Generalized Yang-Baxter deformations in $d = 11$ supergravity

Ilya Bakhmatov

Asia Pacific Centre for Theoretical Physics (Korea)
&
Kazan Federal University (Russia)

Supersymmetries and Quantum Symmetries — SQS'19
Yerevan, 26/08/2019

[1811.09056 IB, E. Musaev] [1906.09052 IB, N.S. Deger, E. Musaev, E. Ó Colgáin, M. Sheikh-Jabbari]

AdS/CFT and deformations

Strings/supergravity on $\mathrm{AdS}_5 \times \mathbb{S}^5$ $\Longleftrightarrow$ $\mathcal{N} = 4$ SYM

$\downarrow$ $\downarrow$

Deformations of $\mathrm{AdS}_5 \times \mathbb{S}^5$ $\Longleftrightarrow$ Field theory deformations

- deformations of CFT;
- non-commutative field theory;
- supergravity deformations : TsT, YB, generalization to $d = 11$;
- moving beyond $\mathrm{AdS}_5 \times \mathbb{S}^5$.

TsT: a prototype deformation

U(1)×U(1) isometry: adapted coordinates (x^1, x^2)
[Lunin, Maldacena; Frolov'05]

T-duality x^1	$\Longrightarrow$	shift $x^2 \to x^2 + \eta x^1$	$\Longrightarrow$	T-duality x^1

$$ds^2 = \frac{-dt^2 + dx_3^2 + dr^2}{r^2} + \frac{r^2}{r^4+\eta^2}(dx_1^2 + dx_2^2) + ds^2(\mathbb{S}^5),$$

$$B = \frac{\eta}{r^4+\eta^2}\, dx_1 \wedge dx_2, \qquad e^{2(\Phi-\Phi_0)} = \frac{r^4}{r^4+\eta^2}.$$

Dual to non-commutative deformations of SYM
[Hashimoto, Itzhaki; Maldacena, Russo'99]

Yang-Baxter deformations: generalizing TsT

Yang-Baxter deformed AdS_5 σ-model in the coset formulation:
[Klimčík'02; Delduc, Magro, Vicedo'13]

$$S = -\frac{1}{4}(\gamma^{ab} - \epsilon^{ab}) \int_{-\infty}^{\infty} d\tau \int_0^{2\pi} d\sigma \, \mathrm{Str}\left[A_a \, \mathrm{d} \circ \frac{1}{1 - \eta R_g \circ \mathrm{d}} A_b \right],$$

$$A_a = g^{-1}\partial_a g, \; g \in SO(4,2); \quad \mathrm{d} \text{ projects onto } \mathfrak{so}(4,2)/\mathfrak{so}(4,1).$$

$\exists$ Lax pair $\implies$ classical integrability
κ-symmetry preserved by the deformation

$$R_g(X) = g^{-1} R(gXg^{-1}) g, \quad X \in \mathfrak{so}(4,2)$$

R satisfies the homogeneous classical Yang-Baxter equation,

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = 0, \quad X, Y \in \mathfrak{so}(4,2).$$

r-matrix parameterisation of the YB deformation

Parameterise R in terms of the **r-matrix**:

$$R(X) = \mathrm{Tr}_2[r(1 \otimes X)] = \sum_{\alpha,\beta} r^{\alpha\beta} b_\alpha \mathrm{Tr}[b_\beta X],$$

$$r = \frac{1}{2} \sum_{\alpha,\beta} r^{\alpha\beta} b_\alpha \wedge b_\beta, \quad \{b_\alpha\} = \mathrm{bas}\, \mathfrak{so}(4,2), \quad [b_\alpha, b_\beta] = f_{\alpha\beta}{}^\gamma b_\gamma.$$

classical Yang-Baxter equation in terms of $r^{\alpha\beta}$:

$$f_{\delta\epsilon}{}^{[\alpha} r^{\beta|\delta|} r^{\gamma]\epsilon} = f_{\delta\epsilon}{}^{\alpha} r^{\beta\delta} r^{\gamma\epsilon} + f_{\delta\epsilon}{}^{\gamma} r^{\alpha\delta} r^{\beta\epsilon} + f_{\delta\epsilon}{}^{\beta} r^{\gamma\delta} r^{\alpha\epsilon} = 0.$$

Two views of Yang-Baxter deformations

Standard narration:

- r-matrix solving the CYBE is **an input**;
- σ-model in **coset formalism** is deformed by an (r-matrix-dependent) operator;
- deformed background is a solution to (generalised) supergravity;
- deformed backgrounds can be obtained via TsT or nonabelian T-duality.

Our narration:

- Start with a bi-Killing $r \sim k \wedge k$; **not fixing the CYBE**;
- Deformation is the **open-closed string map**, essentially matrix inversion;
- CYBE is sufficient for the deformed background to solve supergravity;
- procedure works for **non-coset geometries**, no (obvious) relation to integrability;
- supergravity solution generation.

Our approach for $d = 10$

Initial data:

- background G_{mn}, Φ, RR fields
- isometry algebra $[k_\alpha, k_\beta] = f_{\alpha\beta}{}^\gamma k_\gamma$

Procedure:

- Specify a bi-vector deformation: $\beta^{mn} = r^{\alpha\beta} k_\alpha^m k_\beta^n$
- New metric and 2-form: $g_{mn} + b_{mn} = (G^{mn} + \beta^{mn})^{-1}$
- Dilaton transformation $e^{-2\phi} |\det g_{mn}|^{1/2} = e^{-2\Phi} |\det G_{mn}|^{1/2}$

Note:

- Generalized supergravity criterion $I^m = \nabla_k \beta^{mk} \neq 0$
- Prescription for the RR fields exists (Page forms)
- Generalization for initial $B_{mn} \neq 0$ exists

Statement for $d = 10$ type II supergravity

Deformed background g_{mn}, b_{mn}, ϕ, etc. is a solution **if** the r-matrix satisfies the Yang-Baxter equation:

$$f_{\delta\epsilon}{}^{[\alpha} r^{\beta|\delta|} r^{\gamma]\epsilon} = 0.$$

For any supergravity solution there exists a deformation, such that the field equations reduce to the classical Yang-Baxter equation.

Yang-Baxter deformations and DFT

Double Field Theory: manifestly T-duality covariant form of supergravity [Tseytlin'90; Siegel'93; Hohm, Hull, Zwiebach'10]

Doubled coordinates: $X^M = (x^m, \tilde{x}_m)$

Fundamental variable: generalized metric

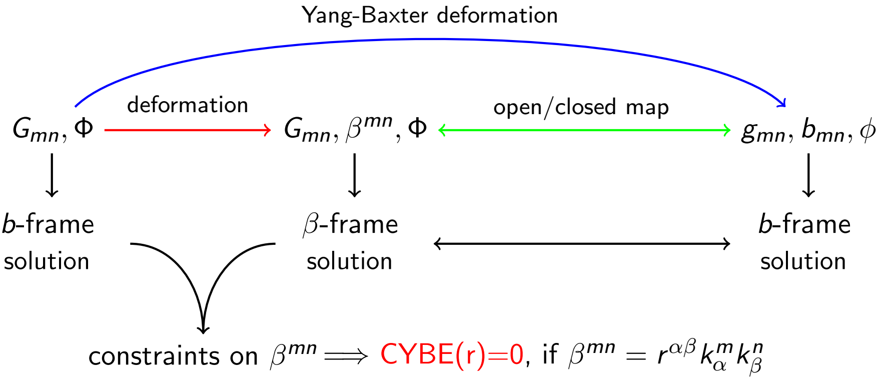
$$\begin{bmatrix} G & \beta G^{-1} \\ G^{-1}\beta & G^{-1} + \beta G \beta \end{bmatrix} = \mathcal{H}_{MN} = \begin{bmatrix} g + bg^{-1}b & gb \\ bg & g^{-1} \end{bmatrix}$$

Action ($d = \phi + \frac{1}{4}\log g_{mn}$):

$$S = \int dx\, d\tilde{x}\, e^{-2d} \Big(\frac{1}{8}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\partial_N \mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{KL}\partial_L \mathcal{H}^{MN}\partial_N \mathcal{H}_{KM} - \\ -2\partial_M d \partial_N \mathcal{H}^{MN} + 4\mathcal{H}^{MN}\partial_M d \partial_N d \Big).$$

The open-closed string map $g_{mn} + b_{mn} = (G^{mn} + \beta^{mn})^{-1}$ can be realized as a frame change in DFT.

Yang-Baxter deformations and DFT



Equations of motion in the β-frame [Andriot, Betz'13]

$$\boxed{\mathcal{R} - 4(\partial\Phi)^2 + 4\nabla^2\Phi} - \frac{1}{2}R^2 = 4(\beta^{mr}\partial_r\Phi + I^m)^2 - \hat{\mathcal{R}} + G_{mn}\hat{\nabla}^m(\beta^{nr}\partial_r\Phi + I^n);$$

$$\boxed{\mathcal{R}_{pq} + 2\nabla_p\partial_q\Phi} + \frac{1}{4}R_p{}^{mn}R_{qmn} = \hat{\mathcal{R}}_{(pq)} - 2\hat{\nabla}_{(p}\big(\beta_{q)r}\nabla^r\Phi + I_{q)}\big);$$

$$e^{2\Phi}\hat{\nabla}^m(e^{-2\Phi}R_{mrp}) + 2I^mR_{mrp} = 2e^{-2\Phi}\nabla^q(e^{2\Phi}\nabla_{[p}\beta_{r]q}) - 4\mathcal{R}_{[p}{}^s\beta_{r]s} - e^{2\Phi}\nabla^m(e^{-2\Phi}\nabla_m\beta_{rp}) + 8G_{n[p}\nabla_{r]}(\beta^{nq}\partial_q\Phi)$$

Notation:

$$\hat{\mathcal{R}} = G_{mn}\hat{\mathcal{R}}^{mn}, \quad \hat{\Gamma}^{mn}_p = \nabla^{(m}\beta^{n)}{}_p - \frac{1}{2}\nabla_p\beta^{mn} + \beta^{mq}\Gamma^n{}_{pq},$$

$$\hat{\mathcal{R}}^{mn} = -\beta^{pq}\partial_q\hat{\Gamma}^{mn}_p + \beta^{mq}\partial_q\hat{\Gamma}_p{}^{pn} + \hat{\Gamma}_p{}^{mn}\hat{\Gamma}_q{}^{qp} - \hat{\Gamma}_p{}^{qm}\hat{\Gamma}_q{}^{pn},$$

$$I^m = \nabla_k\beta^{km} \equiv -\hat{\Gamma}_k{}^{km}, \quad R^{mnp} = 3\beta^{q[m}\nabla_q\beta^{np]},$$

$$\hat{\nabla}^m V^p = -\beta^{mn}\partial_n V^p - \hat{\Gamma}_n{}^{mp}V^n, \quad \hat{\nabla}^m V_p = -\beta^{mn}\partial_n V_p + \hat{\Gamma}_p{}^{mn}V_n.$$

$d = 11$ supergravity

$$\text{strings} \to \text{membranes}$$
$$(g_{mn}, b_{mn}) \to (g_{mn}, C_{mnk})$$

Obvious obstacles:

- no integrability in membrane worldvolume theory setup;
- no "open-closed membrane" map;

However

- **Exceptional field theory** (ExFT) provides geometrical setup that parallels the $d = 10$ DFT case;
- We propose a concrete $d = 11$ solution deformation method based on a frame change in the $SL(5)$ ExFT.

$SL(5)$ ExFT setup

Exceptional Field Theory: manifestly U-duality covariant form of $d = 11$ supergravity [Hull'07; Berman, Perry'10]

Extended coordinates: (x^μ, Y^M), $\mu \in \{1, \ldots, 7\}$, $M \in \{1, \ldots, 10\}$
$SL(5)$ U-duality group: $11 = 7 + 4$ split.

The **generalized metric** in two frames:

$$e^{-\frac{\phi}{2}} \begin{bmatrix} g^{-\frac{1}{2}} g_{ab} & V_a \\ V_b & g^{\frac{1}{2}}(1 + V^2) \end{bmatrix} = m_{mn} = e^{-\frac{\Phi}{2}} \begin{bmatrix} G^{-\frac{1}{2}}(G_{ab} + W_a W_b) & W_a \\ W_b & G^{\frac{1}{2}} \end{bmatrix}$$

encodes degrees of freedom

$$V^a = \frac{1}{3!} \frac{1}{\sqrt{g}} \epsilon^{abcd} C_{bcd}, \qquad W_a = \frac{1}{3!} \sqrt{G}\, \epsilon_{abcd} \Omega^{bcd}.$$

We propose a tri-vector deformation $\Omega^{abc} = \rho^{\alpha\beta\gamma} k_\alpha{}^a k_\beta{}^b k_\gamma{}^c$

Deformation prescription in $d = 11$

Initial data:

- background fields $G_{\mu\nu}$, G_{ab}, $C_{\mu\nu\rho}$, C_{abc} **(7 + 4 split)**
- isometry algebra $[k_\alpha, k_\beta] = f_{\alpha\beta}{}^\gamma k_\gamma$

Procedure:

- Specify a tri-vector deformation: $\Omega^{abc} = \rho^{\alpha\beta\gamma} k_\alpha{}^a k_\beta{}^b k_\gamma{}^c$, $W = \star_4 \Omega$;
- New external metric $g_{\mu\nu} = (1 + W_a W^a)^{1/3} G_{\mu\nu}$;
- New internal metric $g_{ab} = (1 + W_a W^a)^{-2/3} (G_{ab} + W_a W_b)$;
- New 3-form $c_{abc} = (1 + W_a W^a)^{-1} \Omega_{abc}$.

Note:

- This replaces the open-closed string map of $d = 10$ theory;
- Powers of $1/3$ and $-2/3$ agree with uplift of a TsT.

Examples

Deformation agrees with the sequence:
(reduction to $d = 10$) — (TsT) — (uplift back to $d = 11$).

Geometry with a flat 3-torus:

$$\mathrm{d}s^2_{(11)} = \mathrm{d}s^2(M_7) + G_{zz}\mathrm{d}z^2 + \delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j, \qquad \Omega = \gamma\,\partial_{x^1}\wedge\partial_{x^2}\wedge\partial_{x^3},$$
$$\mathrm{d}\tilde{s}^2_{(11)} = K^{-1/3}\left[\mathrm{d}s^2(M_7) + G_{zz}\mathrm{d}z^2\right] + K^{2/3}\delta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j.$$

Supersymmetry preserving deformation of $\mathrm{AdS}_4\times\mathbb{S}^7$
[Lunin, Maldacena'05]
deform along the $U(1)^3 < U(1)^4 < SO(8)$

Recovering YB

Geometry with a Lorentzian 3-torus

$$\mathrm{d}s^2_{(11)} = \eta_{ab}\mathrm{d}x^a\mathrm{d}x^b + \mathrm{d}s^2(M_7), \quad \Omega = (\tau^i M_{ij} \wedge P^j) \wedge \partial_3.$$

Deformation ($\tau^i = (\alpha, \beta, \gamma) = \mathrm{const}$):

$$\begin{aligned} \mathrm{d}\tilde{s}^2_{(11)} &= K^{2/3}\left[\eta_{ij}\,\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}z^2 - W^2\right] + K^{-1/3}\mathrm{d}s^2(M_7), \\ K &= \left[1 + (\gamma x^1 - \beta x^2)^2 - (\gamma x^0 - \alpha x^2)^2 - (\beta x^0 - \alpha x^1)^2\right]^{-1}. \end{aligned}$$

This deformed background

- is not a solution in $d = 11$ unless $\tau^i = 0$ (trivial);
- naively reduces to a solution of generalized sugra if $(\tau^0)^2 - (\tau^1)^2 - (\tau^2)^2 = 0$ with $I^i = 2\tau^i$

Conclusions

- T-duality covariant description of $d = 10$ supergravity
- open-closed string map

$\Longrightarrow$ Supergravity knows about S-matrix symmetries in the string σ-model

- U-duality covariant description of $d = 11$ supergravity
- frame change map for $d = 11$

$\Longrightarrow$ Supergravity might know about symmetries of the membrane worldsheet theory

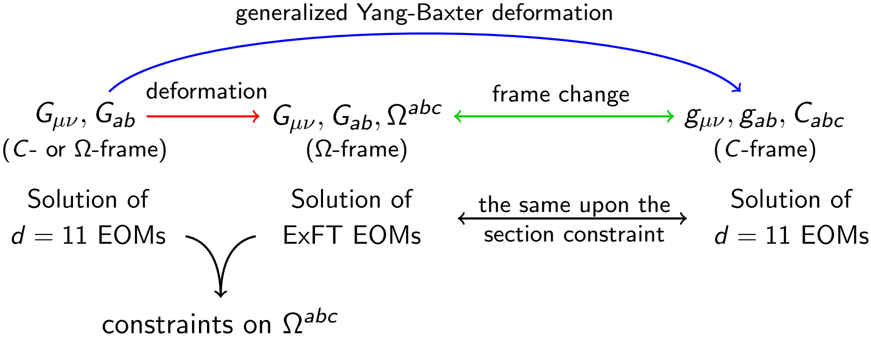
? general form of constraints for Ω from ExFT equations of motion;

? algebraic picture of tri-vector deformations;

? the role of higher simplex equations;

? generalized $d = 11$ supergravity and κ-symmetry of the membrane;

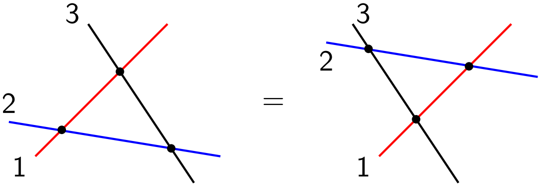
? study deformations that are not uplifts of YB or TsT from $d = 10$.


generalized Yang-Baxter deformation
deformation
frame change
$G_{\mu\nu}, G_{ab}$
(C- or Ω-frame)
$G_{\mu\nu}, G_{ab}, \Omega^{abc}$
(Ω-frame)
$g_{\mu\nu}, g_{ab}, C_{abc}$
(C-frame)
Solution of $d = 11$ EOMs
Solution of ExFT EOMs
the same upon the section constraint
Solution of $d = 11$ EOMs
constraints on Ω^{abc}

- algebra of functions $\{x^m, x^n\} = \beta^{mn}$

$$\begin{aligned} &\{f, g\} = \beta^{mn}\partial_m f \partial_n g, \\ &\{x^m, \{x^n, x^k\}\} + cyclic = 0 \quad \Longrightarrow \quad R^{mnk} = 0 \end{aligned} \tag{1}$$

- quantum Yang-Baxter equation



$$\begin{aligned} &R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \\ &R_{12}(u) = \mathrm{id} + \hbar r_{12}(u). \\ &[r_{12}(u-v), r_{13}(u)] + [r_{13}(u), r_{23}(v)] + [r_{12}(u-v), r_{23}(v)] = 0. \end{aligned} \tag{2}$$

Generalizations of the Yang-Baxter equation

- 3-algebra associated with M2-branes $\{x^a, x^b, x^c\} = \Omega^{abc}$ [Bagger,Lambert]

$$\begin{aligned} &\{f, g, h\} = \Omega^{abc}\partial_a f \partial_b g \partial_c h, \\ &\{x^a, x^b, \{x^c, x^d, x^e\}\} + cyclic = 0 \quad \not\Rightarrow \quad R^{a,bcde} = 0 \end{aligned} \tag{3}$$

- quantum 3-simplex equation [Zamolodchikov, Frenkel, Moore]

$$\begin{aligned} &R_{123}R_{124}R_{134}R_{234} = R_{234}R_{134}R_{124}R_{123}, \\ &[r_{123}, r_{124}] + [r_{123}, r_{134}] + [r_{124}, r_{134}] + [r_{123}, r_{234}] + [r_{134}, r_{234}] + [r_{124}, r_{23 \end{aligned} \tag{4}$$

- scattering of membranes?..

Yang-Baxter deformations

Coset formulation of the Yang-Baxter deformed $AdS_5 \times S^5$ σ-model:

$$\mathcal{L} = \mathrm{Tr}\left[A\,P^{(2)} \circ \frac{1}{1 - 2\eta R_g \circ P^{(2)}} A\right], \quad A = -g^{-1} dg, \quad g \in SO(4,2), \tag{5}$$

where

$$P^{(2)}(X) = \eta^{mn}\mathrm{Tr}[X\,\mathbf{P}_m]\,\mathbf{P}_n, \quad X \in \mathfrak{so}(4,2), \quad \mathbf{P}_m \in \mathrm{bas}\,\frac{\mathfrak{so}(4,2)}{\mathfrak{so}(4,1)}, \tag{6}$$
$$R_g(X) = g^{-1}R(gXg^{-1})g,$$

R is an antisymmetric operator satisfying the homogeneous classical Yang-Baxter equation,

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = 0, \quad X, Y \in \mathfrak{so}(4,2). \tag{7}$$

Examples of YB deformed backgrounds: $r = \frac{1}{2} P_1 \wedge P_2$

Starting from $AdS_5 \times S^5$:

$$ds^2_{\text{open}} = \frac{1}{z^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dz^2) + ds^2(S^5), \tag{8}$$

using the abelian r-matrix $r = \frac{1}{2} P_1 \wedge P_2$:

$$\begin{aligned} ds^2 &= \frac{-dt^2 + dx_3^2 + dz^2}{z^2} + \frac{z^2}{z^4 + \eta^2}(dx_1^2 + dx_2^2) + ds^2(S^5), \\ B &= \frac{\eta}{z^4 + \eta^2} dx_1 \wedge dx_2, \quad e^{2\Phi} = g_s^2 \frac{z^4}{z^4 + \eta^2}. \end{aligned} \tag{9}$$

[Hashimoto, Itzhaki; Maldacena, Russo]

Generalised IIB supergravity

[Arutyunov, Frolov, Hoare, Roiban, Tseytlin]

For less trivial r-matrices, deformed background is not a solution of supergravity.

$$
\begin{aligned}
R_{\mu\nu} &= \frac{1}{4} H_{\mu\rho\sigma} H_{\nu}{}^{\rho\sigma} - \nabla_\mu X_\nu - \nabla_\nu X_\mu, \\
R &= \frac{1}{12} H^2 - 4\nabla_\mu X^\mu + 4 X_\mu X^\mu, \\
\frac{1}{2}\nabla^\rho H_{\rho\mu\nu} &= X^\rho H_{\rho\mu\nu} + \nabla_\mu X_\nu - \nabla_\nu X_\mu.
\end{aligned}
\tag{10}
$$

There are also deformed field equations for the RR sector.

I^μ is a Killing vector

$$
X_\mu = \partial_\mu \Phi + I^\nu (G_{\nu\mu} + B_{\nu\mu}) \tag{11}
$$

$AdS_2 \times S^2 \times T^6$ example

The original background:

$$ds^2 = \frac{-dt^2 + dr^2}{r^2} + d\theta^2 + \sin^2\theta d\phi^2. \tag{12}$$

Homogeneous Ansatz for the deformation parameter:

$$\Theta^{tr} = \Theta_1(t,r), \qquad \Theta^{\theta\phi} = \Theta_2(\theta,\phi). \tag{13}$$

Find the deformed metric $g_{\mu\nu} = (G^{-1} + \Theta)^{-1}{}_{(\mu\nu)}$, solve the Einstein equations:

$$\begin{aligned} \Theta_1 &= c_1 tr + c_2 r(t^2 - r^2) + c_3 r + c_4 r^2, \\ \Theta_2 &= c_5 \cos\phi + c_6 \sin\phi + c_7 \cot\theta + \frac{c_8}{\sin\theta}. \end{aligned} \tag{14}$$

CYBE emerges as the constraints

$$c_1^2 - 4c_2c_3 = 0 = c_5^2 + c_6^2 + c_7^2, \qquad c_4 = 0 = c_8. \tag{15}$$

Lessons from the $AdS_2 \times S^2$ example

- We have not assumed that $\Theta \sim K_i \wedge K_j$ from the beginning, rather this is enforced by the field equations.
- Denote AdS_2 Killing vectors,

$$K_1 = -t\partial_t - r\partial_r, \quad K_2 = -\partial_t, \quad K_3 = -(t^2 + r^2)\partial_t - 2tr\partial_r. \quad (16)$$

- Then $\Theta = (c_1 tr + c_2 r(t^2 - r^2) + c_3 r)\partial_t \wedge \partial_r$ is fixed to be equal to the Killing bivector r-matrix:

$$r = -c_3 K_1 \wedge K_2 + \frac{c_1}{2} K_2 \wedge K_3 - c_2 K_3 \wedge K_1, \quad (17)$$

- and the constraint on c_1, c_2, c_3 is nothing but the $\mathfrak{sl}(2)$ classical YB equation:

$$f_{ij}{}^a r^{ib} r^{jc} + f_{ij}{}^c r^{ia} r^{jb} + f_{ij}{}^b r^{ic} r^{ja} = 0. \quad (18)$$

Bianchi III spacetime

$$ds^2 = -(a_1 a_2 a_3\, e^{-2\Phi} dt)^2 + (a_1 dx)^2 + (a_2 dy)^2 + (a_3 e^x dz)^2, \qquad \Phi = \lambda t,$$
$$a_1 = a_3 = \frac{p_1}{\sinh(p_1 t)} e^{-\frac{1}{2}p_2 t + \lambda t}, \quad a_2 = e^{\frac{1}{2}p_2 t + \lambda t}, \qquad 4p_1^2 = p_2^2 + 4\lambda^2. \tag{19}$$

Isometry algebra:

$$K_1 = \partial_x - z\partial_z, \quad K_2 = \partial_y, \quad K_3 = \partial_z, \qquad [K_1, K_3] = K_3. \tag{20}$$

The most general r-matrix

$$r = \alpha K_1 \wedge K_2 + \beta K_2 \wedge K_3 + \gamma K_3 \wedge K_1 \tag{21}$$

is a solution to the CYBE provided $\alpha\gamma = 0$.

Bianchi III spacetime

When $\gamma = 0$, it can be checked that the deformed geometry

$$
\begin{aligned}
g_{\mu\nu} dx^\mu dx^\nu &= -(a_1 a_2 a_3 e^{-2\lambda t} dt)^2 + \frac{1}{[1+\alpha^2 a_2^2(a_1^2 + z^2 e^{2x} a_3^2)]} \Big[a_1^2 dx^2 \\
&\quad + a_2^2 dy^2 + a_3^2 e^{2x} dz^2 + \alpha^2 e^{2x} a_1^2 a_2^2 a_3^2 (z dx + dz)^2 \Big], \\
b &= -\frac{\alpha a_2^2}{[1+\alpha^2 a_2^2(a_1^2 + z^2 e^{2x} a_3^2)]} (a_1^2 dx \wedge dy + z e^{2x} a_3^2 dy \wedge dz), \\
\Phi &= \lambda t - \frac{1}{2} \log[1+\alpha^2 a_2^2(a_1^2 + z^2 e^{2x} a_3^2)],
\end{aligned} \tag{22}
$$

is a solution to supergravity.

Comments

- Our map is defined for any spacetimes (not only for cosets);
- The classical Yang-Baxter equation is an output rather than the input.
- Solving field equations for $(g_{\mu\nu}, b_{\mu\nu})$ for an arbitrary Θ is intractable; for practical purposes we assumed that $\Theta = r^{ij} K_i K_j$.

The above examples seemed compelling to us, but they do not constitute a proof that CYBE always emerges.

Perturbative proof

Assume that the NC parameter is an arbitrary Killing bivector,

$$\Theta^{\mu\nu} = r^{ij} K_i^\mu K_j^\nu, \qquad r^{ij} = -r^{ji}. \tag{23}$$

Expand everything in powers of Θ:

$$\begin{aligned} g_{\mu\nu} &= G_{\mu\nu} + \Theta_\mu{}^\alpha \Theta_{\alpha\nu} + \mathcal{O}(\Theta^4), \\ B_{\mu\nu} &= -\Theta_{\mu\nu} - \Theta_{\mu\alpha}\Theta^{\alpha\beta}\Theta_{\beta\nu} + \mathcal{O}(\Theta^5), \\ \phi &= \Phi + \frac{1}{4}\Theta_{\rho\sigma}\Theta^{\rho\sigma} + \mathcal{O}(\Theta^4). \end{aligned} \tag{24}$$

Write down the field equations.

First order: $I^\mu = \nabla_\nu \Theta^{\nu\mu}$. Second order: CYBE

$$\begin{aligned} K_i^\alpha K_k^\beta \nabla_\alpha K_{\beta m} \left(f_{l_1 l_2}{}^m r^{i l_1} r^{k l_2} + f_{l_1 l_2}{}^k r^{m l_1} r^{i l_2} + f_{l_1 l_2}{}^i r^{k l_1} r^{m l_2} \right) \\ + \left(\Theta^{\beta\gamma}\Theta^{\alpha\lambda} + \Theta^{\alpha\beta}\Theta^{\gamma\lambda} + \Theta^{\gamma\alpha}\Theta^{\beta\lambda} \right) R_{\beta\gamma\alpha\lambda} = 0. \end{aligned} \tag{25}$$

Divergence condition

Some YB deformations result in solutions of generalised supergravity, which are specified by a Killing vector field I^μ
[Arutyunov, Frolov, Hoare, Roiban, Tseytlin]

The recipe for construction of the vector field I:

$$I^\mu = \nabla_\nu \Theta^{\nu\mu}. \tag{26}$$

- Directly relates open and closed string pictures;
- The condition has been attributed to the preservation of the Λ-symmetry in the generalised supergravity;
- Initially observed for examples, but later has been proven perturbatively (i.e. also follows from the field equations).

Can also be viewed as a bulk-boundary relationship for the NC parameter (supporting the holographic NC idea).

The previous example was a simple coset model.

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2(d\zeta^2 + \sin^2\zeta d\chi^2). \qquad (27)$$

Fix Θ to be a Killing bivector (this involves ∂_t and the three vectors on the sphere):

$$\begin{aligned} \Theta^{t\zeta} &= -\epsilon\cos\chi + \lambda\sin\chi, \\ \Theta^{t\chi} &= \delta + \cot\zeta\,(\epsilon\sin\chi + \lambda\cos\chi), \\ \Theta^{\zeta\chi} &= \alpha\cos\chi - \beta\cot\zeta + \gamma\sin\chi. \end{aligned} \qquad (28)$$

The field equations precisely match the CYBE.