VERY HIGH MULTIPLICITY PHYSICS

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LAW OF LARGE NUMBERS AND ASYMPTOTIC OF PHASE SPACE INTEGRAL

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Abstract. The law of large numbers is used for estimation of the longitudinal phase space integral for big value of particle numbers. A fully completed analytical expression of the phase space integral is received.

1 Introductions

In my talk I report on statistical method of estimation of the phase space integral in the case when number of particles $n- > n_{max}$. This work is small part of the big program that realize in Sissakian - Manjavidze group at JINR and devoted to the statistical description of inelastic processes at high energy, including processes with very high multiplicity.

Thermodynamical approach give as the possibility of full description of complicated system by the limited number of parameters. We use conception of thermodynamical equilibrium that means assumption of uniform distribution of energy over all degrees of freedom and fluctuations are the Gauss type.

Pioneering works of the use thermodynamical description of hadron processes belong to Fermi and Landau. Its main assumption consist in treatment the inelastic hadron collisions as dissipation of kinetic energies of colliding particles.

Interest in the events with very high multiplicity catenate with following. Fermi-Landau model badly described processes with medium multiplicity from the limitations of Non-Abelian gauge symmetries. The role of the last will be weaken when the number of particles tends to the n_{max}. Therefore, it is important to investigate the phenomenological indications of thermalization.

So, it is actually the program of building fast events generator for correctly describing processes with very high multiplicity.

2 Asymptotic of Phase Space Integral

We propose a new method of estimation of the phase space integral at $n \rightarrow n_{max} = \sqrt{s}/m$:

$$Z_n = \int \left\{ \prod_{i=1}^{n} \frac{d^3 k_i}{2\sqrt{k_i^2 + m^2}} \right\} \delta^4\Big(P - \sum k_i\Big) \; f_n(k_1, \ldots, k_n) \qquad (1)$$

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where f_n is the amplitude module square, $P \equiv (E, 0, 0, 0)$ is the momentum 4-vector, i.e. we work in the center of mass (CM) system. Such a type integrals arise when the topological cross-sections are calculated. We will examine the simplest case of separable amplitude f_n:

$$f_n(k_1, \ldots, k_n) = \prod_{i=1}^{n} \exp\left(-r_0^2 k_{t,i}^2\right), \tag{2}$$

where $k_{t,i}$ is a transverse momentum and r_0 is a phenomenological cutting parameter. This choice means assumption that the secondaries are produced independently from each other. Limitations to small values of transverse momentum imposed by the dynamics on the collision amplitude are compatible with the present experimental evidence.

The theory of calculations of integrals (1) have a long history [1-10].

The dominant problem descended from the presence of Dirac delta function in the integrand. In order to avoid this difficulty Kajantie and Karimaki [6] introduce Fourier transformation for δ-function and use a saddle point method for the calculating of the Fourier transform. Lurcat and Mazur [1] use Laplace transformation for integrand, normalize it and then interpret it as a frequency function. The last approximated by the Edgeworth series keeps the first few terms only. The analogous technique with small modifications for a special case of f_n was used by Krzywicki and Bilash [2-4]. There are series of works, where in calculations used the method of Monte Carlo (MC) [8-10]. There is a need to note that the preceding theories represent themselves an algorithm of numerical calculations.

A base of our method consists of the expansion of (1) by terms of the universally independent functions $T_s(n)$. Then we will use the law of large numbers for $T_s(n)$. On this way we receive for Z_n completely analytical expression.

Our interest focused on the asymptotic behavior of Z_n when $n \to n_{max}$. This case corresponds to the situation when produced particles have the small momenta. We may neglect the momentum conservation law at $n \to n_{max}$ and leave only energy conservation law:

$$\delta^4\left(P - \sum_{i=1}^{n} k_i\right) \to \delta\left(E - \sum_{i=1}^{n} \sqrt{k_i^2 + m^2}\right) . \tag{3}$$

Performing integration over spherical angles, we get to the expression depending from the longitudinal $k_{z,i}$ and transverse $k_{t,i}$ momentum :

$$Z_n(E) = (\pi/2)^n \int \left\{ \prod_{i=1}^{n} \frac{d(k_{t,i}^2)dk_z}{\sqrt{k_{t,i}^2 + k_{z,i}^2 + m^2}} e^{-r_0^2 k_{t,i}^2} \right\} \times$$
$$\times \delta\left(E - \sum_{i=1}^{n} \sqrt{k_{t,i}^2 + k_{z,i}^2 + m^2} \right) . \quad (4)$$

Then we introduce the particle energy as the independent variable. We receiving result:

$$Z_n(E) = (\pi/r_0)^n [m(n_{max} - n)]^{n-1} \times$$
$$\times \left\{ \prod_{i=1}^{n} \int_0^1 dy_i F(r_0 m \sqrt{(n_{max} - n)y_i((n_{max} - n)y_i + 2)}) \right\} \delta\left(1 - \sum y_i\right) \quad (5)$$

where F(x) is the Dawson integral:

$$F(x) = e^{-x^2} \int_0^x e^{t^2} dt . \quad (6)$$

Let us represent integral (6) in the following form:

$$F(r_0 m \sqrt{(n_{max} - n)y((n_{max} - n)y + 2)}) = \sqrt{y} \exp\{a_0 + a_1 y + a_2 y^2 + \ldots\} \quad (7)$$

where the coefficients

$$a_s = \frac{1}{2\pi i} \int_C d\zeta \zeta^{-s-1} \ln \left\{ F(r_0 m \sqrt{(n_{max} - n)\zeta((n_{max} - n)\zeta + 2)}) / \sqrt{\zeta} \right\} . \quad (8)$$

After the substitution (7) into eq. (5) we find:

$$Z_n(E) = (\pi/r_0)^n [m(n_{max} - n)]^{n-1} \left\{ \prod_{i=1}^{n} \int_0^1 dy_i \sqrt{y_i} \right\} \delta\left(1 - \sum y_i\right) \times$$
$$\times \exp\{n[a_0 + a_1 T_1(n) + a_2 T_2(n) + \ldots]\} \quad (9)$$

where $T_s(n)$ is an universal function:

$$T_s(n) = \frac{1}{n} \sum_{i=1}^{n} y_i^s . \quad (10)$$

Since we wish to calculate our integral (9) by the method of MC it is reasonable to investigate the statistical distributions of functions (10) with the

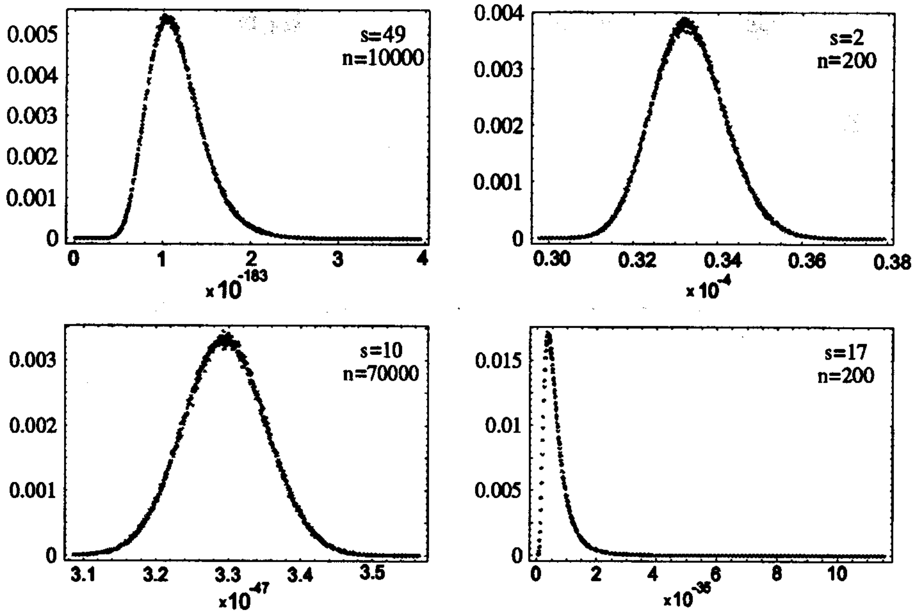

Figure 1: Examples of statistical distributions of $T_s(n)$ functions

constraint $\sum y_i = 1$. The calculations were realized for the different values of s and n. As can be seen from Fig.1 for relatively small values of s the distribution tends to the normal type but for relatively big values of s the distribution tends to the Poisson's one.

We cannot receive the exact expression for statistical distribution of functions $T_s(n)$. But we find the good approximation for their extremum:

$$\frac{1}{s+1}\left(\frac{2}{n}\right)^s. \tag{11}$$

We find also the limits of the area where is grouped more significant values of $T_s(n)$:

$$\left(\frac{1}{s+1}\left(\frac{1.7}{n}\right)^s, \frac{1}{s+1}\left(\frac{2.3}{n}\right)^s\right). \tag{12}$$

Let us imagine that we calculate our integral (9) by the MC method. On every step we must randomly select the group of nonnegative numbers $y_1, y_2, \ldots, y_n$ with the constraint $\sum y_i = 1$ and substitute their to eq. (9). As a result, the maximal number of items must contain as a functions $T_s(n)$ values $(2/n)^s/(s+1)$. Consequently, if we neglect a small contributions terms we may change in (9) functions $T_s(n)$ by the corresponding values $(2/n)^s/(s+1)$ and carry out exponent from integrand expression (9).

After this procedure the remainder integral has a form:

$$\left\{\prod_{i=1}^{n}\int_0^1 dy_i\sqrt{y_i}\right\}\delta\left(1-\sum_{i=1}^{n}y_i\right). \tag{13}$$

It is easily calculable if we pass to the hyperspherical coordinates:

$$\begin{aligned} y_1 &= \rho\cos^2(\varphi_{n-1})\ldots\cos^2(\varphi_2)\cos^2(\varphi_1) \\ y_2 &= \rho\cos^2(\varphi_{n-1})\ldots\cos^2(\varphi_2)\sin^2(\varphi_1) \\ &\vdots \\ y_{n-1} &= \rho\cos^2(\varphi_{n-1})\sin^2(\varphi_{n-2}) \\ y_n &= \rho\sin^2(\varphi_{n-1}). \end{aligned} \tag{14}$$

The result looks as follows:

$$\left\{\prod_{i=1}^{n}\int_0^1 dy_i\sqrt{y_i}\right\}\delta\left(1-\sum_{i=1}^{n}y_i\right)=\frac{(\Gamma(3/2))^{n-1}}{\Gamma(3n/2)}. \tag{15}$$

The final expression has a form:

$$Z_n(E)=\frac{\pi^{(3n-1)/2}}{r_0\Gamma(3n/2)}\left[\frac{m(n_{max}-n)}{2r_0}\right]^{n-1}e^{nW(2)} \tag{16}$$

where

$$W(t)=\frac{n}{t}\int_0^{t/n} ln\left[\frac{1}{\sqrt{y}}F\left(r_0m\sqrt{(n_{max}-n)y((n_{max}-n)y+2)}\right)\right]dy. \tag{17}$$

Fig.2 demonstrates three graphics of $\ln(Z_n(E))/n$ for various values of parameter t in (17) - 1.7, 2.0, 2.3. As is shown in the graphic observable differences take place for small n (< 5000). Thus, we find an area of validity of our equation (16).

Generalization for another form of f_n (2) is not a complicated procedure. It is interesting to calculate the limit $r_0 \to 0$:

$$Q_n(E)=\lim_{r_0\to 0}Z_n(E). \tag{18}$$

For it we use eq.(5) and because $\lim_{x\to 0}F(x)/x=1$ receive:

$$Q_n(E)=\pi^n m^{2n-1}(n_{max}-n)^{\frac{3}{2}n-1}\times$$
$$\times\left\{\prod_{i=0}^{n}\int_0^1 dy_i\sqrt{y_i((n_{max}-n)y_i+2)}\right\}\delta(1-\sum y_i). \tag{19}$$

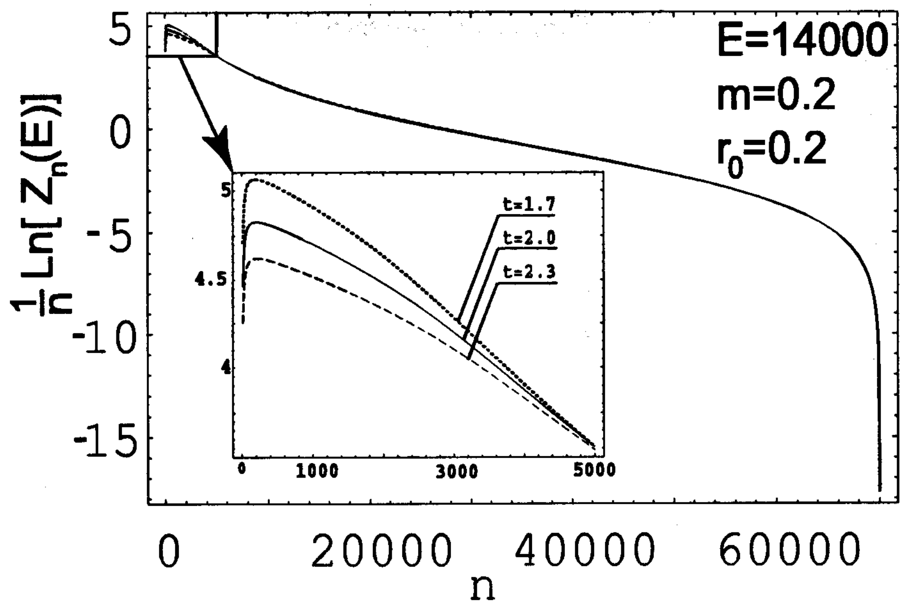

Figure 2: Behavior of $\ln(Z_n(E))/n$ for various values of parameter t

Then we expand the square root in (19) into the series:

$$\sqrt{y_i((n_{max}-n)y_i+2)} = \exp\{\sum_{j=0}^{\infty} a_j y_j\} \tag{20}$$

where:

$$a_0 = \frac{1}{2}\ln(2); a_j = (-1)^{j+1}\frac{1}{2j}(\frac{y}{2})^j . \tag{21}$$

Using the analogous method we receive the following expressions for the "normalized" phase space integral $\overline{Z_n(E)}$:

$$\overline{Z_n(E)} = Z_n(E)/Q_n(E) = \left\{ \frac{1}{r_0 m\sqrt{2(n_{max}-n)}} \exp^{W(2)-U(2)} \right\}^n \tag{22}$$

where

$$U(t) = \frac{1}{2}[(2-t)\tau/(2t) + (1+1/\tau)\ln(1+\tau) - 1] \tag{23}$$

and

$$\tau = \frac{t(n_{max}-n)}{2n} .$$

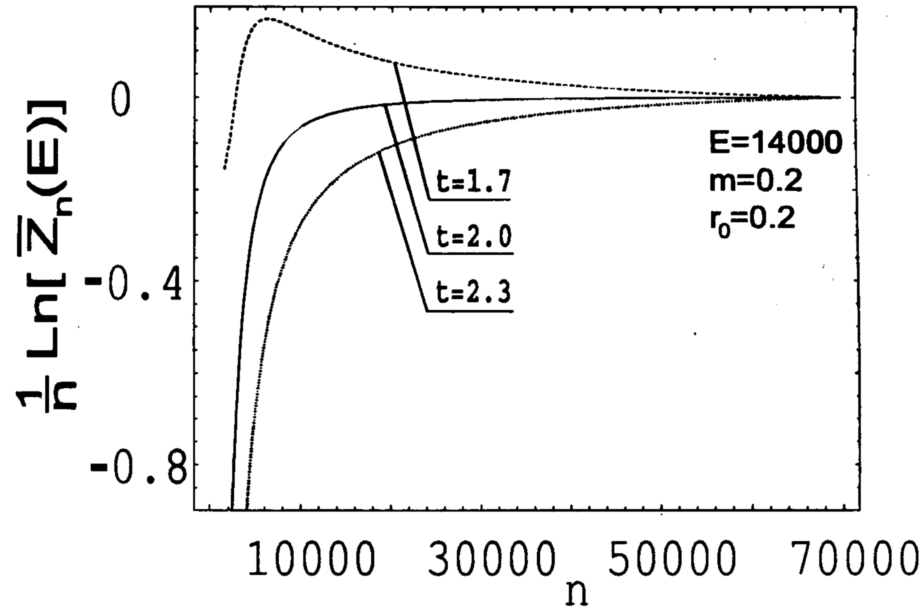

Figure 3: Behavior of $\ln(\overline{Z_n(E)})/n$ for various values of parameter t

In Fig.3 you can see the behavior of $\overline{Z_n(E)}$ for various values of the parameter t in (17) - 1.7, 2.0, 2.3.

3 Conclusion

Method of estimation phase space integral by use the law of large numbers allows to receive an analytical expression. The last is very significant when we build the fast event generator.

As would be expected, the dependence on the cutting parameter r_0 vanishes at the asymptotic on multiplicity $n \rightarrow n_{max}$ but it is essential in case when n have a finite value.

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