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# LAW OF LARGE NUMBERS AND ASYMPTOTIC OF PHASE-SPACE INTEGRAL

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The law of large numbers is used for estimation of the longitudinal phase-space integral for big value of particle numbers. A fully completed analytical expression of the phase-space integral is received.

We propose a new method of estimation of the phase-space integral at $n \to n_{\max} = \sqrt{s}/m$:

$$Z_n = \int \left\{ \prod_{i=1}^{n} \frac{d^3 k_i}{2\sqrt{k_i^2 + m^2}} \right\} \times \tag{1}$$
$$\times\, \delta^4 \left( P - \sum k_i \right) f_n(k_1, \ldots, k_n),$$

where $f_n$ is the amplitude module squire, $P \equiv \equiv (E, 0, 0, 0)$ is the total momentum. We work in the center-of-mass (CM) system. The integrals of such a type arise when the topological cross sections are calculated. We will examine the simplest case when $f_n$ looks as follows:

$$f_n(k_1, \ldots, k_n) = \prod_{i=1}^{n} \exp\left(-r_0^2 k_{t,i}^2\right), \tag{2}$$

where $k_{t,i}$ is a transverse momentum of $i$th particle and $r_0$ is a phenomenological transverse radius. This choice means the assumption that the secondaries are produced independently from each other. The transverse momentum cutoff is compatible with the present experimental data. The attempts of calculations of the integrals (1) have a long history [1–10]. Having the factorized amplitude (2), the dominant problem descended from the energy–momentum conservation $\delta$ function in (1). In order to avoid this difficulty Kajantie and Karimaki [1] introduce Fourier transformation for $\delta$ function and use a saddle point method for the calculating of the Fourier transform. Lurcat and Mazur [2] use Laplace transformation for integrand, normalizes it and then interprets it as a frequency function. The latter was approximated by the Edgeworth series leaving only the first few terms. The analogous technique with small modifications for a special case of $f_n$ was used by Krzywicki as well as Bilash [3–5]. A number of attempts use the method of Monte Carlo (MC) [8–10]. It is necessary to underline that all noted above approaches present the algorithm of numerical calculations.

The basis of our method consists of the expansion of (1) in terms of the universally independent functions. Then we will use the law of large numbers for their estimation. In this way we find for (1) the completely analytical expression. It is important to build the fast generator of events if $n \to n_{\max} \gg 1$.

The produced particles have small momenta at $n \to n_{\max}$. One can neglect the motion of CM frame in this limit. For this reason we neglect the momenta conservation law $\delta$ function:

$$\delta^4 \left( P - \sum_{i=1}^{n} k_i \right) \to \delta \left( E - \sum_{i=1}^{n} \sqrt{k_i^2 + m^2} \right). \tag{3}$$

Performing integration over spherical angles, we come to the expression

$$Z_n(E) = (\pi/2)^n \times \tag{4}$$
$$\times \int \left\{ \prod_{i=1}^{n} \frac{d(k_{t,i}^2) dk_z}{\sqrt{k_{t,i}^2 + k_{z,i}^2 + m^2}} e^{-r_0^2 k_{t,i}^2} \right\} \times$$
$$\times\, \delta \left( E - \sum_{i=1}^{n} \sqrt{k_{t,i}^2 + k_{z,i}^2 + m^2} \right),$$

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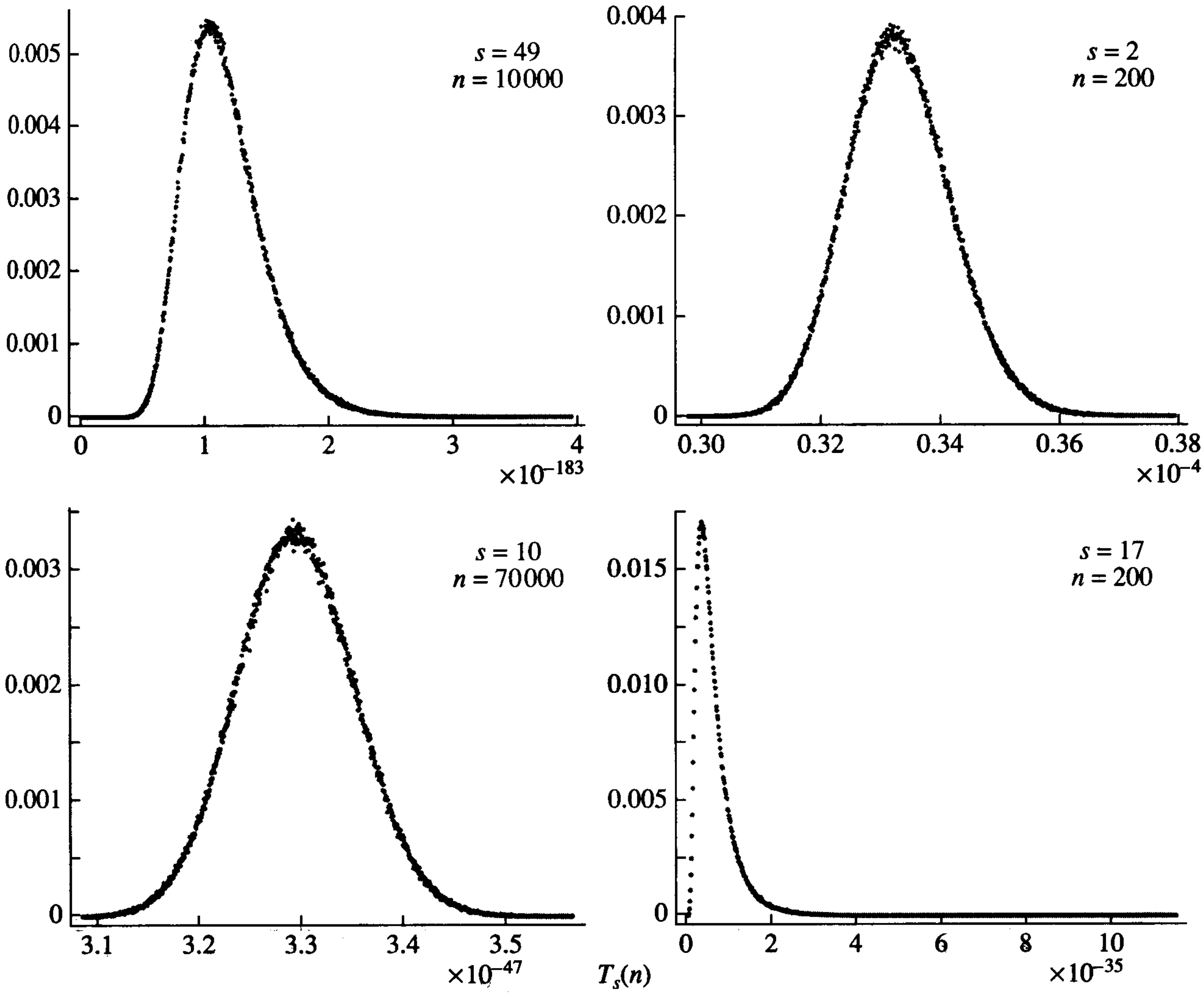


**Fig. 1.** Examples of statistical distribution of $T_s(n)$ functions.

where $k_{z,i}$ is the longitudinal and $k_{t,i}$ is the transverse momentum. Then we introduce the particle energy as the independent variable. As a result:

$$Z_n(E) = (\pi/r_0)^n[m(n_{\max}-n)]^{n-1} \times \tag{5}$$
$$\times \left\{ \prod_{i=1}^{n} \int_0^1 dy_i F\left(r_0 m\sqrt{(n_{\max}-n)y_i((n_{\max}-n)y_i+2)}\right)\right\} \delta\left(1-\sum y_i\right),$$

where $F(x)$ is the Dawson integral:

$$F(x) = e^{-x^2}\int_0^x e^{t^2}dt. \tag{6}$$

The Dawson integral can be presented by the following form:

$$F\left(r_0 m\sqrt{(n_{\max}-n)y((n_{\max}-n)y+2)}\right) = \sqrt{y}\exp\{a_0 + a_1 y + a_2 y^2 + \ldots\}, \tag{7}$$

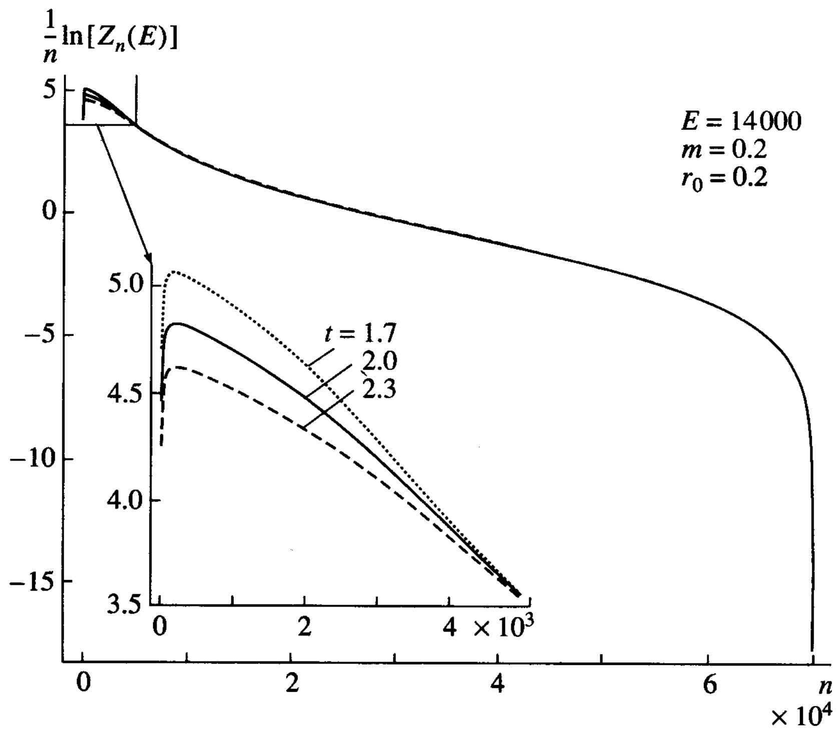


**Fig. 2.** Graphics of $(1/n)\ln[Z_n(E)]$ for various values of parameter $t$.

where the coefficients

$$a_s = \frac{1}{2\pi i}\int\limits_C d\zeta\zeta^{-s-1}\ln\left\{F\left(r_0 m\sqrt{(n_{\max}-n)\zeta((n_{\max}-n)\zeta+2)}\right)/\sqrt{\zeta}\right\}. \quad (8)$$

After the substitution (7) into (5) we receive:

$$Z_n(E) = (\pi/r_0)^n[m(n_{\max}-n)]^{n-1}\times \quad (9)$$
$$\times\left\{\prod_{i=1}^{n}\int\limits_0^1 dy_i\sqrt{y_i}\right\}\delta\left(1-\sum y_i\right)\times$$
$$\times\exp\{n[a_0+a_1T_1(n)+a_2T_2(n)+\ldots]\},$$

where $T_s(n)$ is:

$$T_s(n) = \frac{1}{n}\sum_{i=1}^{n} y_i^s. \quad (10)$$

Since we plan to calculate the integral (9) by the MC method it is reasonable to investigate the statistical distributions of functions (10) with the constrain $\sum y_i = 1$. The calculations were realized for the different values of $s$ and $n$. As it can be seen from Fig. 1 for relatively small values of $s$ the distribution tends to the normal type but for relatively big values of $s$ the distribution tends to the Poissonian one.

We can not find the exact expression for the distribution law of functions (10). Nevertheless, we can find the acceptable approximation for extremum points of the distribution of functions $T_s(n)$:

$$\frac{1}{s+1}\left(\frac{2}{n}\right)^s. \quad (11)$$

We find also the limits of the area, where the mostly significant values of $T_s(n)$ are grouped:

$$\left(\frac{1}{s+1}\left(\frac{1.7}{n}\right)^s, \frac{1}{s+1}\left(\frac{2.3}{n}\right)^s\right). \quad (12)$$

Let us imagine that we calculate our integral (9) by the MC method. At every step we must randomly select the group of nonnegative numbers $y_1, y_2, \ldots, y_n$ with the constrain $\sum y_i = 1$. Then one must substitute these numbers into (9). As a result, the maximal number of items would have the value of $T_s(n)$ coinciding with the expression (11). Consequently, if we neglect small contributions then one may change functions $T_s(n)$ in (9) by the corresponding value (11) and carry out an exponent from the integral (9).

After this procedure the remainder integral has a

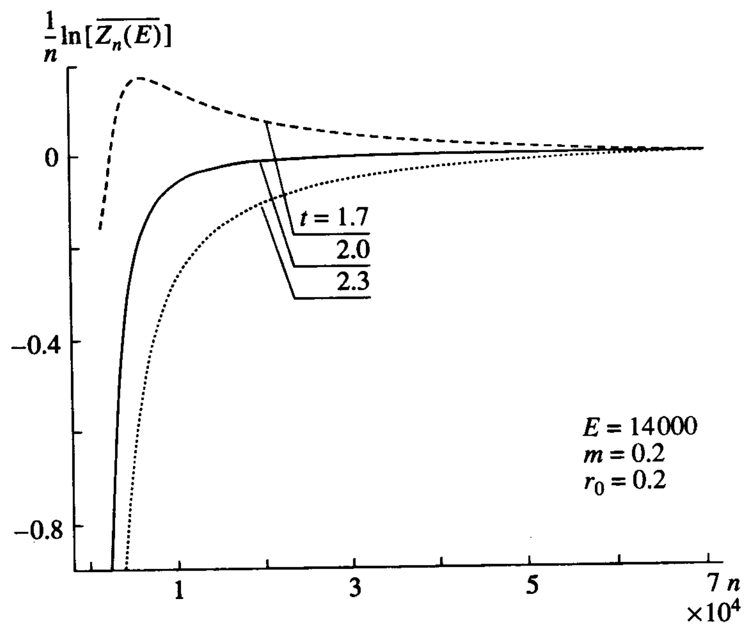


**Fig. 3.** Graphics of $(1/n)\ln[\overline{Z_n(E)}]$ for various values of parameter $t$.

form

$$\left\{\prod_{i=1}^{n}\int_0^1 dy_i\sqrt{y_i}\right\}\delta\left(1-\sum_{i=1}^{n}y_i\right). \qquad (13)$$

It is easily calculable if the hyperspherical coordinates:

$$y_1 = \rho\cos^2(\varphi_{n-1})\ldots\cos^2(\varphi_2)\cos^2(\varphi_1), \qquad (14)$$

$$y_2 = \rho\cos^2(\varphi_{n-1})\ldots\cos^2(\varphi_2)\sin^2(\varphi_1),$$

$$\vdots$$

$$y_{n-1} = \rho\cos^2(\varphi_{n-1})\sin^2(\varphi_{n-2}),$$

$$y_n = \rho\sin^2(\varphi_{n-1}),$$

are introduced. As a result,

$$\left\{\prod_{i=1}^{n}\int_0^1 dy_i\sqrt{y_i}\right\}\delta\left(1-\sum_{i=1}^{n}y_i\right) = \frac{(\Gamma(3/2))^{n-1}}{\Gamma(3n/2)}. \qquad (15)$$

The final expression has a form

$$Z_n(E) = \frac{\pi^{(3n-1)/2}}{r_0\Gamma(3n/2)}\left[\frac{m(n_{\max}-n)}{2r_0}\right]^{n-1} e^{nW(2)}, \qquad (16)$$

where

$$W(t) = \frac{n}{t}\int_0^{t/n}\ln\left[\frac{1}{\sqrt{y}}F\left(r_0m\sqrt{(n_{\max}-n)y((n_{\max}-n)y+2)}\right)\right]dy. \qquad (17)$$

Figure 2 demonstrates three distribution of $(1/n)\ln[Z_n(E)]$ for various values of parameter $t$. As is seen from the graphic the differences have a place for small $n$ ($< 5000$). Thus, we find an area of validity of our Eq. (16).

Generalization for another form of $f_n$ (2) is not a complicated procedure.

It is interesting to calculate the limit $r_0 \to 0$:

$$Q_n(E) = \lim_{r_0\to 0} Z_n(E). \qquad (18)$$

For this purpose, the equation (5) was used and because of the property, $\lim_{x\to 0} F(x)/x = 1$, we receive

$$Q_n(E) = \pi^n m^{2n-1}(n_{\max}-n)^{3(n-1)/2} \times \qquad (19)$$

$$\times\left\{\prod_{i=0}^{n}\int_0^1 dy_i\sqrt{y_i((n_{\max}-n)y_i+2)}\right\}\times$$

$$\times\,\delta\left(1-\sum y_i\right).$$

Then we expand the square root in (19) into the series:

$$\sqrt{y_i((n_{\max}-n)y_i+2)} = \exp\left\{\sum_{j=0}^{\infty} a_j y_j\right\}, \quad (20)$$

where

$$a_0 = \frac{1}{2}\ln 2, \quad a_j = (-1)^{j+1}\frac{1}{2j}\left(\frac{y}{2}\right)^j. \quad (21)$$

Using the analogous method we receive the following expressions for the "normalized" phase-space integral $\overline{Z_n(E)}$:

$$\overline{Z_n(E)} = Z_n(E)/Q_n(E) = \quad (22)$$

$$= \left\{\frac{1}{r_0 m\sqrt{2(n_{\max}-n)}}\exp(W(2)-U(2))\right\}^n,$$

where

$$U(t) = \frac{1}{2}\left[\frac{(2-t)}{4}\frac{(n_{\max}-n)}{n} + \quad (23)\right.$$

$$\left. + \left(1+\frac{2n}{t(n_{\max}-n)}\right)\ln\left(1+\frac{(n_{\max}-n)t}{2n}\right) - 1\right].$$

In Fig. 3 you can see the behavior of $\overline{Z_n(E)}$ for various value of the parameter $t$.

The method of estimation of the phase-space integral using the law of large numbers allows to receive an analytical expression. It is very significant when we build the fast event generator.

As would be expected, the dependence on the catting parameter $r_0$ vanishes at the asymptotic on multiplicity $n \to n_{\max}$ but it is essential in case when $n$ has a finite value.

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