Supersymmetries and Quantum Symmetries 2019
Yerevan, 26/09/2019

THE GEOMETRY BEHIND TOPOLOGICAL AMPLITUDES

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to appear … soon!

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THE GEOMETRY BEHIND TOPOLOGICAL AMPLITUDES

OVERVIEW

TOPOLOGICAL AMPLITUDES

FLUX-TUBE (MELVIN) GEOMETRY

HETEROTIC STRING ON FLUX TUBES

TOPOLOGICAL AMPLITUDES

The topological amplitudes corresponding to higher derivative F-term of the form $F_g W_{2g}$ can be computed in heterotic string as

$$\mathscr{A}_g = \langle V_{\rm grav}^2 \, V_{\rm gph}^{2g-2} \rangle$$

graviton   graviphoton

in a background which preserves N=2 supersymmetries, *i.e.* K3xT^2

TOPOLOGICAL AMPLITUDES

A suitable choice of space-time momenta drastically simplifies the computation of the amplitude, although one is still left with

$$G_g \equiv \left\langle \prod_{i=1}^{g} \int d^2 x_i \, Z^1 \bar{\partial} Z^2 (x_i) \prod_{j=1}^{g} \int d^2 y_j \, \bar{Z}^2 \bar{\partial} \bar{Z}^1 (y_j) \right\rangle$$

To this end, one actually computes the generating function

$$G(\lambda) = \sum_{g=1}^{\infty} \frac{1}{(g!)^2} \left(\frac{\lambda}{\tau_2} \right)^{2g} G_g$$

The advantage of having introduced G is that it can be expressed as the normalised Gaussian functional integral

$$G(\lambda) = \frac{\int [\mathscr{D}Z\mathscr{D}\bar{Z}] \exp\left(-S_0 + \frac{\lambda}{\tau_2}\int d^2x (Z^1\bar{\partial}Z^2 + \bar{Z}^2\bar{\partial}\bar{Z}^1)\right)}{\int [\mathscr{D}Z\mathscr{D}\bar{Z}] \exp(-S_0)}$$

Free action for Z^1 and Z^2

It can be straightforwardly computed (using zeta-function regularisation) to yield

$$G(\lambda) = \left(\frac{2\pi i\lambda\bar{\eta}^3}{\bar{\vartheta}_1(\lambda|\bar{\tau})}\right)^2 e^{-\pi\lambda^2/\tau_2}$$

Back to the amplitude, the generating function of topological amplitudes is

$$F(\lambda) = \sum_{g=1}^{\infty} \lambda^{2g} F_g$$

$$= \int_{\mathscr{F}} \frac{d^2\tau}{\tau_2} F(\bar{\tau}) \sum_{m,n} \left(\frac{2\pi i \lambda \bar{\eta}^3}{\bar{\vartheta}_1(\tilde{\lambda}|\bar{\tau})} \right)^2 e^{-\pi\lambda^2\tau_2} \, q^{\frac{1}{2}|p_L|^2} \, \bar{q}^{\frac{1}{4}|p_R|^2}$$

encodes the contribution of the internal coordinates

$\tilde{\lambda} \propto \lambda p_L \tau_2$

TOPOLOGICAL AMPLITUDES

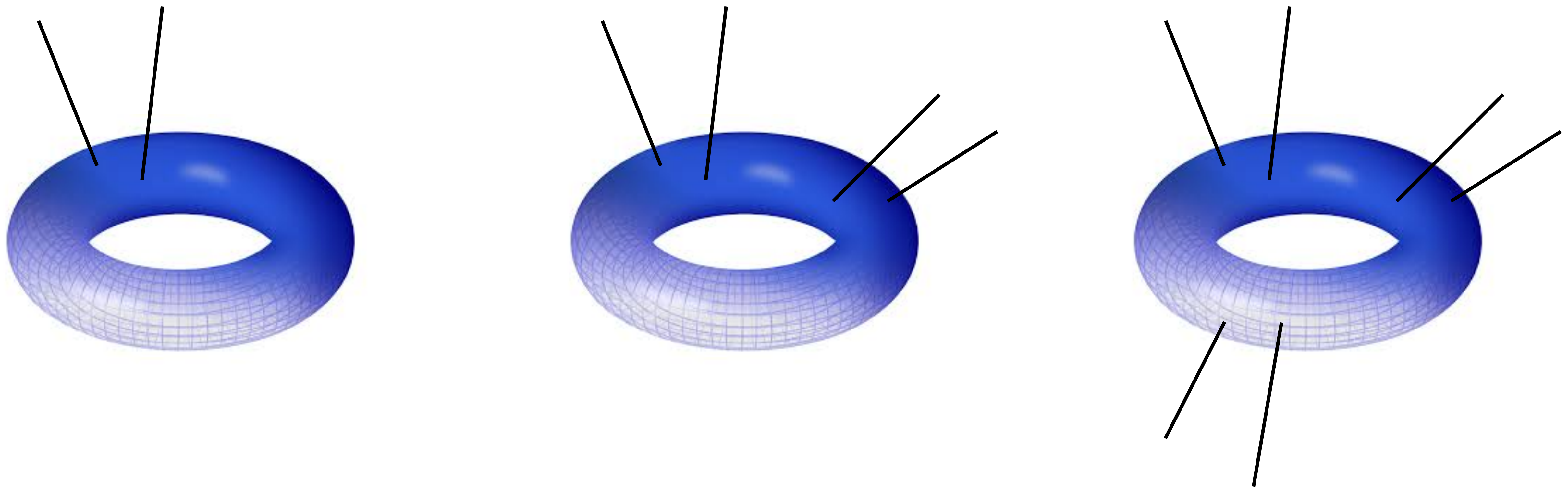
What can/do we "learn" from all this?

$$F(\lambda) = \int [\mathscr{D}Z\mathscr{D}\bar{Z}][\mathscr{D}X_{K3}][\mathscr{D}\lambda_{\text{gauge}}]\,\exp\left(-S_{K3} - S_{\text{gauge}} - S_0 + \frac{\lambda}{\tau_2}\int d^2x(Z^1\bar{\partial}Z^2 + \bar{Z}^2\bar{\partial}\bar{Z}^1)\right)$$

Are topological amplitudes equivalent to
"free energies" of strings on "non-trivial" backgrounds?

TOPOLOGICAL AMPLITUDES

After all …



the vertex operators effectively modify the geometry of space-time!

FLUX-TUBE (MELVIN) GEOMETRY

What is then the geometry behind the topological amplitudes?

The graviphoton vertex operators correspond to anti-self-dual gauge field configurations (in Euclidean space-time)

The generating function, on the other hand, involves (anti-chiral) rotations on the non-compact directions

How to combine these two aspects?

FLUX-TUBE (MELVIN) GEOMETRY

Consider the four-dimensional geometry

$$
\begin{aligned}
ds_4^2 &= d\rho_1^2 + \rho_1^2 \tilde{G}(1 + q_2^2 \rho_2^2) d\varphi_1^2 + d\rho_2^2 + \rho_2^2 \tilde{G}(1 + q_1^2 \rho_1^2) d\varphi_2^2 - 2\tilde{G} q_1 q_2 \rho_1^2 \rho_2^2 d\varphi_1 d\varphi_2 \,, \\
A &= \tilde{G}(q_1 \rho_1^2 d\varphi_1 + q_2 \rho_2^2 d\varphi_2) \,, \\
\phi &= \phi_0 \,, \\
e^{2\sigma} &= \tilde{G}^{-1}
\end{aligned}
$$

$$
\tilde{G}^{-1} = 1 + q_1^2 \rho_1^2 + q_2^2 \rho_2^2
$$

FLUX-TUBE (MELVIN) GEOMETRY

It includes uniform magnetic fields

$$F^{ab} = 2\tilde{G} \begin{pmatrix} 0 & -q_1 & 0 & 0 \\ q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_2 \\ 0 & 0 & q_2 & 0 \end{pmatrix} \qquad \tilde{F}_{ab} = 2\tilde{G} \begin{pmatrix} 0 & -q_2 & 0 & 0 \\ q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_1 \\ 0 & 0 & q_1 & 0 \end{pmatrix}$$

which are (anti)self-dual (at leading order) if $\boldsymbol{q_1=q_2}$

FLUX-TUBE (MELVIN) GEOMETRY

The sigma model associated to this background geometry is

$$\mathscr{L} = \partial\rho_1\bar{\partial}\rho_1 + \rho_1^2(\partial\varphi_1 + q_1\partial y)(\bar{\partial}\varphi_1 + q_1\bar{\partial}y) + (1 \to 2) + \partial y\bar{\partial}y$$

and corresponds to free fields if

$$\varphi_i \to \phi_i = \varphi_i + q_i y$$

$$\mathscr{L} = \sum_{i=1,2}\partial\rho_i\bar{\partial}\rho_i + \rho_i^2\partial\phi_i\bar{\partial}\phi_i + \partial y\bar{\partial}y = \sum_i \partial Z_i\bar{\partial}\bar{Z}_i + \partial y\bar{\partial}y$$

FLUX-TUBE (MELVIN) GEOMETRY

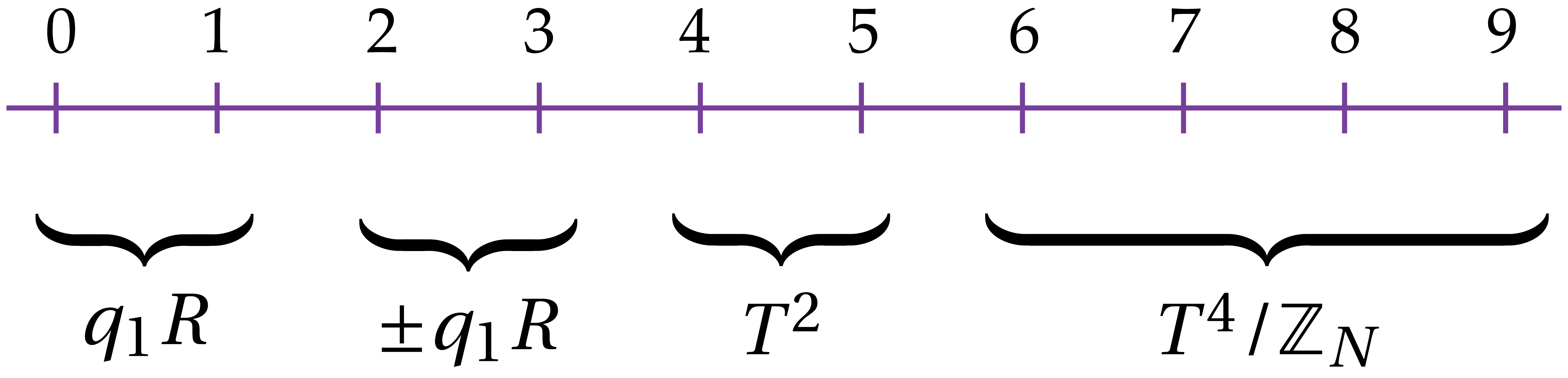
The new angular coordinate is not a *true angular coordinate,*
and therefore Z is not periodic

$$Z_i(\sigma+\pi,\tau) = e^{2\pi i n q_i R}\, Z_i(\sigma,\tau)$$

Z is rotated by the angle $\boldsymbol{2\pi n q_i R}$

The (anti)self-duality of the magnetic field
is equivalent to opposite (equal) rotations of the two planes.

THE GEOMETRY BEHIND TOPOLOGICAL AMPLITUDES



THE HETEROTIC STRING FREE ENERGY

… can be calculated using standard techniques and Riemann identity

$$\mathscr{F}(\lambda) = -\int_{\mathscr{F}} \frac{d^2\tau}{\tau_2} \sum_{g,h} \frac{\left(\sum_{k,l} \bar{\theta}^6\left[{k \atop l}\right] \bar{\theta}\left[{k+g/2 \atop l+h/2}\right] \bar{\theta}\left[{k-g/2 \atop l-h/2}\right]\right) \left(\sum_{\rho,\sigma} \bar{\theta}^8\left[{\rho \atop \sigma}\right]\right)}{\bar{\eta}^{18}\, \bar{\theta}\left[{1/2+g/2 \atop 1/2+h/2}\right] \bar{\theta}\left[{1/2-g/2 \atop 1/2-h/2}\right]}$$

$$\times \sum_{m_i,n_i} \left(\frac{2\pi i \lambda \bar{\eta}^3}{\bar{\vartheta}_1(\tilde{\lambda}|\tau)}\right)^2 e^{-\pi\tilde{\lambda}^2/\tau_2}\, q^{\frac{1}{4}|p_{\rm L}|^2}\, \bar{q}^{\frac{1}{4}|p_{\rm R}|^2}\,,$$

This expression matches the generating function for the topological amplitude

THE HETEROTIC STRING FREE ENERGY

Was it expected?

In his seminal paper, Nekrasov conjectured a relation between the free energy of a N=2 gauge theory and the field-theory limit of the topological amplitude.

$$ds^2 = A\, dz d\bar{z} + g_{IJ}\left(dx^I + \Omega^I{}_K x^K dz + \bar{\Omega}^I{}_K x^K d\bar{z}\right)\left(dx^J + \Omega^J{}_L x^L dz + \bar{\Omega}^J{}_L x^L d\bar{z}\right)$$

The Omega background with a single parameter is the Flux tube (Melvin) geometry considered previously.

GENERALISATION

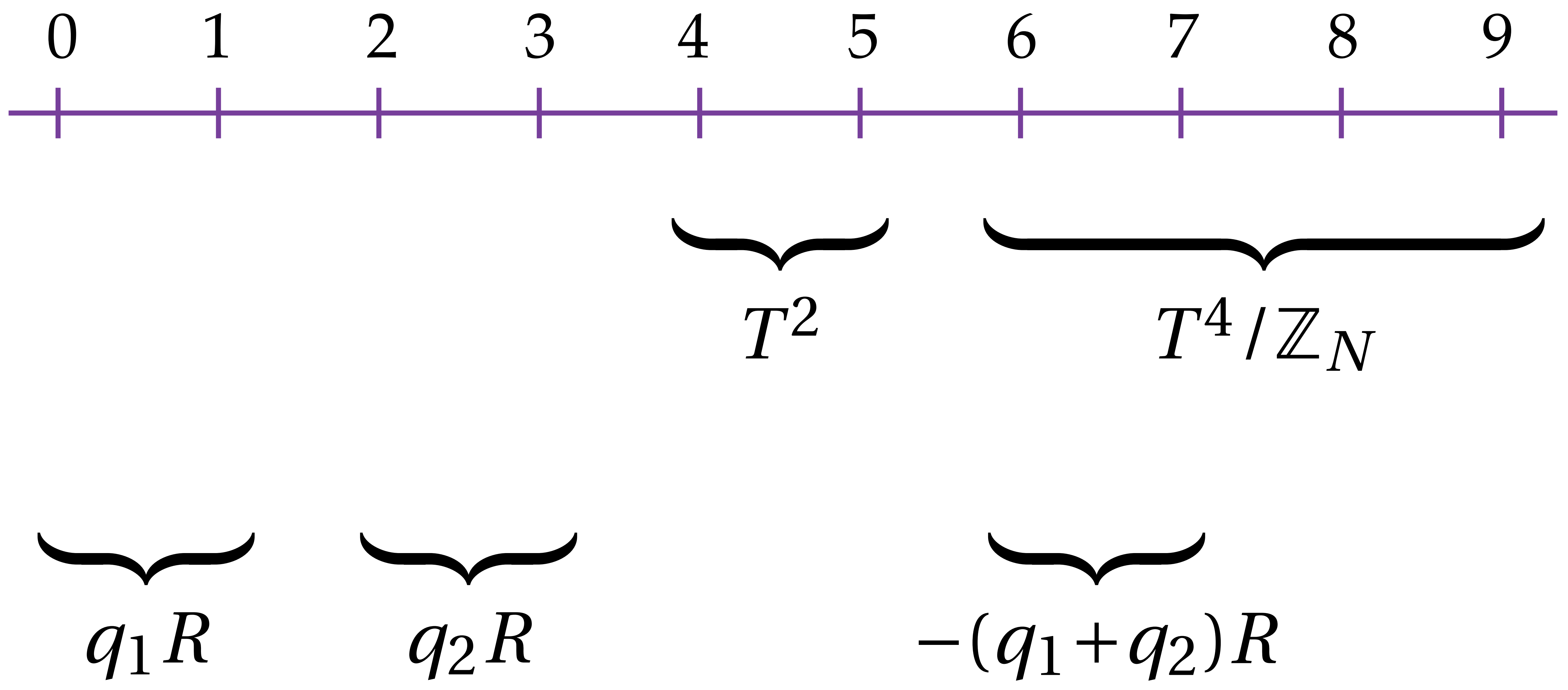
What about the refinement of the topological amplitudes?

$$ds^2 = A\,dz\,d\bar{z} + g_{IJ}\left(dx^I + \Omega^I{}_K x^K dz + \bar{\Omega}^I{}_K x^K d\bar{z}\right)\left(dx^J + \Omega^J{}_L x^L dz + \bar{\Omega}^J{}_L x^L d\bar{z}\right)$$

The two-parameter Omega background needs an action on the SU(2) R-symmetry to preserve supersymmetry

What is the origin of R-symmetry from a Kaluza-Klein perspective?

What about the refinement of the topological amplitudes?



THANK YOU