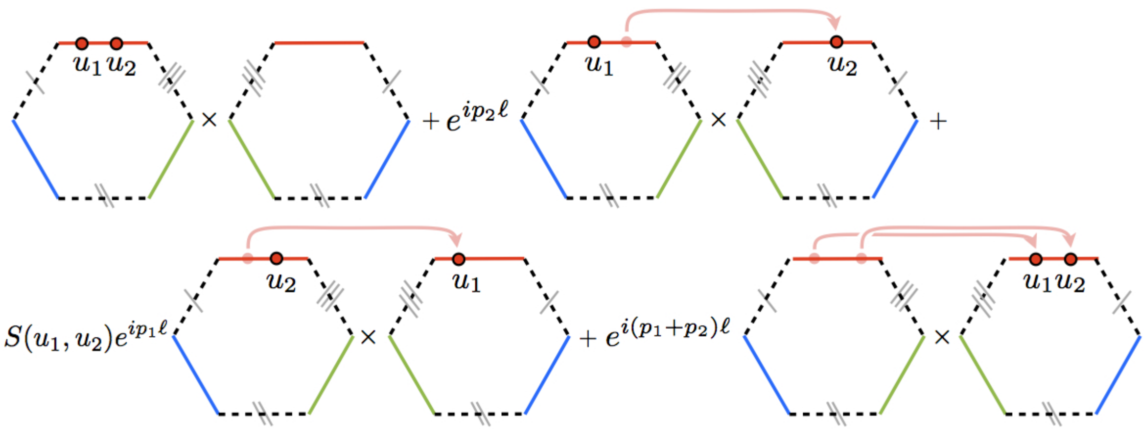
$$\gamma_1 = \sum_{i=1}^{2} \frac{1}{u_i^2 + \frac{1}{4}} .$$

Three-point functions by hexagons [Basso, Komatsu, Vieira (2015)]





- **Input**: **Bethe roots** characterising the three **operators**.
- Split into **top** and **bottom hexagon. Entangled state:**



$$\mathcal{A} = \sum_{\alpha\cup\bar\alpha=\{u_i\}} w(\alpha,\bar\alpha)(-1)^{|\alpha|}\,\mathfrak{h}_{Y\dots Y}(\alpha)\,\mathfrak{h}_{Y\dots Y}(\bar\alpha)$$

$$\mathfrak{h}_{Y_1\dots Y_n} = \Big(\prod_{i<j} h_{ij}\Big)\,\langle\,\phi^1_n\dots\phi^1_1\,|\,S\,|\,\bar\phi^{2'}_1\dots\bar\phi^{2'}_n\,\rangle\,, \qquad h_{12} = \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\,\frac{1-1/(x_1^-x_2^+)}{1-1/(x_1^+x_2^+)}\,\frac{1}{\sigma_{12}}\,,$$

$$x^\pm = x(u\pm i/2),\quad x(u) = \frac{u}{2}\big(1+\sqrt{1-2\,g^2/u^2}\big)\,.$$

- Use **S** on **left or right**: $Y \to \phi^1\bar\phi^{2'}\,,\quad S\,|\bar\phi^{2'}_1\bar\phi^{2'}_2\bar\phi^{2'}_3\rangle = A_{12}A_{13}A_{23}\,|\bar\phi^{2'}_3\bar\phi^{2'}_2\bar\phi^{2'}_1\rangle$. [Beisert (2005)]
- Contract using $\langle\psi^\alpha|\bar\psi^{\dot\beta}\rangle = \epsilon^{\alpha\dot\beta}$, $\langle\phi^a|\bar\phi^{b'}\rangle = \epsilon^{ab'}$.

- **Lüscher** corrections: X_a **bound state** in **mirror kinematics**.

$$\delta\mathcal{A} = \sum_{a>0}\int\frac{du}{2\pi}\,\mu(u)\;\omega(\alpha,\bar\alpha)\;\mathfrak{h}_{XY\dots Y}(u^{n\gamma},\alpha)\;\mathfrak{h}_{Y\dots Y\bar X}(\bar\alpha,u^{-n\gamma})\,, \qquad \mu(u) = \frac{a\,(g^2)^{l+1}}{(2\,u^{+a}\,u^{-a})^{l+2}}+\dots$$

- Mirror measure: **bridge length** as **exponent of $\mathbf{g^2}$**.
- Further **g**-dependence in $\mathbf{x}^\pm$.

An example: $\langle \mathcal{B}_4 \mathcal{O}_2 \mathcal{O}_2 \rangle_{g^0}$

Entangled state:

$$\begin{aligned}\mathcal{A} \;=\; & \mathfrak{h}_f(\{u_1,u_2\})\,\mathfrak{h}_b(\{\}) - e^{ip_2 l_{12}}\,\mathfrak{h}_f(\{u_1\})\,\mathfrak{h}_b(\{u_2\}) \\ & -e^{ip_1 l_{12}}\,S(u_2,u_1)\,\mathfrak{h}_f(\{u_2\})\,\mathfrak{h}_b(\{u_1\}) + e^{i(p_1+p_2) l_{12}}\,\mathfrak{h}_f(\{\})\,\mathfrak{h}_b(\{u_1,u_2\})\end{aligned}$$

The **Bethe** equations imply **simplifications** and guarantee **cyclic invariance**.

For **longitudinal scalars Y**

$$\mathfrak{h}(\{\}) = \mathfrak{h}(\{u_i\}) = 1\,, \qquad \mathfrak{h}(\{u_1,u_2\}) = h(u_1,u_2) S(u_2,u_1)\,, \qquad h(u_1,u_2) = \frac{u_1-u_2}{u_1-u_2-i}\,.$$

Gaudin norm:

$$\mathcal{G} = \det\left(\frac{\partial\phi_j}{\partial u_i}\right)\,, \qquad \phi_j = -i\log\left[\left(\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}}\right)^L \frac{u_k-u_j+i}{u_k-u_j-i}\right]$$

One finds

$$C(4;2,2) = \frac{\mathcal{A}}{\sqrt{\mathcal{G}\,S(u_2,u_1)}} = -\frac{1}{2\sqrt{3}}\,.$$

To match field theory, **scale up** by $\sqrt{L_1 L_2 L_3} = 4$.

Kinematics on a line

Co-moving vacuum [Drukker, Plefka (2009)]

$$\hat{\mathcal{O}}_L = \frac{1}{\sqrt{L N^L}} \mathrm{Tr}(\hat{Z}^L)\,, \qquad \hat{Z} = Z + a^2 \bar{Z} + a(Y - \bar{Y})\,, \qquad x_i^\nu = \delta_3^\nu \, a_i$$

with **a** the **position** along a **line** in Minkowski space.

$SO(6)$ covariantly

$$\hat{Z} = z^\mu \phi_\mu\,, \quad z^\mu = \big((1+a^2), 0, 0, 2a, 0, i(1-a^2)\big)\,.$$

$$\langle \hat{Z}(a_1)\hat{Z}(a_2)\rangle = \frac{(a_1-a_2)^2}{4\pi^2(a_1-a_2)^2} = \frac{1}{4\pi^2}$$

Drop $4\pi^2$:

$$\langle Z(0)\hat{Z}(a_2)\rangle = 1, \qquad \langle Y(0)\hat{Z}(a_2)\rangle = -\frac{1}{a_2}$$

- In the calculation **before**, the points were chosen as $\mathbf{a} = \mathbf{0}, \mathbf{1}, \infty$.
- The three- and four-point functions we construct **now** will be **homogeneous of order -2 in $\mathbf{a_i}$**.

$\langle \mathcal{B}_4 \mathcal{O}_2 \mathcal{O}_2 \rangle_{g^0}$ with residual kinematic dependence

- Clockwise, the **front** hexagon has **vertices 1,2,3**, the **back** one **1,3,2**.
- Excitations can travel from point 1 to point 2 or 3. We use the **rule**

$$\mathfrak{h}_f(\alpha) \to \widehat{\mathfrak{h}_f}(\alpha) = (a_{23})^{|\alpha|}\, \mathfrak{h}_f(\alpha)\,, \qquad \mathfrak{h}_b(\bar{\alpha}) \to \widehat{\mathfrak{h}_b}(\bar{\alpha}) = (a_{32})^{|\bar{\alpha}|}\, \mathfrak{h}_b(\bar{\alpha})\,, \qquad a_{23} = \frac{1}{a_2} - \frac{1}{a_3}\,.$$

Then

$$\hat{\mathcal{A}} = \sum_{U=\alpha\cup\bar{\alpha}} \omega(\alpha, \bar{\alpha}, l_{12})\; \mathfrak{h}_b(\alpha)\, a_{23}^{|\alpha|}\; \mathfrak{h}_t(\bar{\alpha})\, a_{32}^{|\bar{\alpha}|}$$

with $\omega(\{u_2\}, \{u_1\}, l_{12}) = e^{ip_2 l_{12}} S(u_2, u_1)$ etc. — **no minus sign** — and hence

$$G(4;2,2) = \frac{4\,\hat{\mathcal{A}}}{\sqrt{\mathcal{G}\, S(u_2,u_1)}} = -\frac{2}{\sqrt{3}}\, a_{23}^2$$

in agreement with free field theory.

BMN-(BPS)^3 four-point functions

BMN operator at $a_1 = 0$.

We can write all results using

$$a_{23} = \frac{1}{a_2} - \frac{1}{a_3}\,, \qquad a_{34} = \frac{1}{a_3} - \frac{1}{a_4}\,.$$

Notation:

$$G(7^2;2,3,2) = \langle \mathcal{B}_7^2(0)\mathcal{O}_2(a_2)\mathcal{O}_3(a_3)\mathcal{O}_2(a_4)\rangle$$

etc. and

$$G(\ldots) = c * \underline{v}.(a_{23}^2,\ a_{23}a_{34},\ a_{34}^2)\,.$$

e.g.

$$G(4;2,2,2) = 4\sqrt{\tfrac{2}{3}}\,(1,1,1).(a_{23}^2,\ a_{23}a_{34},\ a_{34}^2)\,.$$

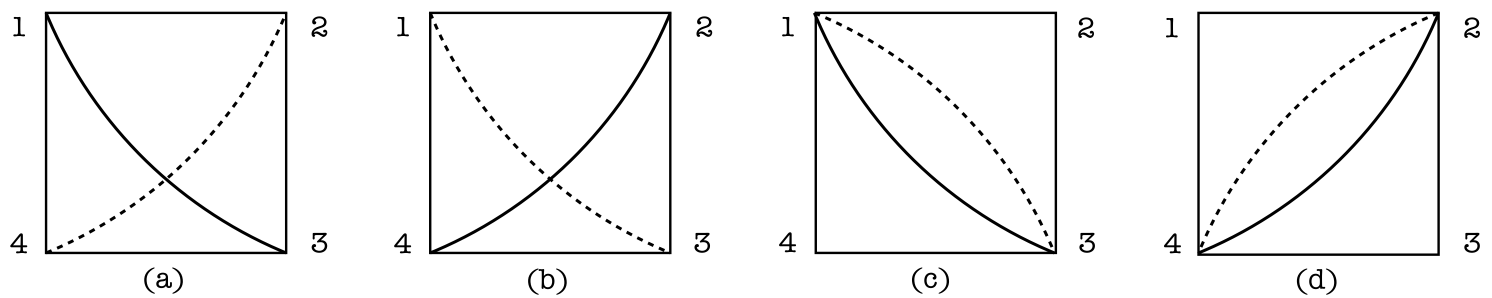
- **Explain these numbers!**

	C	$\underline{\mathbf{v}}$
$G(4;222)$	$4\sqrt{\frac{2}{3}}$	$(1,1,1)$
$G(4;242)$	$\frac{8}{\sqrt{3}}$	$(1,0,1)$
$G(4;233)$	$\sqrt{6}$	$(2,2,3)$
$G(5;232)$	$\sqrt{6}$	$(3,2,3)$
$G(6^{\mp};222)$	$4\sqrt{2}$	$(1,1,1)$
$G(4;235)$	$\sqrt{10}$	$(2,4,5)$
$G(4;244)$	$8\sqrt{\frac{2}{3}}$	$(1,1,2)$
$G(4;343)$	$2\sqrt{3}$	$(3,2,3)$
$G(5;252)$	$3\sqrt{10}$	$(1,0,1)$
$G(5;234)$	$2\sqrt{3}$	$(3,4,7)$
$G(5;333)$	$9\sqrt{6}$	$(1,1,1)$
$G(6^{\mp};242)$	$\frac{4(1\pm\sqrt{5})}{\sqrt{5}}$	$(2,1,2)$
$G(6^{\mp};233)$	$\frac{3(1\pm\sqrt{5})}{\sqrt{10}}$	$(4,4,6\pm\sqrt{5})$
$G(7';232)$	$2\sqrt{6}$	$(2,1,2)$
$G(7'';232)$	$6\sqrt{2}$	$(1,1,1)$

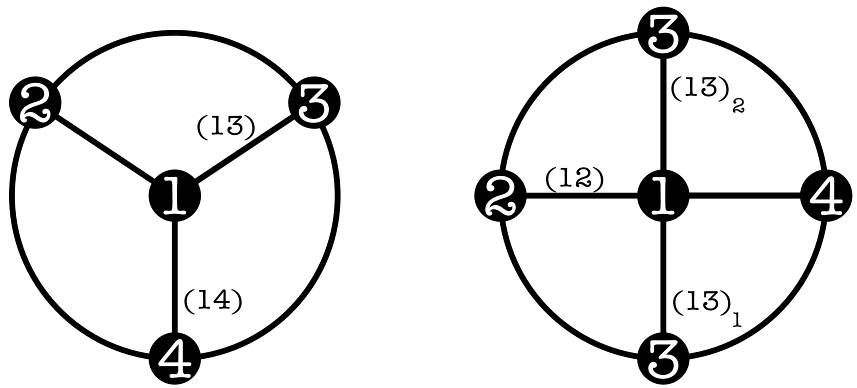
Four points: soft cushions

- **Planar $SU(N)$ tree diagrams**
- Operators: **beads on a ring. Connect on** the surface of a **sphere**.

BMN operator at **point 1**: **Four** classes of graphs with **edge widths/bridge lengths** $\{l_{ij}\}$:



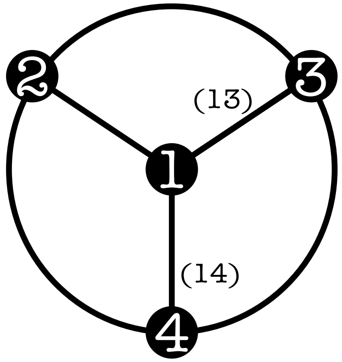
Looking onto point 1:

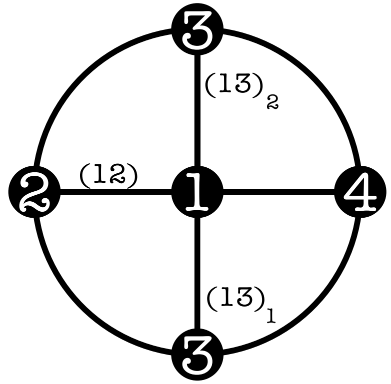


- **Tiling by hexagons** [Basso,Komatsu,Vieira (2015)]

3,4-hexagon amplitudes

Figure from above:





Graph (a): start with magnons between l_{12} and l_{13}, **iterate entangled state** prescription:

$$\hat{\mathcal{A}}^{(3)} = \sum_{U=\alpha\cup\bar{\alpha}} \sum_{\bar{\alpha}=\beta\cup\bar{\beta}} \omega(\alpha,\bar{\alpha},l_{13})\,\omega(\beta,\bar{\beta},l_{14})\;\mathfrak{h}_{123}(\alpha)\,a_{23}^{|\alpha|}\;\mathfrak{h}_{134}(\beta)\,a_{34}^{|\beta|}\;\mathfrak{h}_{142}(\bar{\beta})\,a_{42}^{|\bar{\beta}|}$$

Graph (b): flip $2 \leftrightarrow 3$.

Graph (c): put magnons between l_{14} and l_{13}^{1}, use **three partitions**.

$$\hat{\mathcal{A}}^{(4)} = \sum_{U=\alpha\cup\bar{\alpha}} \sum_{\bar{\alpha}=\beta\cup\bar{\beta}} \sum_{\bar{\beta}=\gamma\cup\bar{\gamma}} \omega(\alpha,\bar{\alpha},l_{13}^{1})\,\omega(\beta,\bar{\beta},l_{12})\,\omega(\gamma,\bar{\gamma},l_{13}^{2}) \ast$$
$$\mathfrak{h}_{143}(\alpha)\,a_{43}^{|\alpha|}\;\mathfrak{h}_{132}(\beta)\,a_{32}^{|\beta|}\;\mathfrak{h}_{123}(\gamma)\,a_{23}^{|\gamma|}\;\mathfrak{h}_{134}(\bar{\gamma})\,(a_{34})^{|\bar{\gamma}|}$$

Graph (d) drops here: $L_2 + L_4 = L_3 + L_1 + 4 \geq 10 \Rightarrow$ at least 16 elementary fields.

Conjecture

$$G\big(\mathcal{B}_{L_1}(0); \mathcal{O}_{L_2}(a_2), \mathcal{O}_{L_3}(a_3), \mathcal{O}_{L_4}(a_4)\big) = \sqrt{\frac{L_1 L_2 L_3 L_4}{\mathcal{G}\, S(u_2, u_1)}} \left[\sum_{l_{13}, l_{14}} c_3(\{l_{ij}\})\, \hat{\mathcal{A}}^{(3)} + \sum_{l_{13}^1, l_{12}, l_{13}^2} c_4(\{l_{ij}\})\, \hat{\mathcal{A}}^{(4)} \right]$$

with some **combinatorial coefficients $\mathbf{c_{3,4}}$** (normally equal to 1, correcting for over-counting).
For ≥ 8 Wick contractions (d) needs to be taken into account, too.

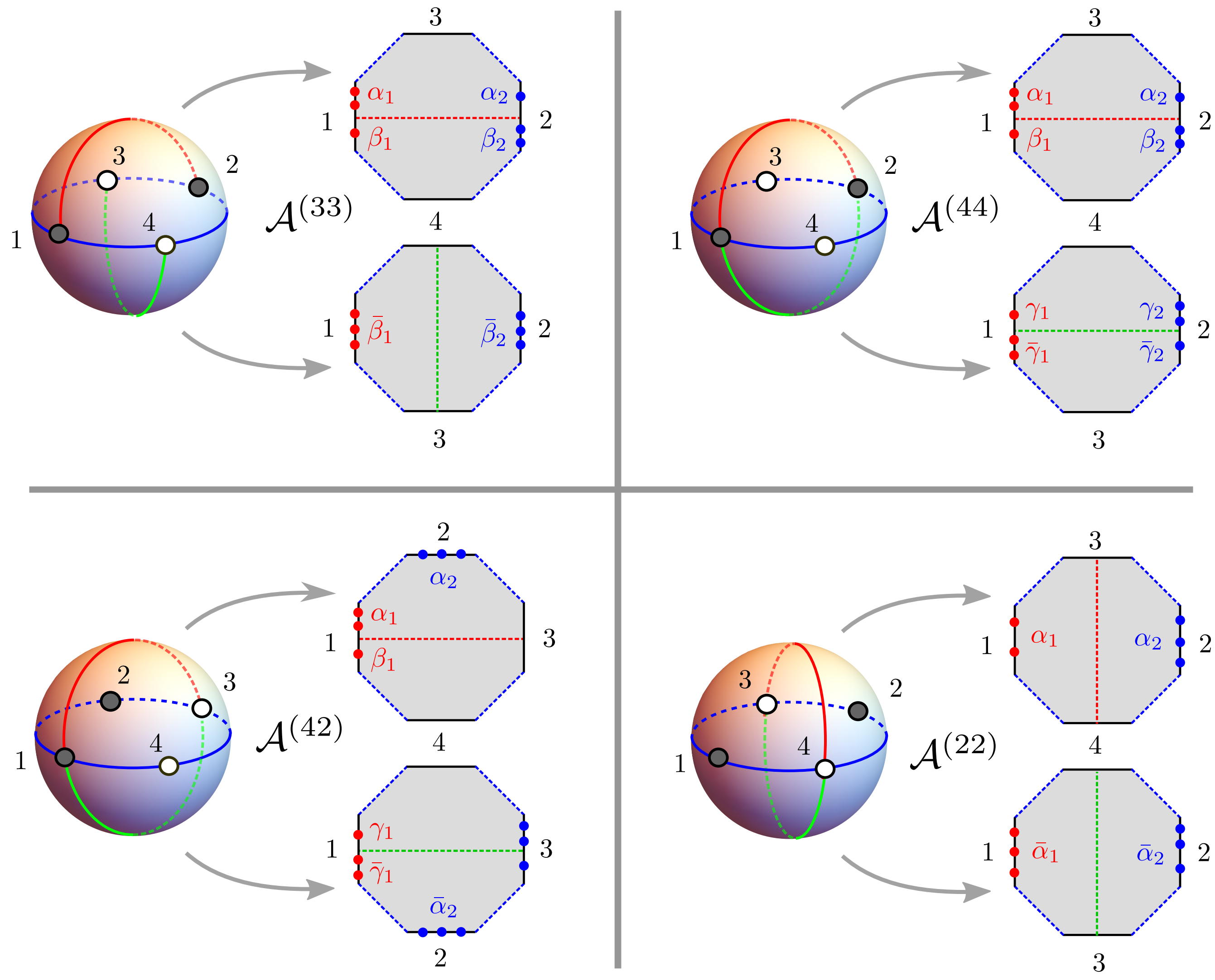
Virtual excitation brought over an **edge of width 0**; let there be a kinematic factor $\mathbf{x^{-iu}}$:

$$\sum_{a>0} \int \frac{du}{2\pi}\, \mu(u)\, x^{-iu} = \frac{g^2}{2} \left(\mathrm{Li}_2 - \log(x) \log(1-x)\right)$$

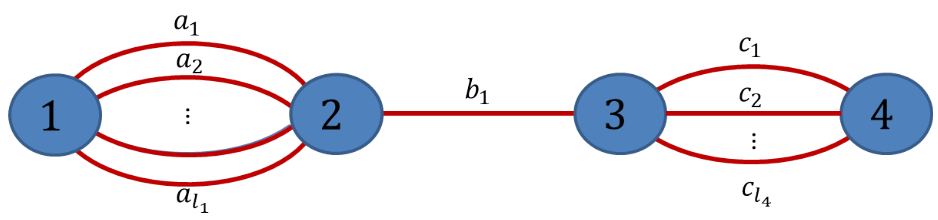
Realisation: **tilt hexagons** into general kinematics by Cartan generators. [Fleury, Komatsu (2016)]

Further tree results

Planar four-point functions with **two scalar BMN** operators:



Colour dressing: to match on field theory we have to **exclude the graph** (BMN at points 1,2):

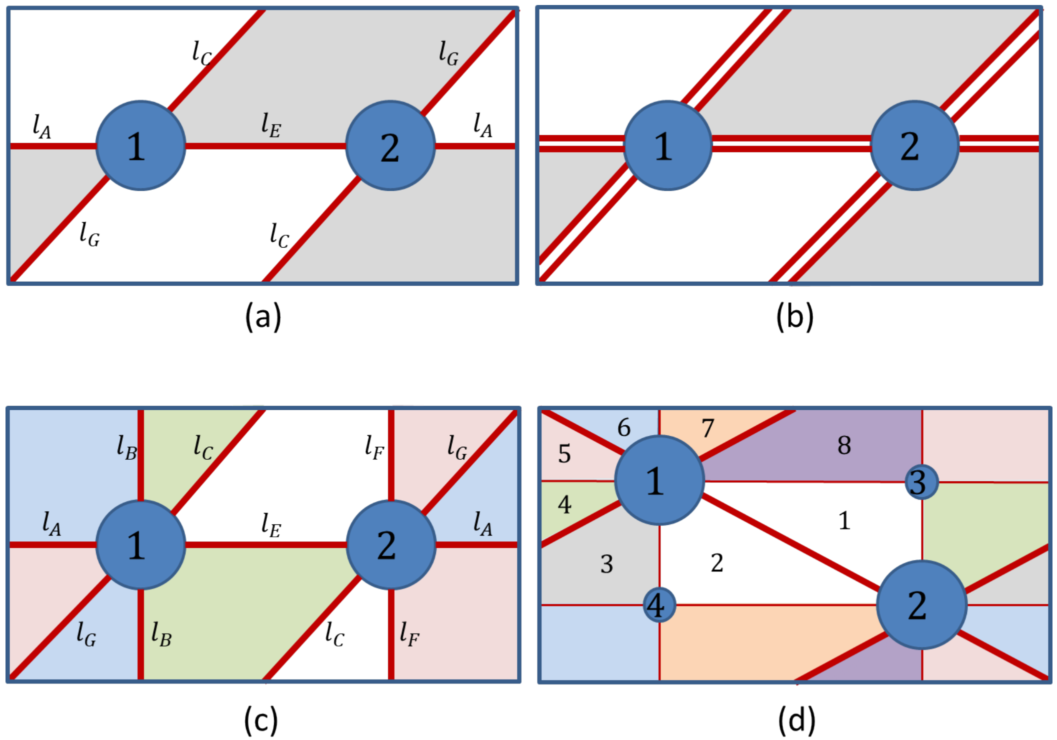


- With $SU(N)$ gauge group, its **colour factor vanishes**.
- **Integrability** captures **combinatorics and flavour**, but **no colour information**!
  For certain applications, colour factors may need to be re-instated.

Double trace operators can be incorporated by **point splitting**, i.e. putting the two traces at distinct points on the figures and taking a coincidence limit in the end.

Torus contribution to the single-trace BMN two-point function:

- On a **torus** two rings (colour traces) can be connected by **four lines**.
- For **multiple lines** we obtain **ribbons**; the associated **colour factors** are easy to read off.
- Tiling by **four hexagons**: rings connected to themselves. **Singular!**
- Inserting an **identity** operator into **each octagon** in (a) we can triangulate as in (d).



Colour factors for ribbon graphs with l_A, l_C, l_E, l_G propagators along the edges A, C, E, G:

$$T_{l_A l_C l_E l_G} = \mathrm{Tr}(T(a_1)\dots T(a_{l_A})\, T(c_1)\dots T(c_{l_C})\, T(e_1)\dots T(e_{l_E})\, T(g_1)\dots T(g_{l_G})) * \\ \mathrm{Tr}(T(a_{l_A})\dots T(a_1)\, T(c_{l_C})\dots T(c_1)\, T(e_{l_E})\dots T(e_1)\, T(g_{l_G})\dots T(g_1))$$

- $\mathbf{T_{ijkl}}$ has complete **permutation symmetry**
- $\mathbf{T_{ij00}} = T_{(i+j)000} \Rightarrow$ genuine **torus** graphs have three or four edges of width > 0.
- $T_{i110} = -N^i + \dots,\ T_{ij10} = 0\, N^{i+j-1} + \dots\ ,\ i, j > 1.$

correlator	Field theory	T_{ijkl}	hexagon amplitude
$\langle \mathcal{B}_4\, \mathcal{B}_4 \rangle\ N^4 a_{12}^4$	$-2 * T_{2110}$	$-N^2 + \ldots$	$\mathcal{A}^8_{2110} = -2$
	$+1 * T_{1111}$	$+N^2 + \ldots$	$\mathcal{A}^8_{1111} = +1$
$\langle \mathcal{B}_5\, \mathcal{B}_5 \rangle\ N^5 a_{12}^4$	$+1 * T_{3110}$	$-N^3 + \ldots$	$\mathcal{A}^8_{3110} = +1$
	$+1 * T_{2111}$	$+N^3 + \ldots$	$\mathcal{A}^8_{2111} = +1$
	$+(1 \pm \sqrt{5}) * T_{4110}$	$-N^4 + \ldots$	$\mathcal{A}^8_{4110} = +(1 \pm \sqrt{5})$
	$+(3 \mp \sqrt{5}) * T_{3111}$	$+N^4 + \ldots$	$\mathcal{A}^8_{3111} = +(3 \mp \sqrt{5})$
$\langle \mathcal{B}_6^{\mp}\, \mathcal{B}_6^{\mp} \rangle\ N^6\, a_{12}^4$	$+\frac{1}{2}(1 \mp \sqrt{5}) * T_{2220}$	$+N^4 + \ldots$	$\mathcal{A}^8_{2220} = +\frac{1}{2}(1 \mp \sqrt{5})$
	$-\frac{1}{2}(1 \mp \sqrt{5}) * T_{2211}$	$+N^4 + \ldots$	$\mathcal{A}^8_{2211} = -\frac{1}{2}(1 \mp \sqrt{5})$
	$+1 * T_{2121}$	$+N^4 + \ldots$	$\mathcal{A}^8_{2121} = +1$
	$+5 * T_{5110}$	$-N^5 + \ldots$	$\mathcal{A}^8_{5110} = +5$
	$+2 * T_{4111}$	$+N^5 + \ldots$	$\mathcal{A}^8_{4111} = +2$
$\langle \mathcal{B}'_7\, \mathcal{B}'_7 \rangle\ N^7 a_{12}^4$	$-1 * T_{3220}$	$+N^5 + \ldots$	$\mathcal{A}^8_{3220} = -1$
	$+1 * T_{3211}$	$+N^5 + \ldots$	$2 * \mathcal{A}^8_{3211} = +1$
	$+2 * T_{3121}$	$+N^5 + \ldots$	$\mathcal{A}^8_{3121} = +2$
	$+\frac{3}{2} * T_{2221}$	$+N^5 + \ldots$	$\mathcal{A}^8_{2221} = +\frac{3}{2}$

Glueing and kinematics

Kinematics in a **plane** — **complexify**:

$$\mathcal{Z} = \eta \cdot \Phi\,, \qquad \eta = \left(\frac{1+\alpha\bar{\alpha}}{2},\, i\frac{1-\alpha\bar{\alpha}}{2},\, i\,\mathrm{Im}\,\alpha,\, i\,\mathrm{Re}\,\alpha,\, 0,\, 0\right)$$

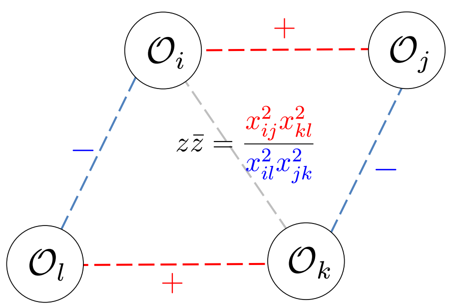
and similarly in configuration space:

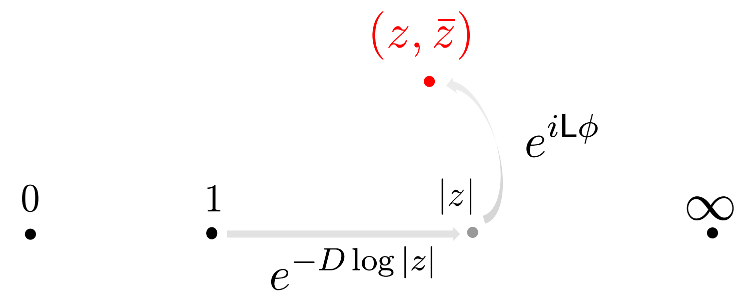
$$x_1 = \{0,\, \tfrac{1}{2}(z+\bar{z}),\, \tfrac{i}{2}(z-\bar{z}),\, 0\}\,, \quad x_2 = \{0,\,0,\,0,\,0\}\,, \quad x_3 = \{0,\,1,\,0,\,0\}\,, \quad x_4 = \{0,\,\infty,\,0,\,0\}\,.$$

Conformal four-point functions depend on **cross ratios:**

$$\frac{x_{12}^2 x_{34}^2}{x_{14}^2 x_{23}^2} = z\bar{z}\,, \qquad \frac{x_{13}^2 x_{24}^2}{x_{14}^2 x_{23}^2} = (1-z)(1-\bar{z})$$

In the left panel below put $x_i = 0,\, x_k = \infty,\, x_l = 1,\, x_j(z\,\bar{z})$. Graphs: [Fleury, Komatsu (2016)]





Left hexagon: **kinematical dependence trivial** (choice of points). **Right** hexagon: **transform**!

Spin chain **vacuum**: $Z = \phi^{12}$, $\bar{Z} = \phi^{34}$ forbidden. **Excitations** in the spin chain picture:

$$\phi^{aa'} \to \phi^a \, \bar{\phi}^{a'} \,, \qquad D^{\alpha\dot{\alpha}} \to \psi^\alpha \, \bar{\psi}^{\dot{\alpha}} \,, \qquad a, \, \alpha, \dot{\alpha} \in \{1,2\}, \; a' \in \{3,4\}$$

- Fermions: $\phi\bar{\psi}$, $\psi\bar{\phi}$
- $psu(2|2)_L \times psu(2|2)_R$ act on $A = (a, \alpha)$, $\bar{A} = (a', \dot{\alpha})$.
- Hexagon: after moving points into a plane, one **diagonal psu(2, 2)** symmetry algebra.

- **Glueing = propagating bound states over an edge**. Imagine inserting unity.

Antisymmetric representation, bound state level **a**: (**symmetric** representation: $\phi \leftrightarrow \psi$)

$$(\psi^1)^{a-k}(\psi^2)^k \,, \qquad (\psi^1)^{a-k-1}(\psi^2)^k \phi^a \,, \qquad (\psi^1)^{a-k-1}(\psi^2)^{k-1} \phi^1 \phi^2$$

The tilting transformation

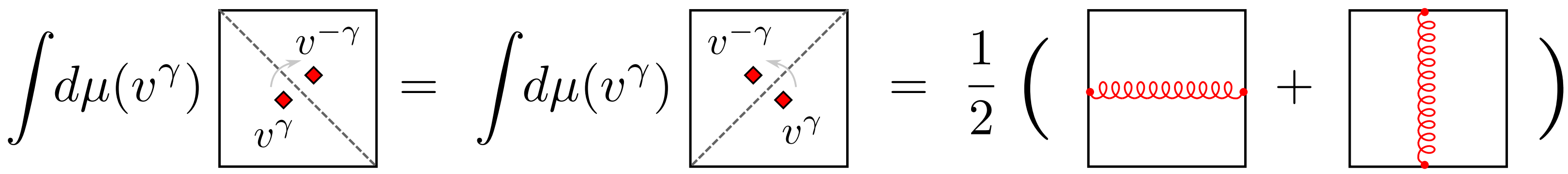
$$1 \to (z, \bar{z}) \quad : \quad \frac{1}{2}(D - J) = E = i \, \tilde{p} = i \, u + \ldots \,, \qquad L = L_1^1 - L_2^2 \,, \quad e^{i\phi} = \sqrt{\frac{z}{\bar{z}}} \,,$$

causes a **weight factor**

$$W(z, \bar{z}) = (z\bar{z})^{-iu} \left(\frac{z}{\bar{z}}\right)^{\frac{L}{2}} \qquad \Rightarrow \qquad (\psi^1)^{a-k}(\psi^2)^k \to (z\bar{z})^{-iu} \left(\frac{z}{\bar{z}}\right)^{\frac{a}{2}-k} (\psi^1)^{a-k}(\psi^2)^k \,,$$

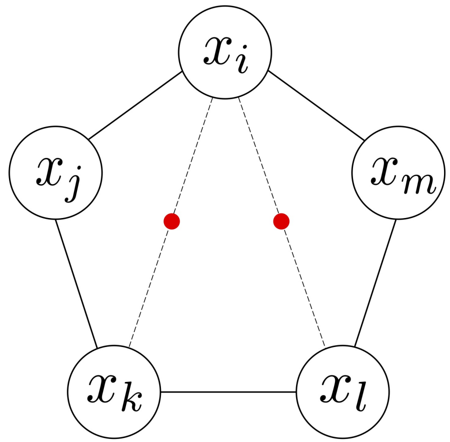
where flavour effects have been suppressed.

One loop, four points: [Fleury, Komatsu (2016)], [Eden, Jiang, le Plat, Sfondrini (2017)]



in terms of Yang-Mills exchanges calculated with $\mathcal{N} = 2$ superfields. **Glueing** = **Feynman**!

Double glueing at five points, one loop: [Fleury,Komatsu (2017)]



- Truncated residue calculation, matched on an ansatz. **Integration procedure?**
- Length changing/brading by e^{ip} dealt with by **"Z-marker averaging"** — consistent?

Particles on different edges $\Rightarrow$ **crossing/mirror transformations**. Use "$\mathbf{3}\gamma$, $\mathbf{1}\gamma$" kinematics

$$x_1^- \to 1/x_1^- \,, \qquad x_2^+ \to 1/x_2^+ \,.$$

Processes to calculate e.g. (leading order in g)

$$I_{\text{two-magnon}} = \sum_{a,b=1}^{\infty} \sum_{k,l=0}^{a-1,b-1} \int \frac{du\, dv\, a\, b\, g^2}{4\pi^2 (u^2 + \frac{a^2}{4})^2 (v^2 + \frac{b^2}{4})^2} W_1\, W_2\, \Sigma^{ab}\, X_k^{k,l}$$

with the **bound state S-matrix** in the **symmetric rep.** [Arutyunov, de Leeuw, Torrielli (2009)]

$$X_n^{k,l} = A \frac{\prod_{j=1}^{n}(a-j) \prod_{j=1}^{k+l-n}(b-j)}{\prod_{j=1}^{k}(a-j) \prod_{j=1}^{l}(b-j) \prod_{j=1}^{k+l}(-i\,\delta u + \frac{a+b}{2} - j)}$$
$$* \sum_{m=0}^{k} \binom{k}{k-m} \binom{l}{n-m} \prod_{j=1}^{m} c_j^+ \prod_{j=1-m}^{l-n} c_j^- \prod_{j=1}^{k-m} d_{k-j+2} \prod_{j=1}^{n-m} \tilde{d}_{k+l-m-j+2} \,,$$

$$c_j^{\pm} = -i\,\delta u \pm \frac{a-b}{2} - j + 1 \,, \quad d_j = a + 1 - j \,, \quad \tilde{d}_j = b + 1 - j \,, \qquad A = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+} \left(\frac{x_1^-\, x_2^+}{x_1^+\, x_2^-} \right)^{\frac{1}{2}}$$

and the **mirror dressing factor** [Beisert, Eden, Staudacher (2006)], [Arutyunov, Frolov (2008)]

$$\Sigma^{ab} = \frac{\Gamma[1 + \frac{a}{2} + i\,u]}{\Gamma[1 + \frac{a}{2} - i\,u]} \frac{\Gamma[1 + \frac{b}{2} - i\,v]}{\Gamma[1 + \frac{b}{2} + i\,v]} \frac{\Gamma[1 + \frac{a+b}{2} - i(u-v)]}{\Gamma[1 + \frac{a+b}{2} + i(u-v)]} \,.$$

- **X** is essentially a $_4\mathbf{F}_3$.
- $_{\mathbf{p+1}}\mathbf{F}_{\mathbf{p}}$ tends to yield **hyperlogs**.
- Four different types of bound states $\Rightarrow$ **16 diagonal elements** of the S-matrix.
- These are sums of **several X's** with coefficients involving $\mathbf{x}^+, \mathbf{x}^-$.
- **Poles** in $(u-v)$ only in the **lower half-plane**.
- Choose **u** in the **upper half-plane**, **v** in the **lower**.
  No contribution from the phase, **poles only from the measure**.
- Note the **momentum factors**

$$e^{\frac{ip}{2}} = \sqrt{\frac{x^+}{x^-}} = \sqrt{\frac{u+i\frac{a}{2}}{u-i\frac{a}{2}}} + O(g)\,.$$

  Braiding introduces compensating **half-powers**. But **which ones?**
- **Some** S-matrix elements **factor** at $u = i\frac{a}{2}$, $v = -i\frac{b}{2}$.

$\mathbf{Y_{11}}$ **sum-integral, derivative** from the residue at $1/(u^-)^2$ falls **onto** $(\mathbf{z_1\bar{z}_1})^{-\mathbf{iu}}$:

$$\tilde{I}_{Y_{11}} = \log(z_1\bar{z}_1)\sum_{a,b=1}^{\infty}\sum_{k,l=0}^{a-1,b-1} z_1^{a-k}\,\bar{z}_1^{k}\,z_2^{-b}\,\bar{z}_2^{-l+b}\frac{\Gamma[a-k+b-l]\qquad\Gamma[1+k+l]}{4\,a\,\Gamma[a-k]\,\Gamma[1+b-l]\,\Gamma[1+k]\,\Gamma[1+l]}$$

Define

$$z\bar{z} = r^2\,, \quad \frac{z}{\bar{z}} = p^2\,. \qquad \Rightarrow \qquad z^{a-k}\bar{z}^k = r^a\,p^{a-2k} \qquad \Rightarrow r\,\frac{\partial}{\partial r}\,z^{a-k}\bar{z}^k = a\;z^{a-k}\bar{z}^k$$

- The **inverse operation** is $\int \mathbf{dr/r}$.
- Comparing to the series the **constant part $\mathbf{O(r^0)}$ must be subtracted**.

Omit $1/a$ from the summand, swap sums over a, k and b, l, send $a \mapsto a+k,\ b \mapsto b+l$:

$$\text{sum} = \frac{-z_1 + \bar{z}_2\,z_1 - \bar{z}_1\,\bar{z}_2\,z_1 + z_1\,z_2 - z_1^2\,z_2}{4(1-\bar{z}_1)(1-\bar{z}_2+\bar{z}_1\bar{z}_2)(1-z_1)(1-z_2+z_1z_2)}$$

Integrating over dr/r and subtracting the constant:

$$\begin{aligned}\tilde{I}_{Y_{11}} &= \frac{\log(z_1\,\bar{z}_1)}{4}\,\frac{z_1(\log[1-\bar{z}_1]-\log[1-z_1])}{\bar{z}_1 - z_1} \\ &+ \frac{\log(z_1\,\bar{z}_1)}{4}\,\frac{\bar{z}_2\,z_1\,z_2(-\log[1-1/\bar{z}_2]+\log[1-\bar{z}_1-1/\bar{z}_2]+\log[1-1/z_2]-\log[1-z_1-1/z_2])}{\bar{z}_1\,\bar{z}_2 - \bar{z}_1\,\bar{z}_2\,z_2 - z_1\,z_2 + \bar{z}_2\,z_1\,z_2}\end{aligned}$$

It will be convenient to relabel

$$\{z_1,\,\bar{z}_1,\,z_2,\,\bar{z}_2\} \to \{z,\,b,\,1/y,\,1/a\}\,.$$

- **Integrating where possible** (e.g. X with a derivative on it), we find a **set of denominators** and an **alphabet** for the hyperlogs.
- **“Symbolism”** allows to fit the rest.

Eight denominators:

$$a-y,\ b-z,\ b\,y-a\,z,\ a-y+b\,y-a\,z,\ b-b\,y-z+a\,z,$$
$$a\,b-a\,b\,y-a\,b\,z-y\,z+a\,y\,z+b\,y\,z,\ a\,b-a\,b\,y-y\,z+a\,y\,z,\ a\,b-a\,b\,z-y\,z+b\,y\,z$$

Letters:

$$1-a,\ a,\ 1-b,\ 1-a-b,\ b,\ 1-y,\ a-y,\ y,\ 1-z,\ b-z,\ 1-y-z,\ z,$$
$$a-y+b\,y-a\,z,\ b-b\,y-z+a\,z,\ a\,b-a\,b\,y-a\,b\,z-y\,z+a\,y\,z+b\,y\,z$$

- The denominators are point **permutations** of $\mathbf{b-z}$. Most of this will be Bloch Wigner.
- The last three **letters** are of $\mathbf{b-z}$ type. For example

$$\mathrm{Li}_2\left(1-\frac{z}{b}\right)-\frac{1}{2}\log^2(b)$$

can be used to generate the part of the symbols with the permuted $b-z$ letters.

Conclusions

- **Hexagon tesselations** compute flavour and combinatorics in tree-level $\mathcal{N} = 4$ correlators. **Colour dressing** is **needed**!
- With it we can do **non-planar integrability, multi-trace operators, general colour groups, multi-point functions . . .**

BPS one-loop five-point process

- The **symmetric rep.** bound state matrix [Arutyunov, de Leeuw, Torrielli (2009)] is in fact an inverse w.r.t. Beisert's convention used for the hexagon.
- **Antisymmetric** rep. $\Rightarrow$ **second inverse**! General result? Simplify matrix.
- Beautiful **integration procedure**, the **alphabet** is **linear** $\Rightarrow$ should work to any order in g.
- Our set of functions consists of the **correct building blocks**, but seems hard to reconcile with the result by $\mathcal{N} = 2$ superfields [Alday, Eden, Korchemsky, Maldacena, Sokatchev (2010)].
- There should be **five Bloch-Wigner dilogs**, so five denominators and no $z - \bar{z}$ in the symbol.
- Conventions? Glitch with the coordinates? Problem of principle?
- **Braiding without averaging?**