Graded Geometry, Tensor Galileons & Duality

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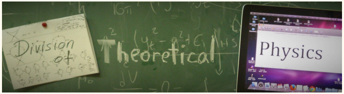

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& with G. Karagiannis, P. Schupp (to appear)

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Motivation from higher derivative interactions

Interacting theories with 2nd order field equations (Galileons, Lovelock, Horndeski &c.)

Simplify, highlight shared features and generalize using a unifying geometric formalism.

At simpler level, kinetic terms for mixed symmetry tensor fields in coordinate free form.

For differential form, use Hodge star. For e.g. (linearized) Einstein-Hilbert or Curtright?

Motivation from electric/magnetic duality

Duals are not unique. Apart from the standard Hodge duality of p- & $(D-p-2)$-forms
Hull ’00-’01, West ’01, de Medeiros, Hull ’02, Boulanger, Cook, Ponomarev ’12, &c.

- Exotic duality of a p-form and a $(D-2, p)$ bipartite tensor.
- Standard duality of the graviton to a $(D-3, 1)$ dual graviton.
- Double duality of the graviton to a $(D-3, D-3)$ field.
- ...
- Infinite chains of dualities. Boulanger, Sundell, West ’15

Dual theories typically related by some first order parent action.

Find a parent action providing a common starting point for all types of dualizations.

Use Graded Geometry

(also more generally motivated by BV/BRST quantization of gauge theories, QP-manifolds and AKSZ σ-models,...)

Motivation from branes in string and M theories

String and M theories contain a host of non-standard, low codimension branes.

West '04; Bergshoeff, Riccioni '10; de Boer, Shigemori '12

Exotic branes source non-geometric fluxes and couple to mixed symmetry tensors.

de Boer, Shigemori '10, Bergshoeff Riccioni '10, A.Ch. Gautason, Moutsopoulos, Zagermann '13; A.Ch., Gautason '14, &c.

Fields in Wess-Zumino terms are typically exotic duals of the graviton or B/C-fields.

A unified and geometric way to think about these branes and couplings? (not in this talk)

Contents

Differential forms as functions

Idea: Tensor fields as functions on a graded supermanifold

A (smooth) supermanifold $\mathcal{M}$ is locally isomorphic to $C^\infty(U) \otimes \bigwedge^\bullet(\mathbb{R}^d)^*$, $U \subset \mathbb{R}^D$.

$\mathbb{Z}_2$-graded geometry, even coordinates x^i and odd coordinates θ^i, (N.B.: Focus on $D = d$)

$$\theta^i \theta^j = -\theta^j \theta^i \ .$$

Identification of functions on graded vector bundles with p-forms or p-vector fields,

$$C^\infty(T[1]M) \simeq \Omega^\bullet(M) \quad \text{and} \quad C^\infty(T^*[1]M) \simeq \Gamma(\wedge^\bullet TM) \ .$$

A function on $T[1]M$ may be expanded (and be related to differential forms) as

$$\omega(x,\theta) = \sum_{k=0}^{D} \frac{1}{k!}\, \omega_{i_1 \dots i_k}(x)\, \theta^{i_1} \dots \theta^{i_k} \ .$$

Integration is defined as usual for Grassmann variables, $\int \mathrm{d}^D\theta\, \theta^1 \theta^2 \dots \theta^D = 1$.

Mixed symmetry tensor fields as functions

For bipartite tensors of degree (p,q), consider functions on $\mathcal{M} = T[1]M \oplus T[1]M$,

$$\omega_{p,q} = \frac{1}{p!q!}\,\omega_{i_1\dots i_p j_1\dots j_q}(x)\,\theta^{i_1}\dots\theta^{i_p}\chi^{j_1}\dots\chi^{j_q}\ .$$

Two separate sets of odd coordinates θ^i and χ^i which mutually commute by convention,

$$\theta^i\theta^j = -\theta^j\theta^i\ ,\quad \chi^i\chi^j = -\chi^j\chi^i\ ,\quad \theta^i\chi^j = \chi^j\theta^i\ .$$

The components of the tensor field have manifest mixed index symmetry

$$\omega_{i_1\dots i_p j_1\dots j_q} = \omega_{[i_1\dots i_p][j_1\dots j_q]}\ .$$

Direct generalization to N-partite mixed symmetry tensors for $\mathcal{M} = \bigoplus^N T[1]M$.

Generalized Hodge duality

- Exterior derivatives $\mathrm{d} : \Omega^{p,q} \to \Omega^{p+1,q}$ and $\widetilde{\mathrm{d}} : \Omega^{p,q} \to \Omega^{p,q+1}$

$$\mathrm{d} = \theta^i \partial_i \quad \text{and} \quad \widetilde{\mathrm{d}} = \chi^i \partial_i \quad \text{with} \quad \mathrm{d}^2 = \widetilde{\mathrm{d}}^2 = 0 \quad \text{and} \quad \mathrm{d}\,\widetilde{\mathrm{d}} = \widetilde{\mathrm{d}}\,\mathrm{d}\,.$$

- Transposition (or exchange of θ and χ, or $\widetilde{\ }$ operation)

$$\omega_{p,q} \mapsto \omega^{\top_{\theta\chi}} \equiv \widetilde{\omega}_{q,p} = \frac{1}{p!q!} \omega_{i_1\dots i_p j_1 \dots j_q} \theta^{j_1} \dots \theta^{j_q} \chi^{i_1} \dots \chi^{i_p} \;.$$

- Hodge star operator for bipartite tensor fields, $\star : \Omega^{p,q} \to \Omega^{D-p,D-q}$ for $p+q \leq D$,
  A.Ch., Khoo, Roest, Schupp '17

$$(\star\,\omega)_{D-p,D-q} = \frac{1}{(D-p-q)!}\, \eta^{D-p-q}\, \widetilde{\omega}_{q,p} \qquad (\eta = \eta_{ij}\, \theta^i \chi^j)\,.$$

Full vs. partial Hodge duality

Define partial Hodge star operations $* : \Omega^{p,q} \to \Omega^{D-p,q}$ and $\widetilde{*} : \Omega^{p,q} \to \Omega^{p,D-q}$ as

cf. de Medeiros, Hull '02

$$
\begin{aligned}
*\,\omega &= \frac{1}{(D-p)!}\int_{\psi} \omega^{\top\theta\psi}\,(\eta^{\top\chi\psi})^{D-p} \propto \epsilon^{i_1\dots i_p}{}_{i_{p+1}\dots i_D}\,\omega_{i_1\dots i_p j_1\dots j_q}\,\theta^{i_{p+1}}\dots\theta^{i_D}\chi^{j_1}\dots\chi^{j_q}\ ,\\
\widetilde{*}\,\omega &= \frac{1}{(D-q)!}\int_{\psi} \omega^{\top\chi\psi}\,(\eta^{\top\theta\psi})^{D-q} \propto \epsilon^{j_1\dots j_q}{}_{j_{q+1}\dots j_D}\,\omega_{i_1\dots i_p j_1\dots j_q}\,\theta^{i_1}\dots\theta^{i_p}\chi^{j_{q+1}}\dots\chi^{j_D}\ ,
\end{aligned}
$$

ψ^i being an auxiliary set of odd coordinates. The combination $*\widetilde{*}$ is different than $\star$:

$$\star\,\omega = *\widetilde{*}\,\overline{\omega}\ ,$$

$$\overline{\omega} := (-1)^{\epsilon}\sum_{n=0}^{\min(p,q)} \frac{(-1)^n}{(n!)^2}\,\eta^n\,\mathrm{tr}^n\,\omega\ , \qquad \epsilon = (D-1)(p+q)+pq+1\ .$$

Very welcome that $\star$ also encodes all traces of the mixed symmetry tensor.

Dual operations and GL(D) irreducibility

The partial Hodge stars define dual operations to the exterior derivatives and trace,
cf. de Medeiros, Hull '02

$$\mathrm{d}^{\dagger} := (-1)^{1+D(p+1)} * \mathrm{d} * : \Omega^{p,q} \to \Omega^{p-1,q} , \qquad (\text{and} \quad \widetilde{\mathrm{d}}^{\dagger}) ,$$
$$\sigma := (-1)^{1+D(p+1)} * \mathrm{tr} * : \Omega^{p,q} \to \Omega^{p+1,q-1} , \qquad (\text{and} \quad \widetilde{\sigma}) .$$

A bipartite tensor is $GL(D)$-irreducible (denote $\omega_{[p,q]}$) if and only if for $p \geq q$ it satisfies

$$\sigma\,\omega = 0 \quad \text{and} \quad \widetilde{\omega} = \omega \quad \text{for} \quad p = q .$$

Irreducible bipartite tensors are obtained via the action of a Young projector $\mathcal{P}_{[p,q]}$, s.t.

$$\omega_{[p,q]} = \mathcal{P}_{[p,q]}\,\omega_{p,q} .$$

The explicit form of this projector is de Medeiros '03

$$\mathcal{P}_{[p,q]} = \begin{cases} \mathbb{I} + \sum_{n=1}^{q} c_n(p,q)\widetilde{\sigma}^n\sigma^n , & p \geq q \\ \mathbb{I} + \sum_{n=1}^{p} c_n(q,p)\sigma^n\widetilde{\sigma}^n , & p \leq q \end{cases} , \quad c_n(p,q) = \frac{(-1)^n}{\prod_{r=1}^{n} r(p-q+r+1)} .$$

Kinetic terms

Kinetic terms in Minkowski spacetime $\mathbb{R}^{1,D-1}$ may be written in a unified form,

$$\boxed{\mathcal{L}_{\text{kin}}(\omega_{p,q}) = \int_{\theta,\chi} \mathrm{d}\omega \star \mathrm{d}\omega\,.}$$

The component form is obtained directly performing the Berezin integration:

- For $p = q = 0$, scalar $\frac{1}{2}\phi\Box\phi$.
- For $p = 1$, $q = 0$, Maxwell $-\frac{1}{4}F_{ij}F^{ij}$.
- For $p = q = 1$, linearized Einstein-Hilbert

$$\mathcal{L}_{\text{LEH}}(h_{[1,1]}) = -\frac{1}{4}h^i{}_i\Box h^j{}_j + \frac{1}{2}h^k{}_k\partial_i\partial_j h^{ij} - \frac{1}{2}h_{ij}\partial^j\partial_k h^{ik} + \frac{1}{4}h_{ij}\Box h^{ij}.$$

- For $p = 2$, $q = 1$, Curtright

$$\begin{aligned}\mathcal{L}_{\text{kin}}(\omega_{[2,1]}) \quad = \quad & \frac{1}{2}\Big(\partial_i\omega_{jk|l}\partial^i\omega^{jk|l} - 2\partial_i\omega^{ij|k}\partial^l\omega_{lj|k} - \partial_i\omega^{jk|i}\partial^l\omega_{jk|l} - \\ & - 4\omega_i{}^{j|i}\partial^k\partial^l\omega_{kj|l} - 2\partial_i\omega_j{}^{k|j}\partial^i\omega^l{}_{k|l} + 2\partial_i\omega_j{}^{i|j}\partial^k\omega^l{}_{k|l}\Big)\ .\end{aligned}$$

“Galileon” type, higher derivative interaction terms

Galileons were introduced for scalar fields in flat space s.t. they have 2nd order EOMs.
Nicolis, Rattazzi, Trincherini ’08

They are invariant under the characteristic symmetry $\partial_i\phi \to \partial_i\phi + c_i$ and $\phi \to \phi + c$.

More generally, bipartite tensor Galileons may be written in a graded-geometric way:

$$\mathcal{L}_{\text{Gal}}(\omega_{p,q}) = \sum_{n=1}^{n_{\max}} \frac{1}{(D-k_n)!} \int_{\theta,\chi} \eta^{D-k_n}\, \omega\, (\mathrm{d}\widetilde{\mathrm{d}}\,\omega)^{n-1} (\mathrm{d}\widetilde{\mathrm{d}}\,\widetilde{\omega})^n, \quad k_n = (p+q+2)n - 1\,.$$

The symmetries are found using a higher version of the Poincaré lemma, s.t. $\mathrm{d}\widetilde{\mathrm{d}}\delta\omega = 0$:

$$\delta\omega_{p,q} = \mathrm{d}\kappa_{p-1,q} + \widetilde{\mathrm{d}}\kappa_{p,q-1} + c_{i_1\dots i_p k_0 k_1\dots k_q}\theta^{i_1}\dots\theta^{i_p}x^{k_0}\chi^{k_1}\dots\chi^{k_q}\,,$$

c being a constant, totally antisymmetric tensor. For $p = q$ there is an enhancement:

$$\mathcal{L}_{\text{Gal}}(\omega_{[p,p]}) = \sum_{n=1}^{n_{\max}} \frac{1}{(D-k_n)!} \int_{\theta,\chi} \eta^{D-k_n}\, \omega\, (\mathrm{d}\widetilde{\mathrm{d}}\,\omega)^n, \quad k_n = (p+1)n + p\,.$$

A controlled way to find and write terms like...

...for scalars in 4D Nicolis, Rattazzi, Trincherini '08

$$n=2: \qquad \partial^i\phi\,\partial^j\phi\,\partial_i\partial_j\phi - \partial^i\phi\,\partial_i\phi\,\Box\phi$$

$$n=3: \qquad -(\Box\phi)^2\,\partial_i\phi\,\partial^i\phi + 2\,\Box\phi\,\partial_i\phi\,\partial_j\phi\,\partial^i\partial^j\phi + \\ +\partial_i\partial_j\phi\,\partial^i\partial^j\phi\,\partial_k\phi\,\partial^k\phi - 2\,\partial_i\phi\,\partial^i\partial^j\phi\,\partial_j\partial_k\phi\,\partial^k\phi$$

$$n=4: \qquad -(\Box\phi)^3\,\partial_i\phi\,\partial^i\phi + 3\,(\Box\phi)^2\,\partial_i\phi\,\partial_j\phi\,\partial^i\partial^j\phi + \\ +3\,\Box\phi\,\partial_i\partial_j\phi\,\partial^i\partial^j\phi\,\partial_k\phi\,\partial^k\phi - 6\,\Box\phi\,\partial_i\phi\,\partial^i\partial^j\phi\,\partial_j\partial_k\phi\,\partial^k\phi + \\ -2\,\partial_i\partial^j\phi\,\partial_j\partial^k\phi\,\partial_k\partial^i\phi\,\partial_l\phi\,\partial^l\phi - 3\,\partial_i\partial_j\phi\,\partial^i\partial^j\phi\,\partial_k\phi\,\partial_l\phi\,\partial^k\partial^l\phi + \\ +6\,\partial_i\phi\,\partial^i\partial^j\phi\,\partial_j\partial_k\phi\,\partial^k\partial^l\phi\,\partial_l\phi$$

Non-triviality and generalizations

- Whether or not these terms have a dynamical footprint is easy to check:
  - Bound on field appearances: $k_n \leq D$; or, $n_{\text{max}}^{(p,q)} = \left\lfloor \frac{D+1}{p+q+2} \right\rfloor$ and $n_{\text{max}}^{[p,p]} = \left\lfloor \frac{D-p}{p+1} \right\rfloor$.
  - Evenophilia of total degree $p+q$: $(\mathrm{d}\widetilde{\mathrm{d}}\,\omega_{p,q})^2|_{p+q=\text{odd}} = 0 = (\mathrm{d}\widetilde{\mathrm{d}}\,\widetilde{\omega}_{q,p})^2|_{p+q=\text{odd}}$.

- A number of generalizations exist, elegantly captured in the graded formalism:
  cf. Deffayet, Deser, Esposito-Farese '09, Deffayet, Esposito-Farese, Vikman '09
  - Multiple species; allows Galileons with odd total degree too.
  - Field equations up to second order.
  - Curved space; e.g. Horndeski for scalar (more tricky for bipartite tensors).

Standard duality and parent actions

- A first order Lagrangian for the dualization of a $(p-1)$-form to a $(D-p-1)$-form is

$$\mathcal{L}_{\mathrm{P}}(F_p, \lambda_{p+1}) = -\frac{1}{2(p+1)!} F_{i_1 \dots i_p} F^{i_1 \dots i_p} - \frac{1}{(p+1)!} \lambda^{i_1 \dots i_{p+1}} \partial_{i_1} F_{i_2 \dots i_{p+1}} \, .$$

λ-EOM $\rightsquigarrow$ Bianchi identity for F_p $\rightsquigarrow$ 2nd order $\mathcal{L}$ for a $(p-1)$-form potential ω_{p-1}.
F-EOM $\rightsquigarrow$ Duality relation $\rightsquigarrow$ 2nd order $\mathcal{L}$ for a dual potential $\widehat{\omega}_{D-p-1} = *\lambda_{p+1}$.

- For the standard dualization of the graviton $h_{[1,1]}$, a 1st order Lagrangian looks like
West '01

$$\mathcal{L}_{\mathrm{P}}(f_{2,1}, \lambda_{3,1}) = f_{ij}{}^j f^{ik}{}_k - \frac{1}{2} f_{ijk} f^{ikj} - \frac{1}{4} f_{ijk} f^{ijk} + \frac{1}{2} \lambda_{ijkl} \partial^i f^{jkl} \, .$$

λ-EOM $\rightsquigarrow$ Bianchi identity for $f_{2,1}$ $\rightsquigarrow$ LEH (the antisymmetric part cancels out)
f-EOM $\rightsquigarrow$ Duality relation $\rightsquigarrow$ $\mathcal{L}$ for the dual graviton $\hat{\omega}_{[D-3,1]} = *\widehat{\lambda}_{3,1}$ s.t. $\mathrm{tr}\,\widehat{\lambda} = 0$.

Exotic duality and parent actions

- For the exotic dualization of e.g. a 2-form, the starting point is
  Boulanger, Cook, Ponomarev '12, Bergshoeff, Hohm, Penas, Riccioni '16

$$\mathcal{L}_{\mathrm{P}}(Q_{1,2},\lambda_{2,2}) = -\frac{1}{6}\,Q_{i|jk}Q^{i|jk} + \frac{1}{3}\,Q_{i|}{}^{ij}Q^{k|}{}_{kj} + \frac{1}{2}\,\lambda_{ij|kl}\partial^{i}Q^{j|kl}\,.$$

λ-EOM $\rightsquigarrow$ the same 2nd order action as in the standard case, for the 2-form.
Q-EOM $\rightsquigarrow$ a dual theory for a $(D-2,2)$ potential $\widehat{\omega}_{[D-2,2]} = *\widehat{\lambda}_{2,2}$ s.t. $\mathrm{tr}\,\widehat{\lambda} = 0$.

- Also double dual graviton, duals for Curtright and higher $(p,1)$ tensors &c.

A unified treatment of all these dualizations?

A universal first order action

A single two-parameter parent Lagrangian simultaneously accounting for

- the standard and exotic duals for any differential p-form, and
- the standard and double standard duals for any bipartite tensor of type $(p, 1)$.

$$\boxed{\mathcal{L}_{\mathrm{P}}^{(p,q)}(F,\lambda) = \int_{\theta,\chi} F_{p,q} \star \mathcal{O}\, F_{p,q} + \int_{\theta,\chi} \mathrm{d}F_{p,q} * \widetilde{*}\, \lambda_{p+1,q}} \quad \text{for} \quad D \geq p+q+1\,.$$

- F and λ are independent $GL(D)$-reducible bipartite tensors.
- $\mathcal{O} = \mathcal{O}^{(p,q)}$ is a (known in closed form) operator acting on (p, q) tensors s.t.

$$\mathcal{O}\, \mathrm{d}\omega_{p-1,q} = \mathrm{d}\omega_{[p-1,q]} + \widetilde{\mathrm{d}}(\dots)\,.$$

  Role: Yield the kinetic term for *irreducible* potential $\omega_{[p-1,q]}$ upon taking λ-EOM.

- E.g. $\mathcal{O}^{(2,1)} = \mathbb{I} - \frac{1}{2}\,\widetilde{\sigma}\,\sigma$ (graviton), $\mathcal{O}^{(3,1)} = \mathbb{I} - \frac{1}{3}\,\widetilde{\sigma}\,\sigma$, $\mathcal{O}^{(2,2)} = \frac{4}{3}\,\mathbb{I} - \frac{1}{3}\,\sigma\,\widetilde{\sigma}$ (Curtright)

Domains of applicability

For four domains of values, this Lagrangian yields all dual theories, in particular

see also poster by Georgios Karagiannis

p	q	Original field	Dual field	Duality type
$\in [1, D-1]$	0	$[p-1, 0]$	$[D-p-1, 0]$	Standard
$\in [2, D-2]$	1	$[p-1, 1]$	$[D-p-1, 1]$	Standard
1	$\in [1, D-2]$	$[0, q]$	$[D-2, q]$	Exotic
2	$\in [2, D-3]$	$[1, q]$	$[D-3, q]$	Standard

- All component forms stem from this single starting point.
- All necessary cancellations follow from general identities.
- Extremal case $p = 0$ also relevant for deformations related to domain walls.

Epilogue & Outlook

- Graded geometry, even at its simplest, offers an elegant way to unify different $\mathcal{L}$'s
- Geometric expressions highlighting shared features. Generalized Hodge star $\star$
- General treatment of (many, all in certain domain) standard and exotic dualizations

- Extend to the infinite chain of dualities of Boulanger, Sundell, West '15
- Sources, exotic branes; unified approach to their effective actions
  cf. Bergshoeff, Kleinschmidt, Musaev, Riccioni '19
- Higher gauge theory approach to mixed symmetry tensors á la Grützmann, Strobl '14

THANKS

Back-up slides

An alternative form for kinetic and mass terms

In the irreducible case, define a graded geometric analog of a gen'd Einstein tensor,
cf. Hull '01

$$E_{[p,q]} := (-1)^{(D-1)(p+q)} * \widetilde{*} \, \mathrm{d} \star \mathrm{d} \, \omega_{[p,q]} \, .$$

Then an alternative form of the kinetic term is

$$\mathcal{L}_{\text{kin}}(\omega_{[p,q]}) = \int_{\theta,\chi} \omega_{[p,q]} * \widetilde{*} E_{[p,q]} \, .$$

Mass terms take also a unified form,

$$\mathcal{L}_{\text{mass}}(\omega_{p,q}) = m^2 \int_{\theta,\chi} \omega \star \omega \, .$$

E.g., for $p = q = 1$ this is the familiar Fierz-Pauli term, $m^2 \left(h^{ij} h_{ij} - (h^i{}_i)^2 \right)$.

Generalizations of Galileons

- Multiple species

$$\mathcal{L}(\omega_0, \ldots, \omega_n) = \frac{1}{(D-k)!} \int_{\theta,\chi} \eta^{D-k} \omega_0^{(p_0,q_0)} \prod_{j=1}^{n} \mathrm{d}\tilde{\mathrm{d}}\, \omega_j^{(p_j,q_j)} \,.$$

$$\sum_{k=0}^{n} p_k = \sum_{k=0}^{n} q_k = k - n \,.$$

- Generalized Galileons (up to second order, polynomial)

$$\mathcal{L}(\omega_0, \ldots, \omega_n) = \frac{1}{(D-k)!} \int_{\theta,\chi} \eta^{D-k} \prod_i \omega_i^{(p_i,q_i)} \prod_j \mathrm{d}\omega_j^{(p_j,q_j)} \prod_k \tilde{\mathrm{d}}\omega_k^{(p_k,q_k)} \prod_l \mathrm{d}\tilde{\mathrm{d}}\omega_l^{(p_l,q_l)} \,.$$

Finding $\mathcal{O}$

The operator $\mathcal{O}$ has the role of selecting the irreducible field. The requirement is

$$\mathcal{O}\,\mathrm{d}\omega_{p-1,q} \overset{!}{=} \mathrm{d}\omega_{[p-1,q]} + \widetilde{\mathrm{d}}(\dots)\ .$$

We find (recall that $c_n(p,q) = \frac{(-1)^n}{\prod_{r=1}^{n} r(p-q+r+1)}$)

$$\mathcal{O} = \begin{cases} \mathbb{I} + \sum_{n=1}^{q} c_n(p-1,q)\,\widetilde{\sigma}^n\,\sigma^n\,, & p \geq q+1 \\ \mathbb{I} + \sum_{n=1}^{p-1} c_n(q,p-1)\left(\sigma^n\,\widetilde{\sigma}^n + \sum_{k=1}^{n}(-1)^k \prod_{m=0}^{k-1}(n-m)^2 \sigma^{n-k}\,\widetilde{\sigma}^{n-k}\right)\,, & p < q+1 \end{cases}\ .$$

N.B.: For the domains of interest, only one term in the sum is relevant.

In fact, the domains are such that solving for λ with this $\mathcal{O}$ leads to the 2nd order theory

$$\mathcal{L}^{(p,q)}_{\lambda\text{-on-shell}} = \int_{\theta,\chi} \mathrm{d}\omega_{[p-1,q]} \star \mathrm{d}\omega_{[p-1,q]}\ .$$

This guarantees that the first side of the duality is correctly obtained.

Comments on the dualization

Establishing the duality requires varying with respect to $F_{p,q}$. We first show that

$$\int_{\theta,\chi} \delta\,(F \star \mathcal{O}F) = 2\int_{\theta,\chi} \delta F \star \mathcal{O}F\,.$$

The F-variation then yields a duality relation, and $\mathcal{O}^{-1}$ is needed to solve it. We find

$$\begin{aligned}(\mathcal{O}^{(p,1)})^{-1} &= \mathbb{I} - \widetilde{\sigma}\,\sigma\,,\\ (\mathcal{O}^{(2,q)})^{-1} &= b_1\,\mathbb{I} + b_2\,\sigma\,\widetilde{\sigma} + b_3\,\sigma^2\,\widetilde{\sigma}^2\,,\end{aligned}$$

or trivial for the rest of the cases; b coefficients are given by

$$b_1 = \frac{q+1}{q+2}\,, \qquad b_2 = \frac{q+1}{2(q+2)}\,, \qquad b_3 = -\frac{q+1}{2q(q+2)}\,.$$

Further comments on the dualization

- Domain I: straightforward (dual field is a differential form).
- Domain II: decompose the Lagrange multiplier

$$\lambda_{p+1,1} = \widehat{\lambda}_{p+1,1} + \eta\,\mathring{\lambda}_{p,0}\,, \qquad \mathrm{tr}\,\widehat{\lambda} = 0\,.$$

  Define $\widehat{\omega} = *\widehat{\lambda}$ (irreducible dual field). The dual $\mathcal{L}$ depends only on $\widehat{\omega}$.
- Domain III: decompose the Lagrange multiplier

$$\lambda_{2,q} = \widehat{\lambda}_{2,q} + \eta\mathring{\lambda}_{1,q-1}\,, \qquad \mathrm{tr}\,\widehat{\lambda} = 0\,.$$

  Define $\widehat{\omega} = *\widehat{\lambda}$. The dual $\mathcal{L}$ depends not only on $\widehat{\omega}$, but also on $\mathring{\lambda}$. The correct dual EOM is obtained by taking a suitable trace:

$$\mathrm{tr}^{q+1}\,\mathrm{d}\widetilde{\mathrm{d}}\,\widehat{\omega}_{[D-2,q]} = 0\,.$$

- Domain IV: decompose the Lagrange multiplier

$$\lambda_{3,q} = \widehat{\lambda}_{3,q} + \eta\mathring{\lambda}_{2,q-1}\,, \qquad \mathrm{tr}\,\widehat{\lambda} = 0\,.$$

  Define $\widehat{\omega} = *\widehat{\lambda}$. The dual $\mathcal{L}$ depends not only on $\widehat{\omega}$, but also on $\mathring{\lambda}$. The correct dual EOM is obtained by taking a suitable trace:

$$\mathrm{tr}^{q}\mathrm{d}\widetilde{\mathrm{d}}\widehat{\omega}_{[D-3,q]} = 0\,.$$