VERY HIGH MULTIPLICITY HADRON PROCESSES

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Abstract

The paper contains a description of a first attempt to understand the extremely inelastic high-energy hadron collisions, when the multiplicity of produced hadrons considerably exceeds its mean value. Problems with existing model predictions are discussed. The real-time finite-temperature S-matrix theory is built to have a possibility to find model-free predictions. This allows to take the statistical effects into consideration and build the phenomenology. The questions to experiment are formulated at the very end of the paper.

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1. Introduction

The intuitive feeling that hadron matter should be maximally perturbed in the high-energy extremely inelastic collisions was the main reason of our effort to consider such processes. We had hoped to observe new dynamical phenomena, or new degrees of freedom, unattainable in other ordinary hadron reactions. This paper presents a first attempt to describe the particularity of considered processes, to give a review of existing models prediction and, at the end, we will offer the field-theoretical formalism for hadron inelastic processes.

Thus, considering hadrons mean multiplicity $\bar{n}(s)$ as a natural scale of the produced hadrons multiplicity n at given CM energy $\sqrt{s}$, we would assume that

$$n \gg \bar{n}(s) . \tag{1.1}$$

At the same time we wish to have

$$n \ll n_{\max} = \sqrt{s}/m , \tag{1.2}$$

where $m \simeq 0.1$ Gev is the characteristic hadron mass. The last restriction is introduced to weaken the unphysical constraints from the finite, for given s, phase space volume. We should assume therefore that s is high enough.

The multiple production is the process of colliding particles where kinetic energy is dissipated into the mass of produced particles [1]. Then one may validate that the entropy $\mathscr{S}$ accedes its maximum in the domain (1.1) since the multiplicity n characterizes the rate of stochastization, i.e. the level of incident energy dissipation over existing (free) degrees of freedom.

There is also another quantitative definition of our reactions. Let $\varepsilon_{\max}$ be the energy of the fastest particle in the given frame and let E be the total incident energy in the same frame. Then the difference $(E - \varepsilon_{\max})$ is the energy spent on production of the less energetic particles. It is useful to consider the inelasticity coefficient

$$\kappa = \frac{E - \varepsilon_{\max}}{E} = 1 - \frac{\varepsilon_{\max}}{E} \leq 1 . \tag{1.3}$$

It defines the portion of spent energy. Therefore, we wish to consider processes with

$$1 - \kappa \ll 1 . \tag{1.4}$$

So, the produced particles have comparatively small energies.

This property may be used for experimental triggering of our processes. Indeed, using the energy conservation law,

$$n(1 - \kappa) > 1 . \tag{1.5}$$

Following (1.2) we will assume that

$$1 - \kappa \gg \frac{m}{E} . \tag{1.6}$$

Therefore, the kinetic energy of the particles produced in our processes cannot be arbitrarily small.

Using thermodynamical terminology, we wish to investigate the production and properties of comparatively 'cold' multi-hadron (mostly of π-mesons) state. We would like to note from the very beginning that we have only a qualitative scenario of such states which may be produced and the review of the corresponding why's may be considered as the main purpose of this paper. At the end of this paper (Appendix K) we will describe principal features of possible field-theoretical solution of our problem.

The absence of any authentic experimental information concerning discussed processes should be noted. Moreover, actually the hadron inelastic interactions with a set peculiar to them of unsolved theoretical problems will be considered. Nevertheless, we suggest to work in this field in spite of these difficulties because the system with extremal properties may be more transparent since the asymptotics always simplify a picture. We would demonstrate this idea and will try to put it in the basis of developed theoretical methods.

The absence of experimental information about such high inelastic hadron processes is the consequence of the smallness of corresponding cross sections. Besides this, it was unclear for what purpose the experimental efforts should be done. We would like to convince the reader that the discussed problem is interesting and important. For instance, we will discuss a possibility that asymptotics over n may replace in a definite sense the asymptotics over $\sqrt{s}$. A short address to experimentalists will be given in Section 4.1.2.

We hope that the paper would be useful both for theorists and experimentalists. For this reason the main text of this paper will contain only the qualitative discussion of the problems and results. The quantitative proof, formulation of pure theoretical methods, etc. are added in the appendixes. Considered extremal problem is a good theoretical laboratory and is described in the appendixes and theoretical methods may be applied for other physical problems.

We would like to point out that a special technique was built for the problem discussed, see Section 2.

- *Having the very high multiplicity (VHM) state it is natural to use the thermodynamical methods.* We will offer for this purpose the real-time S-matrix interpretation of thermodynamics. It can be shown in what quantitative conditions it will coincide with simpler canonical imaginary-time Matsubara formalism. We will give also the generalization of the real-time finite-temperature perturbation theory in the case of local temperature $T = 1/\beta$ distribution, when $\beta = \beta(x, t)$. This will allow to use the thermodynamical description if the system is far from equilibrium.
- *The particle spectrum in the VHM region is soft. It is just a situation when the collective phenomena should be important.* To describe these phenomena, the decomposition on correlators will be adopted. The origin of this decomposition lies in Mayer's 'group decomposition'. In multiple production physics this decomposition is known also as the 'multi-component description' [2]. It is based on the idea that the multiple production process may include various mechanisms.

In Section 3 we will investigate model predictions for VHM region. We would like to note two main conclusions:

- Existing multiperipheral-type models are unable to describe the VHM region.
- The infrared region of the pQCD becomes important even if constraint (1.2) is taken into account.

2. Qualitative inside of the problem

In Section 2.1 we will try to formulate the phenomenology of our problem, i.e. the way the VHM processes may be described and what type of phenomena one may expect. The importance of thermodynamical methods will become evident and we will offer in Section 2.2 a general description of the corresponding formalism.

It is important to note that we may classify the possible asymptotics over n. We will find that there exist only three classes of asymptotics. This will simplify consideration definitely restricting the possibilities.

In Section 2.3 we will use the thermodynamical language to give a physical interpretation of these classes of asymptotics. We will see in result that in our choice of the VHM final state this should lead to reorganization of multiple production dynamics: we will get out of the habitual multiperipheral picture in the VHM domain.

Moreover, one may assume that the semiclassical approximation becomes exact in the VHM domain. This naturally leads to the idea to search for such a scheme of calculation which depends on the choice of final state. Quantitative description of this idea may be realized as is described in Appendix K.

2.1. Phenomenology of VHM processes

The VHM production phenomena include two sub-problems. First of all, it is the dynamical problem of incident energy degradation into the secondary particle energies and the second one is the description of the final state.

We will start discussion in Section 2.1.1 from the second part of the problem to explain that the statistical methods are essential for us.

In Section 2.1.2 we will try to outline at least qualitatively the main mechanisms of hadron production. The peculiarity of hadron production phenomena consists in the presence of hidden constraints, the consequence of local nonAbelian gauge symmetry. The constraints may prevent thermalization and the incident energy dissipation is confusing in this case. Just the 'confusing' effect is dominant in the hadron multiple production processes if $n \sim \bar{n}(s)$.

The expected change of dominant mechanism of hadron production is discussed in Section 2.1.3. It is important that, in spite of hidden constraints, the system may freely evolve to define the VHM state. Such a VHM state should be in equilibrium. A formal definition of the 'equilibrium' notion will be given in Section 2.2.2.

The problem described contains small parameters $(\bar{n}(s)/n) \ll 1$ and $(1-\kappa) \ll 1$. To have the possibility of estimation of contributions in accordance with these parameters one should include them into the formalism. This becomes possible if and only if the integral quantities are calculated. So, the multiplicity n is an index only if the multiple production amplitudes $a_n(p_1, \ldots, p_n)$ are considered. But the cross section $\sigma_n(s)$ is a nontrivial *function* of n. We will calculate by this logic mostly integrals of $|a_n|^2$, excluding from consideration the amplitudes, see also Section 2.2, where the first realization of this idea is offered (a *naive* attempt to realize this idea one may find in [3]). This is a general methodological feature of our consideration.

2.1.1. Formulation of the problem

The multiple production cross section $\sigma_n(s)$ falls down rapidly in the discussed very high multiplicity (VHM) domain (1.1) and for this reason the multiplicities $n \sim n_{\max}$ are not accessible experimentally. At the LHC energy $\bar{n}(s) \simeq 100$ is valid and we will assume that $n \sim \bar{n}(s)^2 \simeq 10\,000$ is just the discussed VHM region ($n_{\max} \simeq 100\,000$ at the LHC energy). We will explain later why

$$n \sim \bar{n}(s)^2 \tag{2.1}$$

is chosen for the definition of the VHM region.

Generally speaking, having the state of a large number of particles, it is reasonable to depart from an exact definition of the number n of created particles, their individual energies ε_i, momenta q_i, etc. since they cannot be defined exactly by experiment. Indeed, for instance, full reconstruction of kinematics is a practically impossible task because of neutral particles, neutrinos, the more so as $n \sim 10\,000$ is considered. We suppose that nothing will happen if n is measured with $\Delta n \neq 0$ accuracy since $(\Delta n/n) \ll 1$ is easily attainable in the VHM region. Besides, it is practically impossible to deal with theory which operates by the $N = 3n - 4$ ($\sim 30\,000$!) variables.

Artificial reduction of the set of the necessary variables may lead to a temporary success only. Indeed, the last 30 years of multiple production physics development was based on the inclusive approach [4], when the measured quantities (cross sections) depend on a few dynamical parameters only. But later on the experiment and its fractal analyses show that the situation is not so simple, also as, for instance, for the classical turbulence. So, the event-by-event experimental data show that the particle density fluctuation is unexpectedly large [5] and the fractal dimension D_{f} is not equal to zero [6].

We know that if the fractal dimension is non-trivial, then the system is extremely 'non-regular' [7]. So, $D_{\mathrm{f}} \simeq 0.3$ for the perimeter of Great Britain and $D_{\mathrm{f}} \simeq 0.5$ for Norway. The discrepancy marks the fact that the shore of Norway is much more broken than that of Great Britain [8]. It is noticeable that the fractal dimension D_{f} crucially depends on the type of reaction, incident energy and so on.

It is evident that one may choose from $N = 3n - 4$ an arbitrary finite set of variables to characterize the multiple production process. But the fractal analyses show that such an approach would lead to the same effect as if one may hear, for example, only the first violin of Mahler's music.

So, it is important to understand when the restricted set of dynamical variables will allow to describe the process (state) *completely*. The same problem was solved in statistical physics, where the 'rough' description by a restricted number of (thermodynamical) parameters is a basis of its success, see the discussion of rough variables description, e.g. in the review of Uhlenbeck [9]. We will search for the same solution desiring to build a complete theory of the VHM hadron reactions.

We want to note that just VHM process may be in this sense 'simple': from all evidence, the system becomes 'quiet' in the VHM region and for this reason its 'rough' thermodynamical description is available. It seems natural, therefore, to start investigation of multiple production phenomena from the (extremely rare) VHM processes.

2.1.2. Soft channel of hadron production

The dominant inelastic hadron processes at $n \sim \bar{n}(s)$ are saturated by production of low transverse momentum hadrons [10]. One of the approaches explains this phenomenon by the nonperturbative effect of quarks created from vacuum.

The corresponding dynamics appears as follows. At the expense of transverse kinetic motion color charges may separate at large distances. Nevertheless, the transverse motion is suppressed since separation leads to increasing polarization of vacuum, because of confinement phenomenon. Then, as in QED [11], the vacuum becomes unstable in regard to the tunneling creation of real fermions. Just on their creation, the transverse kinetic energy is spent and, as a result, particles cannot have, with exponential accuracy, high transverse energies. This picture is attractive being simple and transparent, but despite numerous efforts [12], there is no quantitative description of this phenomenon till now. Briefly, the problem is connected with the unknown mechanism of strong colored electric fields formation among distant colored charges, see also [13].

One may use other terms. The soft channel of multiple production means the long-range correlation among hadron colored constituents. Under this special correlation the nonAbelian gauge field theory conservation laws constraints were implied. They are important in dynamics since each conservation law decreases the number of independent degrees of freedom at least on one unity (this may explain why hadrons $\bar{n}(s) \ll n_{\max}(s)$), i.e. it has a nonperturbative effect. Moreover, in the so-called integrable systems each independent integral of motion (in involution) reduces the number of degrees of freedom on two units. As a result there is no thermalization in such systems [14] and the corresponding mean multiplicity $\bar{n}(s)$ should be equal to zero, see Appendix K.

The existence of multiple production, $\bar{n}(s) \gg 1$, testifies to the statement that the thermalization phenomena exist in hadron processes, i.e. the system of Yang–Mills fields is not completely integrable. But the most probable process with $n \sim \bar{n}(s)$ did not lead to the final state with maximal entropy since $\bar{n}(s) \ll n_{\max}$, i.e. the definite restrictions on the dissipation dynamics should be taken into account. Such problems, being intermediate, are mostly complicated ones.

The quantitative theory of these phenomena may lead to deep revision of the main notions of the existing quantum field theory [15,16], see Appendix K. So, the dynamical display of hidden conservation laws of the hadron system are probably unstable since we expect that the system is not completely integrable, solitary field configurations $u_c(x,t)$ [17]. Then the quantum theory should be able to describe quantum excitations of these fields, i.e. to count the fluctuations of 'curved' manifolds. The canonical perturbation theory methods, formulated in terms of creation and annihilation of particles in the external field $u_c(x,t)$, are too complicated, see [18] and references cited therein. For this reason, existing calculations usually do not exceed the semiclassical approximation. We hope that, as described in Appendix K, the quantization scheme would be able to solve this problem (see also the example described there).

Another approach assumes that the special 't-channel' ladder-type Feynman diagrams are able to describe the $n \sim \bar{n}(s)$ region [19]. This approach did not take into account the confinement nonperturbative effects introducing the hardly controllable supposition that the free quarks and gluons may form a complete set of states. Formally this is right, but from all evidence, the decomposition on this Fock basis is realized in the nonAbelian gauge field theories on zero measure [16,20] (see Appendix K). Nevertheless, one may reject this argument assuming that the process is happening at a sufficiently small distance.

Corresponding contribution came from the so-called 'hard Pomeron' [21]. But the intrinsic problems of the accuracy of chosen logarithmic approximations [22], the understanding of the so-called nonlogarithmic corrections [23], of the fate of the infrared divergences remain unsolved till now.

2.1.3. Multiple production as a process of dissipation

So, the multiple production of soft hadron phenomena seems unsolvable on the day-to-day level of understanding and we may with a clear conscience move it away. This is why the hadron inelastic reactions lost some popularity, migrating the last two decades to the class of 'noninteresting' problems. Yet, in a number of modern fundamental experiments, multiple production plays, at least, the role of background to the investigated phenomena and for this reason we should be ready for the quantitative estimation of it.

Our hope to describe such a complicated problem as the multiple production phenomenon in the VHM conditions is based on the following idea. At the very beginning of this century, a couple P. and T. Ehrenfest, had offered a model to visualize Boltzmann's interpretation of the irreversibility phenomena in statistics. The model is extremely simple and fruitful [24]. It considers the two boxes with $2N_b$ numerated balls. Choosing the label of the balls *randomly* one must take the ball with the corresponding label from one box and put it into another one. One may repeat this action an arbitrary number of times t.

Starting from the highly 'nonequilibrium' state with all balls in one box, $N_b \gg 1$, it is seen to be stationary with t tendency to equalization of the number of balls in the boxes (Fig. 1). The stationarity means that the number of balls in the other box rises $\sim t$ at least on an early stage of

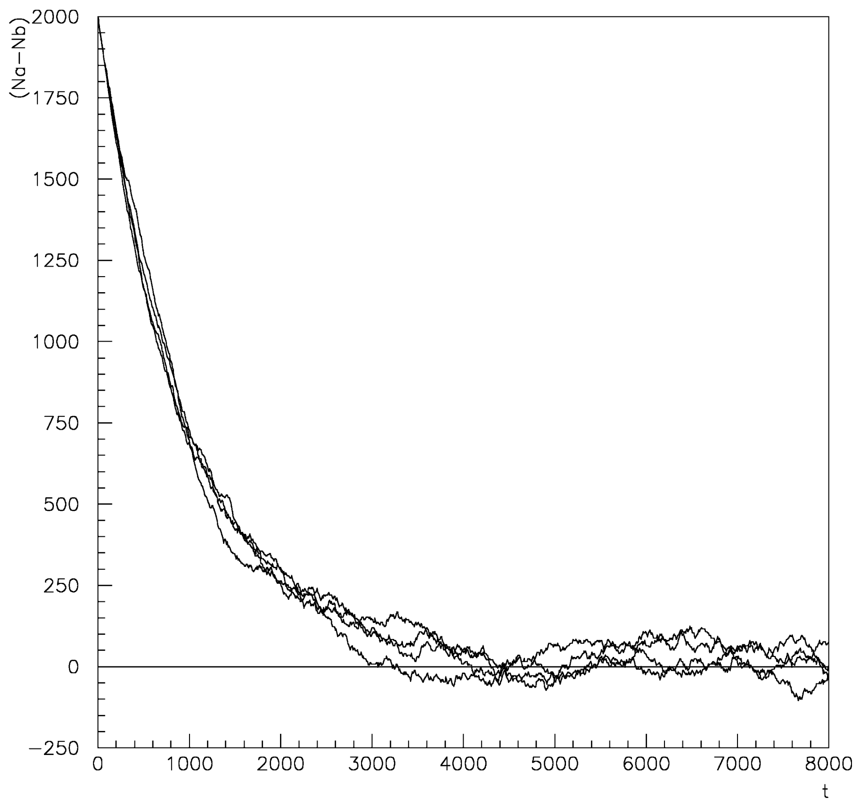

Fig. 1. Predictions of the Ehrenfest model. Four simulations are displayed.

the process. This signifies the presence of an *irreversible*[2] flow (of balls) toward the preferable (equilibrium) state. One can hope [24] that this model reflects a physical reality of nonequilibrium processes with the initial state very far from equilibrium. A theory of such processes with (irreversible) flow toward a state with maximal entropy should be sufficiently simple being close to the stationary Markovian.

The VHM production process may be, at least in an early stage, stationary Markovian. If this is so then one may neglect long-range effects, nonperturbative as well, since they are not Markovian as follows from the experience described in Section 2.1.2.

This is possible if the VHM process is happening so fast (being the short-range phenomenon) that the confinement forces became 'frozen'. It can be shown that the quantitative reason for these phenomena is a fast (exponential) reduction with n of the soft channel contribution into the hadron production process. So, we expect a change of the multiple production dynamics in the VHM region.

Thus, the main input idea consists of two general propositions. The first of them is the following:

(I). *The hadron VHM production processes should be close to the stationary Markovian.*

'Freezing' the confinement constraints, the entropy $\mathscr{S}$ may exceed for given energy $\sqrt{s}$ its available maximum in the VHM domain. Then one can assume that the VHM final state is in 'equilibrium', or is close to it. So,

(II). *The VHM final state should be close to equilibrium* is our second basic proposition.

We would select and appreciate particle physics models in accordance with these propositions.

2.2. S-matrix interpretation of thermodynamics

The field-theoretical description of statistical systems at a finite temperature is based usually on the formal analogy between imaginary time and inverse temperature $\beta = 1/T$ [26]. This analogy is formulated by Schwinger [11] as the 'euclidean postulate' and it assumes that (i) the system is in equilibrium, i.e. it should allow the arbitrary rearrangement of states of temporal sequence in the described process,[3] and (ii) there are no special space-time long-range correlations among states of the process, i.e., for instance, the symmetry constraints should not play a crucial role. We do not know *ad hoc* whether or not to apply the 'euclidean postulate' for given n and s, even if (1.1) is satisfied. For this reason we are forced to formulate the theory in natural real-time terms.

The first important quantitative attempt to build the real-time finite-temperature field theory [27] discovered the formal problem of the so-called 'pinch-singularities'. Further investigation of the theory has allowed to demonstrate the cancellation mechanism of these unphysical singularities [28]. This is attained by doubling the degrees of freedom [29–31]: the Green functions of the theory represent 2×2 matrix [32]. It surely makes the theory more complicated, but the operator formalism of the thermo-field dynamics [32] shows the unavoidable character of this complication.

[2] One can say that the opposite flow is never seen. 'What never? No never! What never? Well, hardly ever'. This dialog was taken from [25].

[3] In other terms, one may have the possibility to apply the ergodic hypothesis.

Notice that the canonical real-time finite temperature field-theoretical description [29,30] of the statistical systems based on the Kubo–Martin–Schwinger (KMS) [29,33,34] boundary condition for a field is

$$\Phi(t) = \Phi(t - \mathrm{i}\beta) . \tag{2.2}$$

It, without fail, leads to the *equilibrium* fluctuation–dissipation conditions [35] (see also [36]). Due to this it cannot be applied in our case, where the dissipation problem is solved. The origin of this boundary condition is shown in Appendix A.

We will use a more natural microcanonical formalism for particle physics.[4] The thermodynamical 'rough' variables are introduced in this approach as the Lagrange multipliers of corresponding conservation laws. The physical meaning of these 'rough' variables is defined by the corresponding equations of state.

We shall use the S-matrix approach which is natural for the description of the time evolution. (The S-matrix description is used also in [38,39].) For this purpose the amplitudes

$$\langle p_1, p_2, \ldots, p_n | q_1, q_2, \ldots, q_m \rangle = a_{nm}(p, q) \tag{2.3}$$

of the m- into n-particles transition will be introduced. The in- and out-states must be composed from mass-shell particles [40]. Using these amplitudes we will calculate

$$R_{nm}(p, q) = |a_{nm}(p, q)|^2 = \langle p_1, p_2, \ldots, p_n | q_1, q_2, \ldots, q_m \rangle \langle q_1, q_2, \ldots, q_m | p_1, p_2, \ldots, p_n \rangle . \tag{2.4}$$

This will lead to the doubling of the degrees of freedom.

The temperature description will be introduced (see also [41]) noting that, for instance,

$$\begin{aligned} \mathrm{d}\Gamma_n &= |a_{nm}(p, q)|^2 \mathrm{d}\Omega_n(p) , \\ \mathrm{d}\Omega_n(p) &= \prod_1^n \frac{\mathrm{d}^3 p_i}{(2\pi)^3 2\varepsilon(p_i)}, \quad \varepsilon(p) = (p^2 + m^2)^{1/2} \end{aligned} \tag{2.5}$$

is the differential measure of the final state. It is a first example where the usefulness of the probability-like quantity $\sim |a_{nm}|^2$ is seen.

Measure (2.5) is defined on the energy–momentum shell

$$\sum_{i=1}^{n} p_i = P . \tag{2.6}$$

It should be underlined that $a_{nm}(p, q)$ are the translationally invariant amplitudes and four equalities:

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{m} q_i \tag{2.7}$$

[4] The statistical methods for particles physics are discussed also in [37].

are obeyed identically. So, Eqs. (2.6) are the constraints and to take them into account one may multiply $d\Gamma_n$ on

$$\prod_{k=1}^{n} e^{i\alpha p_k} ,$$

where α is the time-like 4-vector. It is evident that integration over α with factor $e^{-i\alpha P}$ gives the constraints (2.6).

One may simplify the calculation assuming that all calculations are performed, for example, in the CM frame $P = (E, \mathbf{0})$. Then one may ignore the space components considering $\alpha = (\alpha_0, \mathbf{0})$. This is the equivalent of the assumption that only the energy conservation law is important.

The last step is the substitution $\alpha_0 = i\beta$, where β is our Lagrange multiplier. To define its physical meaning one should solve the equation of state

$$E = -\frac{\partial}{\partial\beta}\ln\int d\Gamma_n \prod_{k=1}^{n} e^{-\beta\varepsilon(p_k)} \equiv -\frac{\partial}{\partial\beta}\ln\int d\Gamma_n(\beta) . \tag{2.8}$$

Such a definition of temperature as the Lagrange multiplier of the energy conservation law is obvious for microcanonical description [33].

The initial-state temperature will be introduced in the same way, taking into account (2.7). So, we will construct the two-temperature theory. It is impossible to use the KMS boundary condition in such a two-temperature description (the equation of state can be applied at the very end of the calculations).

It should be noticed that the 'density matrix' $R_{n,m}(p,q)$, defined in (2.4), describes the 'closed-path motion' in the functional space. So, if

$$\langle p_1, p_2, \ldots, p_n | q_1, q_2, \ldots, q_m \rangle = \langle n, out | e^{iS(\Phi_+)} | m, in \rangle \tag{2.9}$$

and

$$\langle p_1, p_2, \ldots, p_n | q_1, q_2, \ldots, q_m \rangle^* = \langle q_1, q_2, \ldots, q_m | p_1, p_2, \ldots, p_n \rangle$$
$$= \langle m, in | e^{-iS(\Phi_-)} | n, out \rangle \tag{2.10}$$

then, by definition,

$$\Phi_+(\sigma_\infty) = \Phi_-(\sigma_\infty) = \Phi(\sigma_\infty) \tag{2.11}$$

with some 'turning-point' fields $\Phi(\sigma_\infty)$, where σ_∞ is the remote hypersurface. The value of $\Phi(\sigma_\infty)$ specifies the environment of the system. We will show that (2.11) coincides with the KMS boundary condition in some special cases. Here consequences of the vacuum boundary condition:

$$\Phi(\sigma_\infty) = 0 \tag{2.12}$$

are analyzed.

One should admit also that the boundary conditions given below are not unique: one can consider arbitrary organization of the environment of the considered system. The S-matrix interpretation is able to show the way as an arbitrary boundary condition may be adopted. This should extend the potentialities of the real-time finite-temperature field-theoretical methods.

2.2.1. Example

It seems useful to illustrate the above microcanonical approach by the simplest example, see also [41]. By definition, the n particles production cross section

$$\sigma_n(s) = \int d\Omega_n(p)\delta\left(q_1 + q_2 - \sum_{i=1}^{n} p_i\right)|a_n(p,q)|^2 , \tag{2.13}$$

where $a_n(p,q) \equiv a_{n2}(p,q)$ is the ordinary n particle production amplitude in accelerator experiments.

Considering the Fourier transform of energy–momentum conservation δ-function one can introduce the generating function ρ_n, see [41] and references cited therein.[5] We may find in the result that σ_n is defined by the equality

$$\sigma_n(E) = \int_{-i\infty}^{+i\infty} \frac{d\beta}{2\pi} e^{\beta E}\rho_n(\beta),\ E = \varepsilon(q_1) + \varepsilon(q_2) , \tag{2.14}$$

where

$$\rho_n(\beta) = \int\left\{\prod_{i=1}^{n} \frac{d^3p_i e^{-\beta\varepsilon(p_i)}}{(2\pi)^3 2\varepsilon(p_i)}\right\}|a_n|^2 = \int d\Gamma_n(\beta) . \tag{2.15}$$

The most probable value of β in (2.14) is defined by the equation of state (2.8). Inserting (2.15) into (2.14) we find expression (2.13) if the momentum conservation shell is neglected. The last one is possible since the cross sections are always measured in the definite frame.

Let us consider the simplest example of noninteracting particles [41]:

$$\rho_n(\beta) \sim \{2\pi m K_1(\beta m)/\beta\}^n ,$$

where K_1 is the Bessel function. Inserting this expression into (2.8) we can find that in the nonrelativistic case ($n \sim n_{\max}$)

$$\beta_c = \frac{3}{2}\frac{(n-1)}{(\sqrt{s} - nm)} ,$$

i.e., we find the well-known equality

$$E_{\text{kin}} = \tfrac{3}{2}T , \tag{2.16}$$

where $E_{\text{kin}} = (\sqrt{s} - nm)/(n-1)$ is the mean kinetic energy and $T = 1/\beta_c$ is the temperature (the Boltzmann constant was taken to be equal to one).

It is important to note that Eq. (2.8) has a unique real solution $\beta_c(s,n)$ rising with n and decreasing with s [33].

The expansion of the integral (2.14) near $\beta_c(s,n)$ unavoidably gives an asymptotic series with zero convergence radii since $\rho_n(\beta)$ is the essentially nonlinear function of β, see also Section 2.2.2. This

[5] The generating functionals method was developed in [42].

means that, generally speaking, fluctuations in the vicinity of $\beta_c(s,n)$ may be arbitrarily high and in this case $\beta_c(s,n)$ has no physical sense. But if the fluctuations are Gaussian, then $\rho_n(\beta)$ coincides with the partition function of the n particle state and $\beta_c(s,n)$ may be interpreted as the inverse temperature. We will put the observation of this important fact in the basis of our thermodynamical description of the VHM region.

2.2.2. Relaxation of correlations

The notion of 'equilibrium' over some parameter X in our understanding is a requirement that the fluctuations in the vicinity of its mean value, $\bar{X}$, have a Gaussian character. Notice, in this case, that one can use this variable for a 'rough' description of the system. We would like to show now that the corresponding equilibrium condition would have the meaning of the correlations relaxation condition of Bogolyubov [42],[6] see also [43]. Let us define the conditions when the fluctuations in the vicinity of β_c are Gaussian [44]. Firstly, to estimate integral (2.14) in the vicinity of the extremum, β_c, we should expand $\ln \rho_n(\beta+\beta_c)$ over β:

$$\ln \rho_n(\beta+\beta_c) = \ln \rho_n(\beta_c) - \sqrt{s}\beta + \frac{1}{2!}\beta^2 \frac{\partial^2}{\partial \beta_c^2} \ln \rho_n(\beta_c) - \frac{1}{3!}\beta^3 \frac{\partial^3}{\partial \beta_c^3} \ln \rho_n(\beta_c) + \cdots \tag{2.17}$$

and, secondly, expand the exponent in the integral (2.14) over, for instance,

$$\partial^3 \ln \rho_n(\beta_c)/\partial \beta_c^3, \ldots ,$$

etc. In the result, if higher terms in (2.17) are neglected, the kth term of the perturbation series

$$\rho_{n,k} \sim \left\{ \frac{\partial^3 \ln \rho_n(\beta_c)/\partial \beta_c^3}{(\partial^2 \ln \rho_n(\beta_c)/\partial \beta_c^2)^{3/2}} \right\}^k \Gamma\left(\frac{3k+1}{2} \right) . \tag{2.18}$$

Therefore, because of Euler's $\Gamma((3k+1)/2)$ function, the perturbation theory near β_c leads to the asymptotic series. The supposition to define this series formally, for instance, in the Borel sense is not interesting from the physical point of view. Indeed, such a formal solution assumes that the fluctuations near β_c may be arbitrarily high. Then, for this reason, the value of β_c loses its significance: arbitrary values of $(\beta-\beta_c)$ are important in this case.

Nevertheless, it is important to know that our asymptotic series exists in some definite sense, i.e. we can calculate the integral over β by expanding it over $(\beta-\beta_c)$. Therefore, if the considered series is asymptotic, we may estimate it by first term if

$$\partial^3 \ln \rho_n(\beta_c)/\partial \beta_c^3 \ll (\partial^2 \ln \rho_n(\beta_c)/\partial \beta_c^2)^{3/2} . \tag{2.19}$$

One of the possible solutions of this condition is

$$\partial^3 \ln \rho_n(\beta_c)/\partial \beta_c^3 \approx 0 . \tag{2.20}$$

[6] The term 'vanishing of correlations' was used by N.N. Bogolyubov for this phenomenon.

If this condition is satisfied, then the fluctuations are Gaussian with dispersion

$$\sim \{\partial^2 \ln \rho_n(\beta_c)/\partial\beta_c^2\}^{1/2} ,$$

see (2.17).

Let us consider now (2.20) carefully. We will find by computing derivatives that this condition means the following approximate equality

$$\frac{\rho_n^{(3)}}{\rho_n} - 3\frac{\rho_n^{(2)}\rho_n^{(1)}}{\rho_n^2} + 2\frac{(\rho_n^{(1)})^3}{\rho_n^3} \approx 0 , \tag{2.21}$$

where $\rho_n^{(k)}$ means the kth derivative. For identical particles,

$$\rho_n^{(k)}(\beta_c) = n^k(-1)^k \int d\Gamma_n(\beta_c) \prod_{i=1}^{k} \varepsilon(q_i) . \tag{2.22}$$

Therefore, the left-hand side of (2.21) is the 3-point correlator K_3 since $d\Gamma_n(\beta_c)$ is a density of states for given β

$$K_3 \equiv \int d\Omega_3(q) \left(\left\langle \prod_{i=1}^{3} \varepsilon(q_i) \right\rangle_{\beta_c} - 3 \left\langle \prod_{i=1}^{2} \varepsilon(q_i) \right\rangle_{\beta_c} \langle \varepsilon(q_3) \rangle_{\beta_c} + 2 \prod_{i=1}^{3} \langle \varepsilon(q_i) \rangle_{\beta_c} \right) , \tag{2.23}$$

where the index β_c means that averaging is performed with the Boltzmann factor $\exp\{-\beta_c \varepsilon(q)\}$.

Notice, in distinction with Bogolyubov, K_3 is the *energy* correlation function. So, in our interpretation, one can introduce the notion of temperature $1/\beta_c$ if and only if the *macroscopic* energy flows, measured by the corresponding correlation functions, are to die out.

As a result, to have all the fluctuations in the vicinity of β_c Gaussian, we should have $K_m \approx 0$, $m \geq 3$. Notice, as follows from (2.19), that the set of minimal conditions actually appears as follows:

$$|K_m| \ll |K_2|^{m/2}, \ m \geq 3 . \tag{2.24}$$

If the experiment confirms these conditions then, independent of the number of produced particles, the final state may be described with high enough accuracy by one parameter β_c and the energy spectrum of particles is Gaussian. In these conditions one may return to the statistical [1] and the hydrodynamical models [45].

Considering β_c as a physical (measurable) quantity, we are forced to assume that both the total energy of the system, $\sqrt{s} = E$, and the conjugate to it, variable β_c, may be measured simultaneously with high accuracy.

2.2.3. *Connection with Matsubara theory*

We would like to show now that the ordinary big partition function of the statistical system coincides with

$$\sum_{n,m} \int_{(\beta_1, z_1; \beta_2, z_2)} R_{nm}(p, q) = \rho(\beta, z) , \tag{2.25}$$

where $R_{nm}(p,q)$ is defined by (2.4). The summation and integration are performed with constraints that the mean energy of particles in the initial(final) state is $1/\beta_1(1/\beta_2)$. One may interpret $1/\beta$ in the first approximation as the temperature and $z_1(z_2)$ as the activity for initial(final) state.

Direct calculation, see Appendix B, gives the following expression for generating functional:

$$\rho(\beta,z) = e^{-iN(\phi_i^*\phi_j)}R_0(\phi) , \tag{2.26}$$

where the particle number operator ($\hat{\phi}(x) = \delta/\delta\phi(x)$)

$$N(\phi_i^*\phi_j) = -\int dx\, dx'(\hat{\phi}_+(x)D_{+-}(x-x',\beta_2,z_2)\hat{\phi}_-(x') - \hat{\phi}_-(x)D_{-+}(x-x',\beta_1,z_1)\hat{\phi}_+(x')) \tag{2.27}$$

and

$$R_0(\phi) = Z(\phi_+)Z^*(-\phi_-) , \tag{2.28}$$

where $Z(\phi)$ is defined in (B.10):

$$Z(\phi) = \int D\Phi e^{iS(\Phi) - iV(\Phi+\phi)}$$

and, for the vacuum boundary condition $\Phi(\sigma_\infty) = 0$,

$$D_{+-}(x-x',\beta,z) = -i\int d\Omega_1(q)e^{iq(x-x')}e^{-\beta\varepsilon(q)}z(q) , \tag{2.29}$$

$$D_{-+}(x-x',\beta,z) = i\int d\Omega_1(q)e^{-iq(x-x')}e^{-\beta\varepsilon(q)}z(q) , \tag{2.30}$$

are, respectively, the positive and negative frequency correlation functions at $z = 1$.

It is evident that

$$R_{nm}(p,q) = \prod_{k=1}^{n}\left\{e^{\beta_1\varepsilon(p_k)}\frac{\delta}{\delta z_1(p_k)}\right\}\prod_{k=1}^{m}\left\{e^{\beta_2\varepsilon(q_k)}\frac{\delta}{\delta z_2(q_k)}\right\}\rho(\beta,z)\Big|_{z_i=0} . \tag{2.31}$$

Notice, defining $R_{nm}(p,q)$ through the generating functional we extract the Boltzmann factors $e^{-\beta\varepsilon}$ since the energy–momentum conservation δ-functions were extracted from amplitudes $a_{nm}(p,q)$.

We suppose that $Z(\phi)$ may be computed perturbatively. As a result, ($\hat{j} = \delta/\delta j$ is the variational derivative)

$$R(\beta,z) = e^{-iV(-i\hat{j}_+) + iV(-i\hat{j}_-)}e^{i/2\int dx\, dx' j_i(x)D_{ik}(x-x';\,\beta,\,z)j_k(x')} , \tag{2.32}$$

where $D_{ik}(x-x')$ is the matrix Green function. These Green functions are defined on the Mills [46] time contours $C_\pm$ in the complex time plane ($C_- = C_+^*$), see Fig. 2. This definition of the time contours coincides with the Keldysh' time contour [30].

The generating functional (2.32) has the same structure as the generating functional of Niemi and Semenoff [28]. The difference is only in the definition of Green functions D_{ik}. This choice is

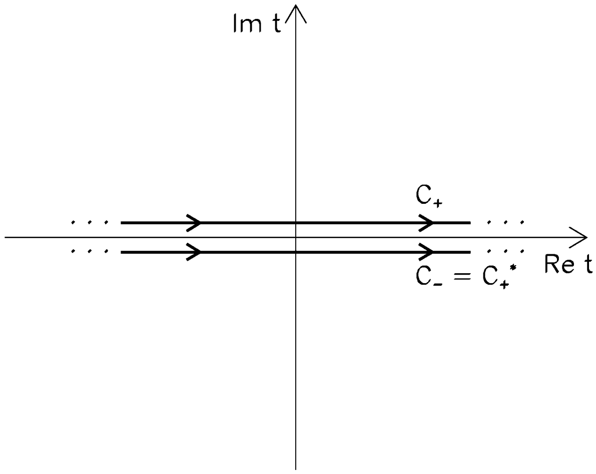

Fig. 2. Keldysh time contour.

a consequence of the boundary condition (B.6). So, if (B.32) is used, then the Green function is defined by Eq. (B.45). Notice also that if $\beta_1 = \beta_2 = \beta$ then a new Green function obeys the KMS boundary condition, see (B.48).

Following Niemi and Semenoff [28] one can write (2.32) in the form

$$\rho(\beta) = \int D_{\mathrm{NS}}\Phi \mathrm{e}^{\mathrm{i}S_{\mathrm{NS}}(\Phi)} , \tag{2.33}$$

where the functional measure $D_{\mathrm{NS}}\Phi$ and the action $S_{\mathrm{NS}}(\Phi)$ are defined on the closed complex time contour C_{NS}, see Fig. 3. The choice of initial time t_{i} and t_{f} is arbitrary. Then one can perform shifts: $t_{\mathrm{i}} \to -\infty$ and $t_{\mathrm{f}} \to +\infty$. In result, (i) if $\beta_1 = \beta_2 = \beta$, (ii) if contributions from imaginary parts C_{+-} and $C_{-\beta}$ of the contour C_{NS} have disappeared in this limit, (iii) if the integral (2.33) may be calculated perturbatively then this integral is a compact form of the representation (2.32).

Notice that the requirements (i)–(iii) are the equivalent of the Euclidean postulate of Schwinger. In this frame one can consider another limit $t_{\mathrm{f}} \to t_{\mathrm{i}}$. Then the C_{NS} contour reduces to the Matsubara imaginary time contour, Fig. 4.

Later on we will use this S-matrix interpretation of thermodynamics. But one should keep in mind that corresponding results will hide assumptions (i)–(iii).

We would like to mention the ambivalent role of external particles in our S-matrix interpretation of thermodynamics. In the ordinary Matsubara formalism the temperature is measured assuming that the system under consideration is in equilibrium with the thermostat, i.e. temperature is the energy characteristics of *interacting* particles. In our definition the temperature is the mean energy of produced, i.e. non-interacting, particles. It can be shown that both definitions lead to the same result.

Explanation of this coincidence is the following. Let us consider the point of particle production as the coordinate of fictitious 'particle'. This 'particle' interacts since the connected contributions

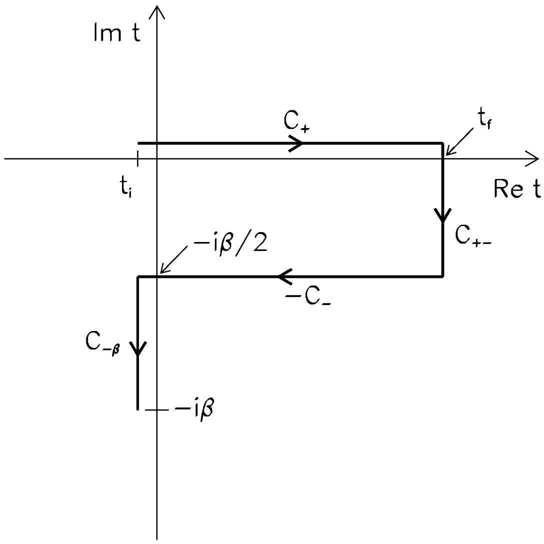

Fig. 3. Niemi–Semenoff time contour.

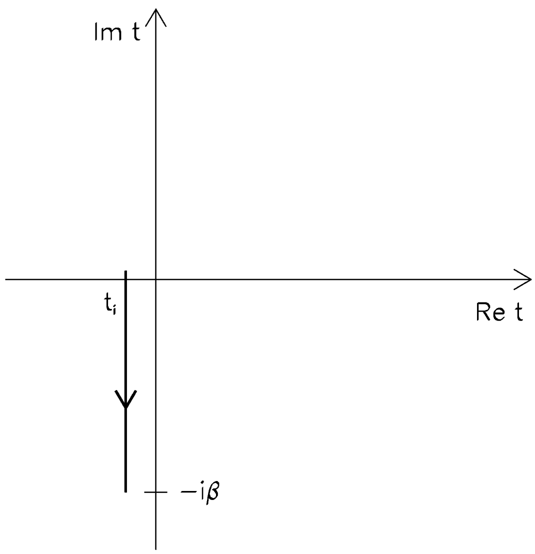

Fig. 4. Matsubara time contour.

into the amplitudes a_{nm} only are considered, and has a momentum equal to the produced particles momentum and so on. The set of these 'particles' forms a system. Interaction among these 'particles' may be described by the corresponding correlation functions, see Section 2.3.3.

Let us consider now the limit $t_f \to t_i$. In this limit (2.33) reduces to

$$\rho(\beta) = \int D_M \Phi e^{-S_M(\Phi)} , \tag{2.34}$$

where the imaginary time measure $D_M\Phi$ and action $S_M(\Phi)$ are defined on the Matsubara time contour. The periodic boundary condition (2.2) should be used for calculating integral (2.34). The rules and corresponding problems in the integral (2.34) can be calculated as described in many textbooks, see also [47].

In the limit considered the time was eliminated in the formalism and the integral in (2.34) performed over all the states of the 'particles' system with the weight $e^{-S_M(\Phi)}$. Notice that the doubling of degrees of freedom has disappeared and our fictitious 'particles' became real ones.

On the other hand, the produced particles may be considered as the probes through which we measure the interacting fields. As was mentioned above, their mean energy defines the temperature, if the energy correlations are relaxed. If even one of the conditions (i)–(iii) is not satisfied then one cannot reduce our S-matrix formalism to the imaginary time Matsubara theory. Then one can ask: is there any possibility, staying in the frame of S-matrix formalism, to conserve the statistics formalism. This question is discussed in Appendix C, where the Wigner functions approach is applied. It may be shown that the formalism may be generalized to describe the kinetic phase of the nonequilibrium process, where the temperature should have the local meaning [49]. The comparison with the 'local equilibrium hypothesis' is discussed at the end of Appendix C.

2.3. Classification of asymptotics over multiplicity

Our further consideration will be based on the model-independent (formal) classification of asymptotics [50].

2.3.1. 'Thermodynamical' limit

We will consider the generating function

$$T(s,z) = \sum_{n=1}^{n_{\max}} z^n \sigma_n(s),\ s = (p_1 + p_2)^2 \gg m^2,\ n_{\max} = \sqrt{s}/m \,. \tag{2.35}$$

This step is natural since the number of particles is not conserved in our problem. So, the total cross section and the averaged multiplicity will be

$$\sigma_{\mathrm{tot}}(s) = T(s,1) = \sum_n \sigma_n(s), \quad \bar{n}(s) = \sum_n n(\sigma_n(s)/\sigma_{\mathrm{tot}}) = \frac{\mathrm{d}}{\mathrm{d}z} \ln T(s,z)\bigg|_{z=1} \,. \tag{2.36}$$

At the same time, the inverse Mellin transform gives

$$\sigma_n = \frac{1}{n!}\frac{\partial^n}{\partial z^n} T(s,z)\bigg|_{z=0} = \frac{1}{2\pi \mathrm{i}}\oint \frac{\mathrm{d}z}{z^{n+1}} T(s,z) = \frac{1}{2\pi \mathrm{i}}\oint \frac{\mathrm{d}z}{z} \mathrm{e}^{(-n\ln z + \ln T(s,z))} \,. \tag{2.37}$$

The essential values of z in this integral are defined by the equation (of state)

$$n = z\frac{\partial}{\partial z} \ln T(z,s) \,. \tag{2.38}$$

Taking into account the definition of the mean multiplicity $\bar{n}(s)$, given in (2.36), we can conclude that the solution of (2.38) z_c is equal to one at $n = \bar{n}(s)$. Therefore, $z > 1$ is essential in the VHM domain.

The asymptotics over n ($n \ll n_{\max}$ is assumed) are governed by the smallest solution z_c of (2.38) because of the asymptotic estimation of the integral (2.37)

$$\sigma_n(s) \propto e^{-n \ln z_c(n,s)} . \tag{2.39}$$

Let us assume that in the VHM region and at high energies, $\sqrt{s} \to \infty$, there exists such a value of $z_c(n,s)$ that we can neglect in (2.35) the dependence on the upper boundary $n_{\max}$. This formal trick with the thermodynamical limit allows to consider $T(z,s)$ as the nontrivial function of z for finite s.

Then, it follows from (2.38) that

$$z_c(n,s) \to z_s \quad \text{at } n \in \text{VHM} , \tag{2.40}$$

where z_s is the leftmost singularity of $T(z,s)$ in the right-half plane of complex z. One can say that the singularity of $T(z,s)$ attracts $z_c(n,s)$ if $n \in$ VHM. We will put this observation in the basis of VHM processes phenomenology.

We would like to underline once more that actually $T(z,s)$ is regular for arbitrary finite z if s is finite. But $z_c(n,s)$ behaves in the VHM domain as if it is attracted by the (imaginary) singularity z_s. And just this $z_c(n,s)$ defines σ_n in the VHM domain. We want to note that actually the energy $\sqrt{s}$ should be high enough to use such an estimation.

2.3.2. Classes and their physical content

One can notice from estimation (2.39) that σ_n weakly depends on the character of the singularity. Therefore it is enough to classify only the possible positions of z_s. We may distinguish the following possibilities:

$$\text{(A) } z_s = \infty : \sigma_n < O(e^{-n}) ,$$

$$\text{(B) } z_s = 1 : \sigma_n > O(e^{-n}) ,$$

$$\text{(C) } 1 < z_s < \infty : \sigma_n = O(e^{-n}) , \tag{2.41}$$

i.e., following this classification, the cross section may decrease faster (A), slower (B), or as (C) an arbitrary power of e^{-n}. It is evident, if all these possibilities may be realized in nature, then we should expect the asymptotics (B).

As was explained in Section 2.2.1, σ_n has the meaning of the n particle partition function in the energy representation. Then $T(z,s)$ should be the 'big partition function'. Taking this interpretation into account, as follows from the Lee–Yang theorem [15], $T(z,s)$ cannot be singular at $|z| < 1$.

At the same time, the direct calculations based on the physically acceptable interaction potentials give the following restriction from above:

$$\text{(D) } \sigma_n < O(1/n) . \tag{2.42}$$

This means that σ_n should decrease faster than any power of $1/n$.

It should be noted that our classification predicts rough (asymptotic) behavior only and did not exclude local increase of the cross section σ_n.

One may notice that

$$-\frac{1}{n}\ln\frac{\sigma_n(s)}{\sigma_{\text{tot}}(s)} = \ln z_c(n,s) + O(1/n) . \tag{2.43}$$

Using thermodynamical terminology, the asymptotics of σ_n is governed by the physical value of the activity $z_c(n,s)$. One can introduce also the chemical potential $\mu_c(n,s)$. It defines the work needed for one particle creation, $\ln z_c(n,s) = \beta_c(n,s)\mu_c(n,s)$, where $\bar{\varepsilon}(n,s) = 1/\beta_c(n,s)$ is the produced particles mean energy. So, one may introduce the chemical potential if and only if $\beta_c(n,s)$ and $z_c(n,s)$ may be used as the 'rough' variables.

Then the above formulated classification has a natural explanation. So, (A) means that the system is stable with reference to particle production and the activity $z_c(n,s)$ is the increasing function of n, the asymptotics (B) may be realized if and only if the system is unstable. In this case $z_c(n,s)$ is the decreasing function. The asymptotics (C) is not realized in equilibrium thermodynamics [52].

We will show that the asymptotics (A) reflects the multiperipheral processes kinematics: created particles form jets moving in the CM frame with different velocities along the incoming particles directions, i.e. with restricted transverse momentum, see Section 3.1.1. The asymptotics (B) assumes condensation-like phenomena, see Section 3.3. The third-type asymptotics (C) is predicted by stationary Markovian processes with the pQCD jets kinematics, see Section 3.2.2. The DIS kinematics may be considered as the intermediate, see Section 3.2.1.

This interpretation of classes (2.41) allows to conclude that we should expect reorganization of production dynamics in the VHM region: the soft channel (A) of particle production should yield a place to the hard dynamics (C), if the ground state of the investigated system is stable with reference to the particle production. Otherwise we will have asymptotics (B).

2.3.3. Group decomposition

Let us consider the system with several correlation scales. For example, in statistics one should distinguish correlation length among particles (molecules) and correlation length among droplets if the two-phase region is considered. In particle physics, one should distinguish in this sense correlation among particles produced as a result of resonance decay and correlations among resonances. In pQCD one may distinguish correlations of particles in jet and correlation among jets.

There exist many model descriptions of this physical picture. In statistics Mayer's group decomposition [25] is well known. In particle physics one should note also the many-component formalism [2].[7] We will consider the generating functions (functionals) formalism [42] considering mostly jet correlations. In many respects it overlaps the above-mentioned approaches.

The generating function $T(z,s)$ may be written in the form

$$\ln T(z,s) = \sum_{k=1}^{\infty} \frac{(z-1)^k}{k!} C_k(s) = \sum_{l=1}^{\infty} z^l b_l \ , \tag{2.44}$$

where the coefficients C_k are the moments of the multiplicity distribution

$$P_n(s) = \sigma_n(s)/\sigma_{\text{tot}}(s) \ . \tag{2.45}$$

[7] The example considered in [48] illustrates this approach.

So,

$$C_1(s) = \sum_n nP_n(s) = \bar{n}(s), C_2(s) = \sum_n n(n-1)P_n(s) - \bar{n}(s)^2 \tag{2.46}$$

and so on. Using the connection with the inclusive distribution functions $f_k(q_1, q_2, \ldots, q_k)$

$$T(z,s) = \sum_{k=1}^{\infty} \frac{(z-1)^k}{k!} \int \mathrm{d}\Omega_k(q) f_k(q_1, q_2, \ldots, q_k; s) \,, \tag{2.47}$$

it is easy to find that

$$C_1(s) = \int \mathrm{d}\Omega_1(q) f_1(q; s) = \bar{f}_1(s) \,,$$

$$C_2(s) = \int \mathrm{d}\Omega_2(q) \{f_2(q_1, q_2; s) - f_1(q_1; s) f_1(q_2; s)\} = \bar{f}_2(s) - \bar{f}_1{}^2(s) \,, \tag{2.48}$$

etc. Generally,

$$\frac{1}{k!} C_k(s) = \sum_{l=1}^{\infty} \frac{(-1)^l}{l} \sum_{\{k\}_l=0}^{\infty} \delta\left(\sum_{i=1}^{l} k_i - k\right) \prod_{i=1}^{l} \left\{\frac{\bar{f}_{k_i}(s)}{k_i!}\right\} , \tag{2.49}$$

where $\{k\}_l = k_1, k_2, \ldots, k_l$ and

$$\bar{f}_k(s) = \int \mathrm{d}\Omega_k(q) f_k(q_1, q_2, \ldots, q_k; s) \,.$$

One may invert formulae (2.49):

$$\frac{1}{l!} \bar{f}_k(s) = \sum_{\{n_k\}=0}^{\infty} \delta\left(\sum_{k=1}^{\infty} k n_k - l\right) \prod_{k=1}^{\infty} \frac{1}{n_k!} \left(\frac{C_k(s)}{k!}\right)^{n_k} \,. \tag{2.50}$$

The Mayer group coefficients b_l in (2.44) have the following connection with C_k:

$$b_l(s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{l!k!} C_{k+l}(s) \,. \tag{2.51}$$

It seems useful to illustrate the effectiveness of the generating function method by the following example. We will consider the transformation (multiplicity $n \to$ activity z) to show the origin of the Koba–Nielsen–Olesen scaling (KNO-scaling).[8]

If $C_m = 0$, $m > 1$, then σ_n is described by the Poisson formulae:

$$\sigma_n(s) = \sigma_{\mathrm{tot}}(s) \mathrm{e}^{-\bar{n}} \frac{\bar{n}(s)^n}{n!} \,. \tag{2.52}$$

It corresponds to the case of absence of correlations.

[8] In a private discussion with one of the authors (A.S.) in the summer of 1973, Koba noted that the main reason for investigation leading to the KNO-scaling was just the generating functional method of Bogolyubov [49].

Let us consider a more weak assumption:

$$C_m(s) = \gamma_m (C_1(s))^m \ , \tag{2.53}$$

where γ_m is the energy-independent constant, see also [53], where a generalization of KNO scaling on the semi-inclusive processes was offered. Then

$$\ln T(z,s) = \sum_{m=1} \frac{\gamma_m}{m!} \{(z-1)\bar{n}(s)\}^m \ . \tag{2.54}$$

To find the consequences of this assumption, let us find the most probable values of z. The equation of state

$$n = z \frac{\partial}{\partial z} \ln T(z,s)$$

has solution $\bar{z}(n,s)$ increasing with n since $T(z,s)$ is an increasing function of z, if and only if, $T(z,s)$ is nonsingular at finite z. As was mentioned above, the last condition has deep physical meaning and practically assumes the absence of the first-order phase transition [51].

Let us introduce a new variable:

$$\lambda = (z-1)\bar{n}(s) \ . \tag{2.55}$$

The corresponding equation of state is as follows:

$$\frac{n}{\bar{n}(s)} = \left(1 + \frac{\lambda}{\bar{n}(s)}\right) \frac{\partial}{\partial \lambda} \ln T'(\lambda) \ . \tag{2.56}$$

So, with $\mathrm{O}(\lambda/\bar{n}(s))$ accuracy, one can assume that

$$\lambda \simeq \lambda_c(n/\bar{n}(s)) \ . \tag{2.57}$$

is essential. It follows from this estimation that such a scaling dependence is rightful at least in the neighborhood of $z=1$, i.e. in the vicinity of main contributions into σ_{tot}. This gives

$$\bar{n}(s)\sigma_n(s) = \sigma_{tot}(s)\psi(n/\bar{n}(s)) \ , \tag{2.58}$$

where

$$\psi(n/\bar{n}(s)) \simeq T(\lambda_c(n/\bar{n}(s))) \exp\{n/\bar{n}(s)\lambda_c(n/\bar{n}(s))\} \leq \mathrm{O}(e^{-n}) \tag{2.59}$$

is the unknown function. The asymptotic estimation follows from the fact that $\lambda_c = \lambda_c(n/\bar{n}(s))$ should be a nondecreasing function of n, as follows from the nonsingularity of $T(z,s)$.

Estimation (2.57) is right at least at $s \to \infty$. The range validity of n, where solution of (2.57) is acceptable, depends on the exact form of $T(z,s)$. Indeed, if $\ln T(z) \sim \exp\{\gamma\lambda(z)\}$, $\gamma = \text{const} > 0$, then (2.57) is right at all values of n and it is enough to have the condition $s \to \infty$. But if $\ln T(z,s) \sim (1 + a\lambda(z))^\gamma$, $\gamma = \text{const} > 0$, then (2.57) is acceptable if and only if $n \ll \bar{n}^2(s)$.

Representation (2.58) shows that just $\bar{n}(s)$ is the natural scale of multiplicity n [54]. This representation was offered first as a reaction on the so-called Feynman scaling for inclusive cross section

$$f_k(q_1, q_2, \ldots, q_k) \sim \prod_{i=1}^{k} \frac{1}{\varepsilon(q_i)} . \tag{2.60}$$

As follows from estimation (2.59), the limiting KNO prediction assumes that $\sigma_n = \mathrm{O}(\mathrm{e}^{-n})$. In this regime $T(z,s)$ should be singular at $z = z_c(s) > 1$. The normalization condition

$$\left.\frac{\partial T(z,s)}{\partial z}\right|_{z=1} = \bar{n}(s)$$

gives: $z_c(s) = 1 + \gamma/\bar{n}(s)$, where $\gamma > 0$ is the constant. Notice, such behavior of the big partition function $T(z,s)$ is natural for stationary Markovian processes described by logistic equations [55]. In the field theory such an equation describes the QCD jets [56].

We wish to generalize expansion (2.44) to take into account the possibility of many-component structure of the multiple production processes [2]. Let us consider particle production through the generation, for instance, of jets. In this case decay of a particle of high virtuality $|q| \gg m$ forms a jet of lower virtuality particles. It is evident that one should distinguish correlation among particles in the jet, and correlation among jets.

Let $\omega_{n_i}(m_i)$ be the *probability* that the ith jet of mass m_i includes n_i particles, $1 \le n_i \le n$, where

$$\sum_{i=1}^{N_j} n_i = n . \tag{2.61}$$

The jets are the result of particles decay. Then let us assume that $\bar{N}_1(m_i, p_i)$ defines the mean number of jets of mass m_i and momentum p_i:

$$\rho_n^{(1)}(\beta) = \sum_{N_j} \frac{1}{N_j} \sum_{\{n\}_{N_j}} \delta\left(\sum_{i=1}^{N_j} n_i - n\right) \int \prod_{i=1}^{N_j} \left\{ \frac{\mathrm{d}m_i}{2m_i} \frac{\mathrm{d}^3 p_i}{(2\pi)^3} \mathrm{e}^{-\beta\varepsilon(p_i)} \bar{N}_1(m_i, p_i) \omega_{n_i}(m_i) \right\} , \tag{2.62}$$

where $\{n\}_{N_j} = (n_1, n_2, \ldots, n_{N_j})$. Notice that the Boltzmann factor $\mathrm{e}^{-\beta\varepsilon}$, where $\varepsilon(p) = m + p^2/2m$ is the jets energy, plays the same role as the corresponding factor in (2.15) and is introduced to take into account the energy conservation law. We consider the VHM domain and for this reason $(p^2/2m) \ll 1$ is assumed.

It is useful to avoid the particles number conservation law (2.61). For this purpose we will introduce

$$\rho^{(1)}(\beta, z) = \sum_n \rho_n^{(1)}(\beta) = \exp\left\{ \int \frac{\mathrm{d}m}{2m} \frac{\mathrm{d}^3 p}{(2\pi)^3} \mathrm{e}^{-\beta\varepsilon(p)} \bar{N}_1(m, p)(t(z,m) - 1) \right\} , \tag{2.63}$$

where

$$t(z,m) = \sum_n z^n \omega_n(m) . \tag{2.64}$$

Comparing (2.63) with (2.44) we may conclude that $t(z,m)$ plays the role of activity of jets. Then the generalization is evident:

$$\rho(\beta,z) = \sum_k \rho^{(k)}(\beta,z)$$
$$= \exp\left\{\sum_k \int \prod_{i=1}^{k} \{\mathrm{d}m_i\, \mathrm{d}\Omega_1(p_i)\mathrm{e}^{-\beta\varepsilon(p_i)}(t(z,m_i)-1)\}\bar{N}_k(m_1,p_1,\ldots,m_k,p_k)\right\}, \tag{2.65}$$

where $\bar{N}_k$ has the same meaning as C_k, i.e. $\bar{N}_k$ is the correlation function of k jets.

2.3.4. Energy-multiplicity asymptotics equivalence

Let us consider the following 'bootstrap' regime when $\rho(\beta,z)$ is defined by the equation

$$\rho(\beta,z) \propto \int \frac{\mathrm{d}m}{2m}\frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{-\beta\varepsilon(p)}\bar{N}_1(m,p)t(z,m)\,. \tag{2.66}$$

Inserting here the strict expression (2.65) we find a nonlinear equation for $t(z,m)$.

The solution of (2.66) assumes that

$$\int \frac{\mathrm{d}m}{2m}\frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{-\beta\varepsilon(p)}\bar{N}_1 t \gg \left\{\int \frac{\mathrm{d}m}{2m}\frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{-\beta\varepsilon(p)}\bar{N}_1 t\right\}^2 \tag{2.67}$$

and

$$\int \frac{\mathrm{d}m}{2m}\frac{\mathrm{d}^3p}{(2\pi)^3}\mathrm{e}^{-\beta\varepsilon}\bar{N}_1 t \gg \int \left(\frac{\mathrm{d}m_1}{2m_1}\frac{\mathrm{d}^3p_1}{(2\pi)^3}\mathrm{e}^{-\beta\varepsilon}t\right)\left(\frac{\mathrm{d}m_2}{2m_2}\frac{\mathrm{d}^3p_2}{(2\pi)^3}\mathrm{e}^{-\beta\varepsilon}t\right)\bar{N}_2\,. \tag{2.68}$$

To solve Eq. (2.66) in the VHM region, where the leftmost singularity over z is important, let us consider the *anzats*

$$t(z,m) = \frac{\varphi(z,m)}{(1-(z-1)a(m))^{\kappa_0}}, \quad \kappa_0 > 0\,, \tag{2.69}$$

where $\varphi(z,m)$ is the polynomial function of z, $\varphi(z=1,m)=1$. Using the normalization condition

$$\bar{n}_j = \frac{\partial}{\partial z}t(z,m)\bigg|_{z=1}\,, \tag{2.70}$$

we can find

$$a(m)\kappa_0 = \bar{n}_j - \varphi'(1,m),\ \varphi'(1,m) \equiv \frac{\partial}{\partial z}\varphi(z,m)\bigg|_{z=1}\,. \tag{2.71}$$

The partition function of the jet $t(z,m)$ defined by *anzats* (2.69) is singular at

$$z_s(m) = 1 + \frac{1}{a(m)}\,. \tag{2.72}$$

This singularity would be significant in the VHM region if $z_s(m)$ is a decreasing function of m. This means an assumption that

$$\frac{\varphi'(1,m)}{\bar{n}_j(m)} \to 0 \quad \text{at } m \to \infty \,.$$

So, in first approximation we will choose

$$a(m) = \bar{n}_j(m) \,. \tag{2.73}$$

This choice may be confirmed by concrete model calculations.

Taking into account the energy conservation law, conditions (2.67) and (2.68) are satisfied if

$$\exp\left\{ -n \frac{\bar{n}_j(s) - \bar{n}_j(s/4)}{\bar{n}_j(s)\bar{n}_j(s/4)} \right\} \ll 1 \tag{2.74}$$

at $n \in$ VHM. Therefore, (2.69) obey Eq. (2.66) with exponential accuracy in the VHM region, i.e. if

$$n \gg \frac{\bar{n}_j(s)\bar{n}_j(s/4)}{\bar{n}_j(s) - \bar{n}_j(s/4)} = \frac{\kappa_0}{z_s(s/4) - z_s(s)} \,. \tag{2.75}$$

We assume here that one can find so large n and s that with exponential accuracy the factors $\sim \bar{N}_k$ do not play an important role. But at low energies condition (1.2) is important and the factors $\sim \bar{N}_k$ should be taken into account.

Notice now an important consequence of our 'bootstrap' solution: it means that we can leave production of the heavy jets only, if $n \in$ VHM. On the other hand, let us choose $n = z_0 \bar{n}_j(s)$, where $z_0 > 1$ is the function of s, and consider $s \to \infty$. Then condition (2.75) defines z_0: if

$$z_0 \gg \frac{\bar{n}_j(s/4)}{\bar{n}_j(s) - \bar{n}_j(s/4)} \,, \tag{2.76}$$

then we are able to obey inequalities (2.67) and (2.68).

The jet mean multiplicity, see Section 3.2.2, is

$$\ln \bar{n}_j(s) \sim \sqrt{\ln s} \,. \tag{2.77}$$

Then

$$\frac{\bar{n}_j(s/4)}{\bar{n}_j(s) - \bar{n}_j(s/4)} = \{e^{\gamma/\sqrt{\ln s}} - 1\}^{-1} \sim \sqrt{\ln s} \ll \bar{n}_j(s) \tag{2.78}$$

at $s \to \infty$. Therefore, (2.76) may be satisfied outside the VHM domain.

Let us compare now the solutions of the equation of state. Inserting (2.69) into (2.38) we can find for a jet of mass $\sqrt{s}$ that

$$z_c^1 = z_s^1 - \frac{\kappa_0}{n} = 1 + \frac{1}{\bar{n}_j(s)} - \frac{\kappa_0}{n} \,. \tag{2.79}$$

The two-jet contribution of the masses $\sim \sqrt{s}/2$ gives

$$z_c^2 = z_s^2 - \frac{\kappa_0}{n} = 1 + \frac{1}{\bar{n}_j(s/4)} - \frac{\kappa_0}{n} \,. \tag{2.80}$$

At arbitrary finite energies $(z_c^2 - z_c^1) > 0$ and, as follows from (2.78), they decrease $\sim (1/\bar{n}_j(s)\sqrt{\ln s})$ with energy.

Noting the normalization condition, $T(z=1,s) = \sigma_{\rm tot}(s)$, and assuming that the vacuum is stable, i.e. $\sigma_n \leq O(e^{-n})$, we can conclude that

– if $n \in$ VHM then z_s attracts z_c, i.e. $z_s \to z_c$, and if $z_c - 1 \ll 1$ then these contributions should be significant in $\sigma_{\rm tot}$;

– if $s \to \infty$, then $z_c - 1 \ll 1$, and if n satisfies inequality (2.75), or if z_0 satisfies inequality (2.76), then the considered contributions are significant in $\sigma_{\rm tot}$.

It is the (energy-multiplicity) asymptotics equivalence principle. One of the simplest consequences of this principle is the prediction that the mean transverse momentum of created particles should increase with multiplicity at sufficiently high energies.

This principle is the consequence of independence of contributions in the VHM domain on the type of singularity in the complex z plane and of the energy conservation law. Just the last one shifts the two-jet singularity to the right and $z_c(s) < z_c(s/4)$.

We would like to mention also that this effect, when the mostly 'energetic population' survives has been described mathematically by Volterra [55]. It is intuitively evident that one may find the 'energetic population' when searching for the VHM one, or, it is the same, giving it a rich supply, i.e. to give the population enough energy. This is our (energy-multiplicity) equivalence ($(\varepsilon - n)$-equivalence) principle.

Notice, if the amount of supply is too high then few populations may grow. This is the case when the difference $(z_c^2 - z_c^1) > 0$ tends to zero at high energies.

We would like to note that singular at finite z partition functions was predicted in the $(\lambda\phi^3)_6$-theory [57], in QCD jets [56], in the generalized Bose–Einstein distribution model [58]. In all of these models decay of the essentially nonequilibrium initial state (highly virtual parton, heavy resonance, etc.) was described.

One may distinguish the phases of the media by a characteristic correlation length. Then the phase transition may be considered as the process of changing correlation length. Our 'bootstrap' solution predicts just such phenomena: at low multiplicities the long-range correlations among light jets are dominant. The 'bootstrap' solution predicts that for VHM processes just the short-range correlations among particles of the heavy jet become dominant. The $(\varepsilon - n)$-equivalence means that this transition is a pure dynamical effect.

3. Model predictions

Multiple production phenomena were first observed more than seventy years ago [59]. During this time vast experimental information was accumulated concerning hadron inelastic interactions, see the review papers [10].

Now we know that at high energy $\sqrt{s}$:

(i) The total cross section $\sigma_{tot}(s)$ of hadron interactions is enhanced almost completely by the inelastic channels;
(ii) The mean multiplicity of produced hadrons $\bar{n}(s)$ slowly (logarithmically) grows with $\sqrt{s}$;
(iii) The interaction radii of hadrons $\bar{b}$ slowly (logarithmically) increase with energy;
(iv) The multiplicity distribution $\sigma_n(s)$ is wider than the Poisson distribution;
(v) The mean value of transverse momentum k of produced hadrons is restricted and is independent of the incident energy $\sqrt{s}$ and produced particle multiplicity n;
(vi) The one-particle energy spectrum $d\sigma \sim d\varepsilon/\varepsilon$.

First of all, (i) means that the high-energy hadron interaction may be considered as an ordinary dissipation process. In this process the kinetic energy of incident particles is spent in produced particle mass formation.

The VHM process takes place in vacuum and then it is assumed in the early stages that the multiple production phenomena reflect a natural tendency of the excited hadron system to get to equilibrium with the environment [1]. In this way one can introduce as a first approximation the model that the excited hadron system evolves without any restrictions. In this model we should have $\bar{n}(s) \sim \sqrt{s}$. The dissipation is maximal in this case and the entropy $\mathscr{S}$ exceeds its maximum. This simple model has definite popularity up to 70-th. But the experimental data (ii) and (iii) prohibit this model and it was forgotten.

Choosing the model we would like to hope that the considered model

- takes into account experimental conditions (i)–(vi) in the $n \sim \bar{n}(s)$ domain;
- has natural asymptotics over multiplicity to the VHM region.

It is necessary to remember also that

- New channels of hadron production may arise in the VHM region.

It is impossible to understand all possibilities without those offered in Section 2.3.2 in the classification of asymptotics.

Thus, we will observe predictions of

- *Multiperipheral models*, distinguishing the soft Pomeron models, see [60].
- *The dual-resonance model* predictions for the VHM region are described also. It can be shown that this models predict asymptotics (A) if $n \gg \bar{n}(s)^2$. Just this result explains why VHM domain is defined by condition (2.1).

We are forced since it allows to include pQCD, forbidden by the multiperipheral models, to consider

- *Hard Pomeron model* production of mini jets. But we will find using Monte Carlo simulations that the pQCD Pomeron is unable to adopt the hard channels of hadron production. Then
- *The deep inelastic processes* for VHM region will be considered to generalize the DGLAP kinematics in the case of heavy QCD jets production. The analysis shows that transition to the

VHM leads to the necessity to include low-x sub-processes. As a result we get out of the range of pQCD validity.

- *Multiple reduction of jets.* We will see, that at very high energies in the VHM region the heavy jets creation should be a dominant process if the vacuum is stable with reference to the particle production.

We will consider also decay of the 'false vacuum' to describe the consequence of

- *Phase transition* in the VHM domain. This channel is hardly seen for $n \sim \bar{n}(s)$ since the confinement constraints may prevent cooling of the system up to phase transitions condition.

3.1. Peripheral interaction

3.1.1. Multiperipheral phenomenology

Later on multiple production physics was developed on the basis of experimental observation (iv). The Regge pole model naturally explains these experimental data and, at the same time, absorbs all experimental information, (i)–(vi). At the very beginning, adopting the Regge poles notion without its microscopical explanation, this description was self-consistent. The efforts to extend the Regge pole model to the relativistic hadron reactions was ended by the Reggeon diagram technique, see [61] and references cited therein, and it was used later to construct the perturbation theory for σ_n [62]. It was shown that the multiplicity distribution is wider than the Poissonian one because of the multi-Pomeron exchanges.

The leading energy asymptotics Pomeron contribution reflects the created particle kinematics described in Appendix D, where the available kinematical scenario in the frame of pQCD is described. So, the longitudinal momentum of produced particles is large and is strictly ordered. At the same time, particles transverse momentum is restricted.

Let us consider the inelasticity coefficient introduced in (1.3) $\kappa = 1 - \varepsilon_{\max}/E < 1$, where $\varepsilon_{\max}$ is the energy of the fastest particle in the laboratory frame. Then the strict ordering of particles in the Pomeron kinematics means that κ is independent of the index of the particle. So, if the fastest particle has the energy $\varepsilon_{\max} \simeq (1-\kappa)\sqrt{s}$, then the following particle should have the energy $\varepsilon_1 \simeq (1-\kappa)\varepsilon_{\max} \simeq (1-\kappa)^2\sqrt{s}$, and so on. Following this law, the $(n-1)$th particle would have the energy $\varepsilon_n \simeq (1-\kappa)^n\sqrt{s}$. In the laboratory frame the energy should degrade to $\varepsilon_n \simeq m$. Inserting here the above-formulated estimation of ε_n we can find that if the number of produced particles is

$$\bar{n}(s) \simeq n_0\xi,\ n_0 = -\ln(1-\kappa)^2 > 0,\ \xi = \ln(s/m^2)\,, \tag{3.1}$$

then we may expect the total degradation of energy. This degradation is the necessary condition noting that the total cross section of slowly moving particles may depend only slightly on energy and it seems necessary for natural explanation of the weak dependence of the hadron cross sections on the energy. This consideration would be Lorentz covariant if one can find the slowly moving particle in an arbitrary frame. The resulting estimation of mean multiplicity has good qualitative experimental confirmation.

Notice that it was assumed when deriving (3.1) that the energy degrades step by step. In other words, if we introduce a time of degradation, then the time $\sim \xi$ is needed for complete degradation of energy. Assuming the random walk in the normal to incident particle plane, we can conclude that the points of particle production are located on a disk (in the moving frame) of radii $\bar{b} \sim \xi^{1/2}$. This means that the interaction radii should grow with the energy of the colliding particles.

If $f(a+b \to c+\cdots)$ is the cross section to observe particle c inclusively in the a and b particles collision, then it was found experimentally that the ratio

$$\frac{f(\pi^+ p \to \pi^- + \cdots)}{\sigma_{\mathrm{tot}}(\pi^+ p)} = \frac{f(K^+ p \to \pi^- + \cdots)}{\sigma_{\mathrm{tot}}(K^+ p)} = \frac{f(pp \to \pi^- + \cdots)}{\sigma_{\mathrm{tot}}(pp)} \tag{3.2}$$

is universal. This may be interpreted as the direct evidence of the fact that the hadron interactions have a large-distance character, i.e. that the interaction radii should be large.

This picture assumes that the probability to have total degradation of energy is

$$\sim \mathrm{e}^{-b^2/4\alpha'\xi} , \tag{3.3}$$

where α' is some dimensional constant (the slope of Regge trajectory) and $\boldsymbol{b}$ is the two-dimensional impact parameter. This formula has also the explanation connected to the vacuum instability with reference to the real particle production in the strong color electric field.

The above picture has natural restrictions. We can assume that each of the produced particles may be the source of above-described t-channel cascade of the energy degradation. This means that in the frame of the Pomeron phenomenology, we are able to describe the production of

$$n < \bar{n}(s)^2 \tag{3.4}$$

particles only. If $n > \bar{n}(s)^2$ then the density of particles in the diffraction disk becomes large and (a) one should introduce short-distance interactions, or (b) rise interaction radii. It will be shown that just (a) is preferable.

We will build the perturbation theory in the phenomenological frames (i)–(vi). Considering the system with variable number of particles the generating function

$$T(z,s) = \sum_n z^n \sigma_n(s) , \tag{3.5}$$

would be useful. One can use also the decomposition:

$$T(z,s) = \sigma_{\mathrm{tot}}(s) \exp\{(z-1)C_1(s) + \tfrac{1}{2}(z-1)^2 C_2(s) + \cdots\} , \tag{3.6}$$

where, by definition,

$$C_1(s) = \bar{n}(s) = \frac{\partial}{\partial z} \ln T(z,s) \bigg|_{z=1} \tag{3.7}$$

is the mean multiplicity,

$$C_2(s) = \frac{\partial^2}{\partial z^2} \ln T(z,s) \bigg|_{z=1} \tag{3.8}$$

is the second binomial momentum, and so on.

Our idea is to assume that all C_m, $m > 1$ may be calculated perturbatively choosing

$$P(0,s) = e^{(z-1)\bar{n}(s)} \tag{3.9}$$

as the Born approximation 'superpropagator'. It is evident that (3.9) leads to Poisson distribution. Then, having in mind (ii), (iii) and (v) we will use the following *anzats*:

$$P(q,s) = e^{\alpha(o) - \alpha' q^2 \ln s} e^{(z-1)\xi} , \tag{3.10}$$

where the transverse momentum $\boldsymbol{q}$ is conjugate to the impact parameter $\boldsymbol{b}$. So, the Born term (3.10) is a Fourier transform of the simple product of (3.9) and (3.3). It contains only one free parameter, the Pomeron intercept $\alpha(0)$. On the phenomenological level it is not important to know the dynamical (microscopical) origin of (3.10).

For our purpose the Laplace transform of P would be useful. If $\bar{n}(s) = n_0 \ln s$, then

$$\mathscr{P}(\omega, q^2) = \int_0^\infty d\xi e^{-\omega\xi} P(q^2, s) = \frac{1}{\omega + \alpha' q^2 + \psi_0(z)} . \tag{3.11}$$

It is the propagator of two-dimensional field theory with mass squared

$$\psi_0(z) = (1-\alpha) + (1-z)n_0,\ n_0 > 0 .$$

Knowing Gribov's Reggeon calculus completed by the Abramovski–Gribov–Kancheli (AGK) cutting rules [63] one can investigate the consequences of this approach.

The LLA approximation of the pQCD [19] gives

$$\Delta = \alpha(0) - 1 = \frac{12 \ln 2}{\pi} \alpha_s \approx 0.55,\ \alpha_s = 0.2 , \tag{3.12}$$

but radiative corrections give $\Delta \approx 0.2$ [64]. We will call this solution as the BFKL model.

The quantitative origin of the restriction (3.4) is the following. The contribution of the diagram with v Pomeron exchange gives, since the diffraction radii increase with s, see (3.7), mean value of the impact parameter decreasing with v:

$$\bar{\boldsymbol{b}}^2 \simeq 4\alpha' \ln(s/m^2)/v = a\alpha' \frac{\bar{n}(s)}{v} ,$$

where $a = 4/n_0$. On the other hand, the number of necessary Pomeron exchanges $v \sim n/\bar{n}(s)$ since one Pomeron gives maximal contribution (with factorial accuracy) at $n \simeq \bar{n}(s)$. As a result,

$$\bar{\boldsymbol{b}}^2 \sim a\alpha' \frac{\bar{n}(s)^2}{n} . \tag{3.13}$$

Therefore, if the transverse momentum of created particles is a restricted quantity, i.e. $\mu_0^2 \bar{\boldsymbol{b}}^2 \sim 1$, where μ_0 is a constant, then the mechanism of particle production is valid up to

$$n \sim \bar{n}(s)^2 . \tag{3.14}$$

Following our general idea, it will be enough for us to find the position of singularity over z. Analysis shows that (3.12) predict the singularity at infinity.

In Appendix E Gribov's Reggeon diagram technique with cut Pomerons is described and (3.14) is derived. It can be shown using this technique that the model with the critical Pomeron, $\Delta = 0$, is inconsistent from the physical point of view [62].

As was mentioned above, the model with $\Delta = \alpha(0) - 1 > 0$ is natural for the pQCD. The concrete value of Δ will not be important for us. We will assume only that

$$0 < \Delta \ll 1 . \tag{3.15}$$

It is evident that the Born approximation (3.10) with $\Delta > 0$ violates the Froissart boundary condition. But it can be shown that the sum of 'eikonal' diagrams[9] solves this problem, see [66] and references cited therein.

The interaction radii may increase with increasing number of produced particles if $\Delta > 0$ and then the restriction (3.14) is not important. In the used eikonal approximation, see Appendix F,

$$z \simeq z_c = 1 + \frac{1}{\bar{n}(s)}\ln\frac{n}{\bar{n}(s)} \tag{3.16}$$

is essential. Then the interaction radii $\bar{\boldsymbol{b}}^2 \sim B^2 \simeq 4\alpha'\xi(\Delta\xi + \ln(n/\bar{n}(s)))$ for these values of z. Note that

$$B^2 \sim \xi^2 \gg \xi , \tag{3.17}$$

even for $n \sim n_{\max} \sim \sqrt{s}$.

Nevertheless, using (3.16) one can find that the cross section decreases faster than any power of e^{-n}:

$$-\ln\left(\frac{\sigma_n(s)}{\sigma_{\mathrm{tot}}(s)}\right) = \frac{n}{\bar{n}(s)}\ln\frac{n}{\bar{n}(s)}(1 + O(\bar{n}(s)/n)) . \tag{3.18}$$

Generally speaking, although this estimation is right in the VHM region, there may be large corrections because of Pomeron self-interactions. But careful analyses show [62] that these contributions cannot drastically change estimation (3.18).

3.1.2. Dual resonance model

The search of dynamical source of the Regge description shows the different dynamical nature of the Regge and Pomeron poles. The established resonance-Regge pole duality, e.g. [67] led to the Veneziano representation of the Regge amplitudes [68]. The Reggeon pole gives the decreasing $\sim s^{-1/2}$ contribution, but careful investigation shows that the mass spectrum of dual to Regge pole resonances increases exponentially. This prediction was confirmed by experiment, see the discussion of this topic in [69].

The field theory development is marked by considerable efforts to avoid the problem of color charge confinement. Notice that the classical string has the same excitation spectrum. A remarkable attempt in this direction is based on the string model, in its various realizations, see, e.g. [70].

[9] The eikonal approximation in a quantum field theory has been developed in [65].

But, in spite of remarkable success (in formalism especially) there is no experimentally measurable prediction of this approach till now, e.g. [71].

We would like to describe in this section production of 'stable' hadrons through decay of resonances [72,73].

Our consideration will use the following assumptions.

A. The string interpretation of the dual-resonance model indicates that the mass spectrum of resonances, i.e. the total number $\rho(m)$ of mass m resonance excitations, grows exponentially

$$\rho(m) = (m/m_0)^{\gamma} e^{\beta_0 m}, \quad \beta_0 = \text{const}, \; m > m_0 \; . \tag{3.19}$$

Note also that the same hadron mass spectrum (3.19) was predicted in the 'bootstrap' approach [69,74]. Moreover, it predicts that

$$\gamma = -\frac{5}{2} \; . \tag{3.20}$$

B. The mass m resonance creation cross section $\sigma^{R}(m)$ has the Regge pole asymptotics

$$\sigma^{R}(m) = g^{R}\frac{m_0}{m}, \; g^{R} = \text{const} \; . \tag{3.21}$$

It has been assumed here that the intercept of the Regge pole trajectory $\alpha^{R} = \frac{1}{2}$. So, only the meson resonances would be taken into account.

C. If $\sigma_n^{R}(m)$ describes the decay of a mass m resonance into the n hadrons, then the mean multiplicity of hadrons

$$\bar{n}^{R}(m) = \frac{\sum_n n\sigma_n^{R}(m)}{\sigma^{R}(m)} \; . \tag{3.22}$$

Following the Regge model,

$$\bar{n}^{R}(m) = \bar{n}_0^{R} \ln\frac{m^2}{m_0^2} \; . \tag{3.23}$$

D. We will assume that there is a definite vicinity of $\bar{n}^{R}(m)$ where $\sigma_n^{R}(m)$ is defined by $\bar{n}^{R}(m)$ only. So, in this vicinity

$$\sigma_n^{R}(m) = \sigma^{R}(m) e^{-\bar{n}^{R}(m)} (\bar{n}^{R}(m))^n / n! \; . \tag{3.24}$$

This is the direct consequence of the Regge pole model, if m/m_0 is high enough.

Following our idea, we will distinguish the 'short-range' correlations among hadrons and the 'long-range' correlations among resonances. The 'connected groups' would be described by resonances and the interactions among them should be described, introducing for this purpose the correlation functions among strings. So, we will consider the 'two-level' model of hadron creation: the first level describes the short-range correlation among hadrons and the second level is connected to the correlations among resonances.

The exact calculations are given in Appendix G.

Comparing A and B solutions we can see the change of attraction points with rising n: at $n \simeq \bar{n}^2(s) = \bar{n}_0^{R} \ln(\sqrt{s}/m_0)$ the transition from (**A**) asymptotics to (**C**) in (2.41) should be seen.

At the same time one should see the strong KNO scaling violation at the tail of the multiplicity distribution.

We have neglected the resonance interactions when deriving these results. This assumption seems natural since at $\bar{n}(s) \ll n < \bar{n}^2(s)$ inequality (G.17) should be satisfied, see the discussion of inequalities (2.67) and (2.68) in Section 2.3.4.

3.2. Hard processes

3.2.1. Deep inelastic processes

The role of soft color partons in the high-energy hadron interactions is the most intriguing modern problem of particle physics. So, the collective phenomena and symmetry breaking in the nonAbelian gauge theories, confinement of colored charges and the infrared divergences of the pQCD are the phenomena just of the soft color particles domain.

It seems natural that the very high multiplicity (VHM) hadron interaction, where the energy of the created particles is small, should be sensitive to the soft color particle densities. Indeed, the aim of this section is to show that even in the hard-by-definition deep inelastic scattering (DIS), see also [21], the soft color particles role becomes important in the VHM region [75].

To describe the hadron production in pQCD terms the parton–hadron duality is assumed. This is natural just for the VHM process kinematics: because of the energy–momentum conservation law, produced (final-state) partons cannot have high relative momentum and, if they were created at small distances, production of $q\bar{q}$ pairs from the vacuum will be negligible (or did not play an important role). Therefore, if the 'vacuum' channel is negligible, only the pQCD contributions should be considered [50,76]. All this means that the multiplicity, momentum, etc. distributions of hadron and colored partons are the same. (This reduces the problem practically to the level of QED.)

Let us consider now n particles (gluons) creation in the DIS [77]. We would like to calculate $D_{ab}(x, q^2; n)$, where

$$\sum_n D_{ab}(x, q^2; n) = D_{ab}(x, q^2) . \tag{3.1}$$

As usual, let $D_{ab}(x, q^2)$ be the probability to find parton b with virtuality $q^2 < 0$ in the parton a of $\sim \lambda$ virtuality, $\lambda \gg \Lambda$ and $\alpha_s(\lambda) \ll 1$. We may always choose q^2 and x so that the leading logarithm approximation (LLA) will be acceptable. One should assume also that $(1/x) \gg 1$ to have the phase space, into which the particles are produced, sufficiently large.

Then $D_{ab}(x, q^2)$ is described by ladder diagrams. From a qualitative point of view this means the approximation of random walk over coordinate $\ln(1/x)$ and the time is $\ln\ln|q^2|$. LLA means that the 'mobility' $\sim \ln(1/x)/\ln\ln|q^2|$ should be large

$$\ln(1/x) \gg \ln\ln|q^2/\lambda^2| . \tag{3.2}$$

But, on the other hand [78],

$$\ln(1/x) \ll \ln|q^2/\lambda^2| . \tag{3.3}$$

See also Appendix D.

The leading contributions, able to compensate the smallness of

$$\alpha_s(\lambda) \ll 1 ,$$

give integration over a wide range $\lambda^2 \ll k_i^2 \ll -q^2$, where $k_i^2 > 0$ is the 'mass' of a real, i.e. time-like, gluon. If the time needed to capture the parton into the hadron is $\sim (1/\Lambda)$ then the gluon should decay if $k_i^2 \gg \lambda^2$. This leads to the creation of (mini)jets. The mean multiplicity $\bar{n}_j$ in the QCD jets is high if the gluon 'mass' $|k|$ is high: $\ln \bar{n}_j \simeq \sqrt{\ln(k^2/\lambda^2)}$.

Raising the multiplicity may (i) raise the number of (mini)jets ν and/or (ii) raise the mean value mass of (mini)jets $|\bar{k}_i|$. We will see that the mechanism (ii) would be favorable.

But increasing the mean value of gluon masses, $|k_i|$, the range of integrability over k_i decreases, i.e. violates the condition (3.2) for fixed x. One can retain the LLA taking $x \to 0$. But this may contradict (3.3), i.e. in any case the LLA becomes invalid in the VHM domain and the next to leading order corrections should be taken into account.

Note that the LLA gives the main contribution, that the rising multiplicity leads to the infrared domain, where the soft gluon creation becomes dominant.

First of all, neglecting the vacuum effects, we introduce definite uncertainty to the formalism. It is reasonable to define the level of strictness of our computations. Let us introduce for this purpose the generating function $T_{ab}(x, q^2; z)$:

$$D_{ab}(x, q^2; n) = \frac{1}{2\pi i} \oint \frac{dz}{z^{n+1}} T_{ab}(x, q^2; z) . \tag{3.4}$$

At large n, the integral may be calculated by the saddle point method. The smallest solution z_c of the equation

$$n = z \frac{\partial}{\partial z} \ln T_{ab}(x, q^2; z) \tag{3.5}$$

defines the asymptotic over n behavior

$$D_{ab}(x, q^2; n) \propto \exp\{-n \ln z_c(x, q^2; n)\} . \tag{3.6}$$

Using the statistical interpretation of z_c as the fugacity it is natural to write

$$\ln z_c(x, q^2; n) = \frac{C_{ab}(x, q^2; n)}{\bar{n}_{ab}(x, q^2)} . \tag{3.7}$$

Notice that the solution of Eq. (3.5) $z_c(x, q^2; n)$ should be an increasing function of n. At first glance this follows from the positivity of all $D_{ab}(x, q^2; n)$. But actually this assumes that $T_{ab}(x, q^2; z)$ is a regular function of z at $z = 1$. This is a natural assumption considering just the pQCD predictions.

Therefore,

$$D_{ab}(x, q^2; n) \propto \exp\left\{ -\frac{n}{\bar{n}_{ab}(x, q^2)} C_{ab}(x, q^2; n) \right\} . \tag{3.8}$$

This form of $D_{ab}(x, q^2; n)$ is useful since usually $C_{ab}(x, q^2; n)$ is a slowly varying function of n. So, for a Poisson distribution $C_{ab}(x, q^2; n) \sim \ln n$. For KNO scaling we have $C_{ab}(x, q^2; n) = \text{const.}$ over n.

We would like to note that, neglecting effects of vacuum polarization, we introduce into the exponent such high uncertainty assuming $n \simeq n_p$ that it is reasonable to perform the calculations with exponential accuracy. So, we would calculate

$$-\bar{\mu}_{ab}(x,q^2;n) = \ln\frac{D_{ab}(x,q^2;n)}{D_{ab}(x,q^2)} = \frac{n}{\bar{n}_{ab}(x,q^2)}C_{ab}(x,q^2;n)(1+\mathrm{O}(1/n)) \ . \tag{3.9}$$

The n dependence of $C_{ab}(x,q^2;n)$ defines the asymptotic behavior of $\bar{\mu}_{ab}(x,q^2;n)$ and calculation of its explicit form would be our aim.

We can conclude, see Appendix H, that our LLA is applicable in the VHM domain till

$$\omega(\tau,z) \ll \ln(1/x) \ll \tau = \ln(-q^2/\lambda) \ , \tag{3.10}$$

where

$$\omega(\tau,z) = \int_{\tau_0}^{\tau} \frac{\mathrm{d}\tau'}{\tau'} w^g(\tau',z) \ . \tag{3.11}$$

and

$$w^g(\tau,z) = \sum_n z^n w_n^g(\tau) \tag{3.12}$$

is the generating function of the multiplicity distribution in a gluon jet. In the frame of constraints (3.10),

$$F^{ab}(q^2,x;w) \propto \exp\{4\sqrt{N\omega(\tau,z)\ln(1/x)}\} \ . \tag{3.13}$$

The mean multiplicity of gluons created in the DIS kinematics

$$\bar{n}_g(\tau,x) = \frac{\partial}{\partial z}\ln F^{ab}(q^2,x;w)|_{z=1} = \omega_1(\tau)\sqrt{4N\ln(1/x)/\ln\tau} \gg \omega_1(\tau) \ , \tag{3.14}$$

where

$$\omega_1(\tau) = \int_{\tau_0}^{\tau} \frac{\mathrm{d}\tau_1}{\tau_1}\bar{n}_j(\tau) \tag{3.15}$$

and the mean gluon multiplicity in the jet $\bar{n}_j(\tau)$ has the following estimation [79]:

$$\ln\bar{n}_j(\tau) \simeq \sqrt{\tau} \ . \tag{3.16}$$

Inserting (3.16) into (3.15),

$$\omega_1(\tau) = \bar{n}_j(\tau)/\sqrt{\tau} \ .$$

Therefore, noting (3.3),

$$\bar{n}_g(\tau,x) \simeq \bar{n}_j(\tau)\sqrt{4N\ln(1/x)/\tau\ln\tau} \ll \bar{n}_j(\tau) \ . \tag{3.17}$$

This means that the considered 't-channel' ladder is important in the narrow domain of multiplicities

$$n \sim \bar{n}_g \ll \bar{n}_j \ . \tag{3.18}$$

So, in the VHM domain $n \gg \bar{n}_g$ one should

(i) Consider the ladder diagrams with a small number of rungs;
(ii) Take into account the multi-jet correlations assuming that increasing multiplicity leads to the increasing number of rungs in the ladder diagram.

To choose one of these possibilities one should consider the structure of $\omega(\tau, z)$ much more carefully. This will be done in Section 3.5.

We can conclude that in the VHM domain, multiplicity production unavoidably destroys the ladder LLA. To conserve this leading approximation one should choose $x \to 0$ and, in result, to get to the multi-ladder diagrams, since in this case $\alpha_s \ln(-q^2/\lambda^2) \sim 1$ and $\alpha_s \ln(1/x) \sim 1$. Such theory was considered in [80].

3.2.2. QCD jets

As was mentioned above, the pQCD description is right if the color particles virtuality is bounded from below, $|q^2| \geq la^2$, where λ is chosen so that $\alpha_s(\lambda^2) \ll 1$. This kinematical restriction leads to the infrared cutoff [81,82] and may essentially influence the particle production in the VHM region. It is a special property of pQCD. Indeed, for example, careful investigation of this question in the asymptotically free $(\varphi^3)_6$-theory [83] shows that this restriction is 'unobservable' since their inclusion takes us beyond the LLA [84]. At the same time, the condition $|q^2| \geq la^2$ essentially shrinks the phase space where particles are produced.

Particle (gluons) distribution in pQCD jets was investigated firstly in [81,85] and it was shown that the generating function is singular at $z_s - 1 \sim (1/\bar{n})$. Let us consider this solution stable with reference to the discussed cutoff.

The explicit formulae for one jet production may be written in the form, see Appendix I:

$$\sigma_n^{(1j)}(M) = a^{(1j)}(M, n) e^{-c_j n/\bar{n}_j(M)}, \quad n \geq \bar{n}_j(M) \ , \tag{3.19}$$

where $a^{(1j)}(n, M)$ is the polynomial function of n, $\bar{n}_j(M)$ is the mean multiplicity in the mass M jet and c_j is a positive constant.

The linear behavior of the exponent in (I.4) over $n/\bar{n}$ has important consequences. So, let us assume that the total energy M is divided into two jets of masses M_1 and M_2 equally: $M_1 = M_2 = M/2$. If, for instance, $M_2 \ll M_1 \simeq M$ then the distribution will coincide with (3.19), but the second jet distribution would renormalize the coefficient $a^{(1j)}$.

Then the multiplicity distribution in the two-jet event would be

$$\sigma_n^{(2j)}(M) = a^{(2j)}(M, n) e^{-c_j n/\bar{n}_j(M/2)} \ , \tag{3.20}$$

where $n_1 + n_2 = n$ is the total multiplicity.

Comparing (3.19) with (3.20) we can see that with exponential accuracy

$$\sim \exp\left\{ -c_j \frac{\bar{n}_j(M) - \bar{n}_j(M/2)}{\bar{n}_j(M)\bar{n}_j(M/2)} n \right\} ,$$

(3.19) would dominate in the VHM domain since the mean multiplicity $\bar{n}_j(M)$ increases with M.

The experimental observation of these phenomena crucially depends on the value of $a^{1j}, a^{2j}, \ldots$ but if (3.19) is satisfied then one can expect that the events in the VHM domain would be enhanced by QCD jets and the mass of jets would have a tendency to be high with growing multiplicity.

The singular at finite z solutions arise in the field theory, when the s-channel cascades (jets) are described [56]. By definition, $T(z,s)$ coincides with the total cross section at $z=1$. Therefore, the nearness of z_c to one defines the significance of the corresponding processes. It is evident that both s and n should be high enough to expect the jets creation.

Summarizing the above estimations, we may conclude that

$$\mathrm{O}(\mathrm{e}^{-n}) \leq \sigma_n < \mathrm{O}(1/n) , \tag{3.21}$$

i.e. the soft Regge-like channel of hadron creation is suppressed in the VHM region in the high-energy events with exponential accuracy.

3.3. Phase transition – condensation

The aim of this section is to find the experimentally observable consequences of collective phenomena in the high-energy hadron inelastic collision [86]. We will pay attention mainly to the phase transitions, leaving out other possible interesting collective phenomena.

The statistics experience dictates that we should prepare the system for the phase transition. The temperature in a critical domain and the equilibrium media are just these conditions. It is evident that they are not a trivial requirement considering the hadrons inelastic collision at high energies.

The collective phenomena by definition suppose that the kinetic energy of particles of media are comparable, or even smaller, than the potential energy of their interaction. It is a quite natural condition noting that, for instance, the kinetic motion may decay even completely at a given temperature T, necessary for the phase transition long-range order. This gives, more or less definitely, the critical domain.

The same idea as in statistics seems natural in the multiple production physics. We will assume (A) that the collective phenomena should be seen just in the very high multiplicity (VHM) events, where, because of the energy–momentum conservation laws, the kinetic energy of the created particles cannot be high.

We will lean at this point on the S-matrix interpretation of statistics [87], see Section 2.2. It is based on the S-matrix generalization of the Wigner function formalism of Carruthers and Zachariazen [39] and the real-time finite-temperature field theory of Schwinger and Keldysh [29,30], see Appendixes A–C.

It was mentioned that the n-particle partition function in this approach coincides with the n-particle production cross section $\sigma_n(s)$ (in the appropriate normalization condition). Then, the cross section $\sigma_n(s)$ can be calculated applying the n-point Wigner function $W_n(X_1, X_2, \ldots, X_n)$. In the relativistic case $X_k = (u,q)_k$ are the 4-vectors. So, the external particles are considered as the 'probes' to measure the state of the interacting fields, i.e. the low mean energy of probes means that the system is 'cold'.

The multiple production phenomena may be considered also as the thermalization process of incident particle kinetic energy dissipation into the created particle mass. From this point of view the VHM processes are highly nonequilibrium since the final state of this case is very far from the initial one. It is known in statistics [24] that such processes aspire to be the stationary Markovian with a high level of entropy production. In the case of complete thermalization, the final state is in equilibrium.

The equilibrium we will classify as the condition in the frame of which the fluctuations of corresponding parameter are Gaussian. So, in the case of complete thermalization, the probes should have the Gaussian energy spectra. In other terms, the necessary and sufficient condition of the equilibrium is the smallness of the mean value of energy correlators [42,87]. From the physical point of view, the absence of these correlators means depression of the macroscopic energy flows in the system.

The multiple production experiment shows that the created particle energy spectrum is far from a Gaussian law, i.e. the final states are far from equilibrium. The natural explanation of these phenomena consists in the presence of (hidden) conservation laws in the interacting Yang–Mills fields: it is known that the presence of sufficient number of first integrals in involution prevents thermalization completely.

Nevertheless, the VHM final state may be equilibrium (B) in the above formulated sense. This means that the forces created by the nonAbelian symmetry conservation laws may be frozen during the thermalization process (remembering its stationary Markovian character in the VHM domain). We would like to take into account that the entropy $\mathcal{S}$ of a system is proportional to the number of created particles and, therefore, $\mathcal{S}$ should tend to its maximum in the VHM region [1].

One may consider following the small parameter $(\bar{n}(s)/n) \ll 1$, where $\bar{n}(s)$ is the mean value of the multiplicity n at a given CM energy $\sqrt{s}$. Another small parameter is the energy of the fastest hadron $\varepsilon_{\max}$. One should assume that in the VHM region $(\varepsilon_{\max}/\sqrt{s}) \to 0$. So, the conditions

$$\frac{\bar{n}(s)}{n} \ll 1, \quad \frac{\varepsilon_{\max}}{\sqrt{s}} \to 0 \tag{3.1}$$

would be considered as the mark of the processes under consideration. We can hope to organize the perturbation theory over them having there small parameters. In this sense VHM processes may be 'simple', i.e. one can use for their description semiclassical methods.

So, considering VHM events one may assume that conditions (A) and (B) are satisfied and one may expect the phase transition phenomena.

The S-matrix interpretation of statistics is based on the following definitions. First of all, let us introduce the generating function [49]

$$T(z,s) = \sum_n z^n \sigma_n(s) . \tag{3.2}$$

Summation is performed over all n up to $n_{\max} = \sqrt{s}/m$ and, at finite CM energy $\sqrt{s}$, $T(z,s)$ is a polynomial function of z. Following our idea, see Section 2.3, let us assume now that $z > 1$ is sufficiently small and for this reason $T(z,s)$ depends on the upper boundary $n_{\max}$ only weakly. In this case one may formally extend summation up to infinity and in this case $T(z,s)$ may be considered as a whole function. This possibility is important since it is the equivalent of the thermodynamical limit and it allows to classify the asymptotics over n in accordance with the position of singularities over z.

Let us consider $T(z)$ as the big partition function, where z is 'activity'. It is known [51] that $T(z)$ should be regular inside the circle of unit radius. The leftist singularity lies at $z = 1$. This singularity is the manifestation of the first order phase transition [51,24,52].

The origin of this singularity was investigated carefully in the paper [24]. It was shown that the position of singularities over z depends on the number of particles n in the system: the two complex conjugated singularities move to the real z-axis with rising n and in the thermodynamical limit, $n = \infty$, they pinch the point $z = 1$ in the first order phase transition case. More general analysis [52] shows that if the system is in equilibrium, then $T(z)$ may be singular only at $z = 1$ and $z = \infty$.

The position of the singularity over z and the asymptotic behavior of σ_n are closely related. Indeed, for instance, inserting into (3.2) $\sigma_n \propto \exp\{-cn^{\gamma}\}$ we find that $T(z)$ is singular at $z = 1$ if $\gamma < 1$. Generally, using the Mellin transformation (2.37) one can find an asymptotic estimation (2.39)

$$\sigma_n \propto \mathrm{e}^{-n \ln z_c(n)}, \quad z_c > 1 , \tag{3.3}$$

where z_c is the smallest solution of the equation of state

$$n = z\frac{\partial}{\partial z}\ln T(z) . \tag{3.4}$$

Therefore, to have the singularity at $z = 1$, we should consider $z_c(n)$ as a decreasing function of n. On the other hand, at constant temperature, $\ln z_c(n) \sim \mu_c(n)$ is the chemical potential, i.e. is the work necessary for creation of one particle. So, the singularity at $z = 1$ means that the system is unstable: the less work is necessary for creation of one more particle if $\mu(n)$ is a decreasing function of n.

The physical explanation of these phenomena is the following, see also [88]. The generating function $T(z)$ has the following expansion:

$$T(z) = \exp\left\{\sum_l z^l b_l\right\} , \tag{3.5}$$

where b_l are known as Mayer's group coefficients [25]. They can be expressed through the inclusive correlation functions and may be used to describe the formation of droplets of correlated particles, see Section 2.3.3. So, if droplets consist of l particles, then

$$b_l \sim \mathrm{e}^{-\beta\xi l^{(d-1)/d}} \tag{3.6}$$

is the mean number of such droplets. Here $\xi l^{(d-1)/d}$ is the surface energy of d-dimensional droplet.

Inserting this estimation into (3.5),

$$\ln T(z) \sim \sum_l \mathrm{e}^{\beta(l\mu - \xi l^{(d-1)/d})}, \quad \beta\mu = \ln z . \tag{3.7}$$

The first term in the exponent $\beta l\mu$ is the volume energy of the droplet and being positive it tries to enlarge the droplet. The second surface term $-\beta\xi l^{(d-1)/d}$ tries to shrink it. Therefore, the singularity at $z = 1$ is the consequence of instability: at $z > 1$ the volume energy abundance leads to unlimited growth of the droplet.

In conclusion we wish to formulate once more the main assumptions.

(I) It was assumed first of all that the system under consideration is in equilibrium. This condition may be naturally reached in the statistics, where one can wait for an arbitrary time till the system reaches equilibrium. Note, in the critical domain, the time of relaxation $t_r \sim (T_c/(T - T_c))^{\nu} \to \infty$, $(T - T_c) \to +0$, $\nu > 0$, T_c is the critical temperature.

We cannot give any guarantee that in the high-energy hadron collisions the final-state system is in equilibrium. The reason for this uncertainty is the finite time the inelastic processes and the presence of hidden (confinement) constraints on the dynamics. But if the confinement forces are frozen in the VHM domain, i.e. the production process is 'fast', then the equilibrium may be reached.

We may formulate the quantitative conditions, when the equilibrium is satisfied [42]. One should have the Gaussian energy spectra of created particles. If this condition is hardly investigated in the experiment, then one should consider the relaxation of 'long-range' correlations. This excludes the usage of relaxation condition for the 'short-range' (i.e. resonance) correlation.

(II) The second condition consists in the requirement that the system should be in the critical domain, where the (equilibrium) fluctuations of the system become high. Having no theory of hadron interaction at high energies we cannot define where the 'critical domain' lies and even whether it exists or not.

But, having the VHM 'cold' final state, we can hope that the critical domain is achieved.

The quantitative realization of this picture is given in Appendix J. It is important to note that the semiclassical approximation used there is rightfully in the VHM domain.

4. Conclusion

4.1. Discussion of physical problems

It seems useful to start the discussion of models by outlining the main problems, from the authors point of view.

A. *Soft color parton problems.* The infrared region of soft color parton interactions is a very important problem of high-energy hadron dynamics. Such fundamental questions as the infrared divergences of pQCD, collective phenomena in the colored particles system and symmetry breaking are the phenomena of the infrared domain.

The standard (most popular) hadron theory considers pQCD at small distances (in the scale of $\Lambda \simeq 0.2$ Gev) as the exact theory. This statement is confirmed by a number of experiments, namely deep-inelastic scattering data, hard jets observation. But the pQCD predictions have a finite range of validity since the nonperturbative effects should be taken into account at distances larger than $1/\Lambda$.

It is natural to assume, building the complete theory, that at large distances the nonperturbative effects *lay on*[10] the perturbative ones. As a result, pQCD loses its predictability *screened* by the nonperturbative effects.

[10] The corresponding formalism has been described, e.g. in [18].

Notice, the pQCD running coupling constant $\alpha_s(q^2) = 1/b\ln(q^2/\Lambda^2)$ becomes infinite at $q^2 = \Lambda^2$ and we do not know what happens with pQCD if $q^2 < \Lambda^2$. There are few possibilities. For instance, there is a suspicion [13] that at $q^2 \sim \Lambda^2$ the properties of theory changed so drastically (being defined on a new vacuum) that even the notions of pQCD *disappeared*. This means that pQCD should be truncated from below on the 'fundamental' scale Λ. It seems natural that this infrared cut-off would influence the soft hadrons emission.

The new possibility is described in Appendix K. This strict formalism allows to conclude that pure pQCD contributions are realized on zero measure, i.e. it is the phenomenological theory only. The successive approach shows that the Yang–Mills theory should be described in terms of (action,angle)-like variables. The last one means that the self-consistent description excludes such notions as the 'gluon'. As a result of this substitution new perturbation theory would be free from infrared divergences, i.e. there is no necessity to introduce the infrared cutoff parameter Λ. (Moreover, in the sector of vector fields (without quarks) the theory is ultraviolet stable.) It seems important for this reason to investigate experimentally just VHM events, where the soft color partons production is dominant.

It is important to try to raise the role of pQCD in the 'forbidden' area of large distances. The VHM processes are at highly unusual condition, where the nonperturbative effects must be negligible.

B. *Dissipation problems.* The highly nonequilibrium states decay (thermalization) which means in pQCD terms that the process of VHM formation should be enhanced, at least in asymptotics over multiplicity and energy, by jets. It is the general conclusion of nonequilibrium thermodynamics and it means that the very nonequilibrium initial state tends to equilibrium (thermalized) as fast as possible.

The entropy $\mathscr{S}$ of a system is proportional to the number of created particles and, therefore, $\mathscr{S}$ should tend to its maximum in the VHM region [1]. But the maximum of entropy testifies also to the equilibrium of the system.

C. *Collective phenomena.* We should underline that the collective phenomena may take place if and only if the particles interaction energy is comparable to the kinetic one. The VHM system considered may be 'cold' and 'equilibrium'. For this reason the VHM state is mostly adopted for investigation of collective phenomena. One of the possible states in which the collective effects, see, e.g. [89], may be important is the 'could colored plasma' [90].

The fundamental interest presents the problem of vacuum structure of Yang–Mills theory. For instance, if the process of cooling is 'fast', since the dissipation process of VHM final-state formation should be as fast as possible, then one may consider formation of vacuum domains with various properties. Then decay of these domains may lead to large fluctuations, for instance, of the isotopic spin.

Another important question is the collective phenomena in the VHM final state. The last one may be created 'perturbatively', for instance, by the formation of heavy jets. Then the color charges should be confined. There are various predictions about this process. One of them predicts that there should be a first-order phase transition.

4.2. Model predictions

Now we can ask: what can models say concerning the above problems?

A. *Soft colored partons production.* The multiperipheral models predict fast decreasing of topological cross section in the VHM domain $\bar{n}(s)^2 \ll n \ll n_{\max}$, $\sigma_n < O(e^{-n})$. At the same time the mean transverse momentum should *decrease* in this domain since the interaction radii should 'increase' with n.

The BFKL Pomeron predicts the same asymptotics, $\sigma_n < O(e^{-n})$, but the pQCD jet predicts $\sigma_n = O(e^{-n})$. The *naive* attempt to insert into the BFKL Pomeron the production of particles via (mini)jets seems impossible.

This 'insertion' can be done into the DIS ladder but investigation of the LLA kinematics in the VHM domain allows the conclusion that the 'low-x' contributions should be taken into account.

All this experience allows the assumption that in the VHM domain no 't-channel ladder' diagrams play sufficient role. This implies the existence of a transition to the processes with jet dominance. The pQCD is unable to predict the transition mechanism.

B. *Transition into 'equilibrium'*. If the t-channel ladders are 'destroyed' in the VHM region, then jets, despite the small factor $O(1/s)$ in the cross sections, are the only mechanism of particle production in the VHM domain. Dominance of heavy jets in the VHM domain may naturally explain the tendency towards equilibrium.

But the description of thermalization in terms of jets of massless gluons production destroys this hope: the jet contribution $\sigma_n = O(e^{-n})$ assumes 'bremsstrahlung' of *soft* gluons [91]. This prevents the equilibrium since ordering without fail introduces the nonrelaxing correlations.

C. *Collective phenomena*. Considering the collective phenomena, we propose to distinguish (a) the collective phenomena connected with the vacuum and (b) the collective phenomena produced in the VHM system. Following the experience of Section 3.3 we can conclude that the signal of vacuum instability is inequality: $\sigma_n > O(e^{-n})$.

Case (b) will not effect the cross section σ_n. But if the system reached equilibrium in the VHM domain then the collective phenomena may be investigated using ordinary thermodynamical methods. For instance, noting that $-\{\ln(T(\beta_c, z)/T(\beta_c, z))\}/\beta_c = \mathscr{F}(\beta_c, z)$ is the free energy one can measure the thermal capacity

$$\frac{\partial}{\partial \beta_c}\mathscr{F}(\beta_c, z) = C(\beta_c, z) . \tag{4.8}$$

Then, comparing capacities of hadron and γ-quanta systems we can say whether or not the phase transition happened.

The connection of the equilibrium and relaxation of correlations is well known [42]. Continuing this idea, if the VHM system is in equilibrium one may assume that the color charges in the pre-confinement phase of VHM event form the plasma. One should note here that the expected plasma is 'cold' and 'dense'. For this reason no long-range confinement forces would act among color charges. Then, being 'cold', in such a system various, collective phenomena may be important.

4.3. Experimental perspectives

The experimental possibilities in the VHM domain are not clear till now. Nevertheless, first step toward formulation of trigger system was done, see [92]. For this reason we would like to restrict ourselves by following two general questions. It seems that these questions are most important, being in the very beginning of VHM theory.

I. *For what values of multiplicity at a given energy the VHM processes become hard*?

The answer to this question depends on the value of the incident energy. If we know the answer then it will appear possible to estimate

- the role of multiperipheral contributions,
- the jet production rate,
- the role of vacuum instabilities.

It seems that the experimental answer to this question is absent since the produced particles are soft, their mean transverse and longitudinal momenta have the same value. In our understanding this question means: the total transverse energy may be extremely high.

It is interesting also to search the heavy jets, i.e. to observe the fluctuations of particle density in the event-by-event experiment, but this program seems vague since, for all evidence, fractal dimensions tend to zero with increasing multiplicities.

II. *For what values of multiplicity does the VHM final state reach equilibrium*?

We hope that having the answer for this question we would be able

- to investigate the status of pQCD,
- to observe the phase transition phenomena directly,
- to estimate the role of confinement constraints.

The equilibrium means that the energy correlation function mean values are small. Another point of interest, for example, is the charge equilibrium, when the mean value of charge correlation functions is small.

Notice, the effect of the phase space boundary may lead to 'equilibrium'. Indeed, if $(p) \simeq m + p^2/2m$ and $p^2 \ll m^2$ then one may neglect the momentum dependence of the amplitudes a_n. In this case the momentum dependence is defined by the Boltzmann exponent $e^{-\beta p^2/2m}$, $\beta \to \infty$ only and we get naturally to the Gaussian law for momentum distribution. Correlators should be small in this case since there are no interactions among particles (a_n are constants). But our question assumes that we investigate the possibility of equilibrium when $n \ll n_{\max}$ and $p^2 \gg m^2$.

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Appendix A. Matsubara formalism and the KMS boundary condition

There are various approaches to build the real-time finite-temperature field theories of Schwinger–Keldysh type (e.g. [47]). All of them use various tricks for analytical continuation of imaginary-time Matsubara formalism to real time [93]. The basis of the approaches is the introduction of the Matsubara field operator

$$\Phi_{\mathrm{M}}(\boldsymbol{x},\beta)=\mathrm{e}^{\beta H}\Phi_S(\boldsymbol{x})\mathrm{e}^{-\beta H}\;, \tag{A.1}$$

where $\Phi_{\mathrm{S}}(\boldsymbol{x})$ is the interaction-picture operator introduced instead of the habitual Heisenberg operator

$$\Phi(\boldsymbol{x},t)=\mathrm{e}^{\mathrm{i}tH}\Phi_S(\boldsymbol{x})\mathrm{e}^{-\mathrm{i}tH}\;.$$

Eq. (A.1) introduces the averaging over the Gibbs ensemble instead of averaging over zero-temperature vacuum states.

If the interaction switched on at the moment t_{i} adiabatically and switched off at t_{f} then there is the unitary transformation

$$\Phi(x,t)=U(t_i,t_{\mathrm{f}})U(t_i,t)\Phi_S(x)U(t,t_i)\;. \tag{A.2}$$

Introducing the complex Mills time contours [46] to connect t_i to t, t to t_{f} and t_{f} to t_i we form a ‘closed-time’ contour C (the end points of the contours *joined* together). This allows to write the last equality (A.2) in the compact form

$$\Phi(x)=T_C\{\Phi(x)\mathrm{e}^{\mathrm{i}\int_C \mathrm{d}^4x' L_{\mathrm{int}}(x')}\}_S\;,$$

where T_C is the time-ordering on the contour C operator.

The generating functional $Z(j)$ of correlation (Green) functions has the form

$$Z(j)=R(0)\langle T_C\mathrm{e}^{\mathrm{i}\int_C \mathrm{d}^4x\{L_{\mathrm{int}}(x)+j(x)\Phi(x)\}_S}\rangle\;,$$

where $\langle\rangle$ means averaging over the initial state.

If the initial correlations have a little effect, we can perform averaging over the Gibbs ensemble. This is the main assumption of the formalism: the generating functional of the Green functions $Z(j)$

has the form in this case

$$Z(j) = \int D\Phi' \langle \Phi'; t_i | \mathrm{e}^{-\beta H} T_C \mathrm{e}^{\mathrm{i}\int_C \mathrm{d}^4x j(x)\Phi(x)} | \Phi'; t_i \rangle$$

with $\Phi' = \Phi'(\boldsymbol{x})$. In accordance with (A.1) we have

$$\langle \Phi'; t_i | \mathrm{e}^{-\beta H} = \langle \Phi'; t_i - \mathrm{i}\beta |$$

and, as a result,

$$Z(j) = \int D\Phi' \mathrm{e}^{\mathrm{i}\int_{C_\beta} \mathrm{d}^4x \{L(x) + j(x)\Phi(x)\}} , \tag{A.3}$$

where the path integration is performed with KMS periodic boundary condition

$$\Phi(t_i) = \Phi(t_i - \mathrm{i}\beta) .$$

In (A.3) the contour C_β connects t_i to t_f, t_f to t_i and t_i to $t_i - \mathrm{i}\beta$. Therefore it contains an imaginary-time Matsubara part t_i to $t_i - \mathrm{i}\beta$. A more symmetrical formulation uses the following realization: t_i to t_f, t_f to $t_\mathrm{f} - \mathrm{i}\beta/2$, $t_\mathrm{f} - \mathrm{i}\beta/2$ to $t_i - \mathrm{i}\beta/2$ and $t_i - \mathrm{i}\beta/2$ to $t_i - \mathrm{i}\beta$ (e.g. [28]). This case also contains the imaginary-time parts of the time contour. Therefore, Eq. (A.3) presents the analytical continuation of the Matsubara generating functional to real times.

One can note that if this analytical continuation is possible for $Z(j)$ then representation (A.3) gives good recipe of regularization of frequency integrals in the Matsubara perturbation theory, e.g. [47]. But it gives nothing new for our problem since the Matsubara formalism is a formalism for equilibrium states only.

Appendix B. Constant temperature formalism

The starting point of our calculations is the n- into m-particles transition amplitude a_{nm}, the derivation of which is the well-known procedure in the Lehmann–Symanzik–Zimmermann (LSZ) reduction formalism [94] framework, see also [95]. The $(n+m)$-point Green function G_{nm} are introduced for this purpose through the generating functional Z_j [96]

$$G_{nm}(x, y) = (-i)^{n+m} \prod_{k=1}^{n} \hat{j}(x_k) \prod_{k=1}^{m} \hat{j}(y_k) Z_j , \tag{B.1}$$

where

$$\hat{j}(x) = \frac{\delta}{\delta j(x)} \tag{B.2}$$

and

$$Z_j = \int D\Phi \mathrm{e}^{\mathrm{i}S_j(\Phi)} . \tag{B.3}$$

The action

$$S_j(\Phi) = S_0(\Phi) - V(\Phi) + \int \mathrm{d}x j(x)\Phi(x) \;, \tag{B.4}$$

where $S_0(\Phi)$ is the free part and $V(\Phi)$ describes the interactions. At the end of the calculations one can put $j = 0$.

To provide the convergence of the integral (B.3) over the scalar field Φ the action $S_j(\Phi)$ must contain a positive imaginary part. Usually for this purpose Feynman's iε-prescription is used. But it is better for us to use the integral on the Mills complex time contour C_+ [46,47]. For example,

$$C_\pm : t \to t + \mathrm{i}\varepsilon,\ \varepsilon \to +0, \quad -\infty \le t \le +\infty \tag{B.5}$$

and after all the calculations return the time contour on the real axis putting $\varepsilon = 0$.

In Eq. (B.3) the integration is performed over all field configurations with standard vacuum boundary condition

$$\int \mathrm{d}^4 x \partial_\mu (\Phi \partial^\mu \Phi) = \int_{\sigma_\infty} \mathrm{d}\sigma_\mu \Phi \partial^\mu \Phi = 0 \;, \tag{B.6}$$

which leads to zero contribution from the surface term.

Let us introduce now field ϕ through the equation

$$-\frac{\delta S_0(\phi)}{\delta \phi(x)} = j(x) \tag{B.7}$$

and perform the shift $\Phi \to \Phi + \phi$ in integral (B.3), conserving boundary condition (B.6). Considering ϕ as the probe field created by the source

$$\phi(x) = \int \mathrm{d}y G_0(x-y) j(y) \;,$$

$$(\partial^2 + m^2)_x G_0(x-y) = \delta(x-y) \;, \tag{B.8}$$

only the connected Green function G^{c}_{nm} will be of interesting to us. Therefore,

$$G^{\mathrm{c}}_{nm}(x,y) = (-i)^{n+m} \prod_{k=1}^{n} \hat{j}(x_k) \prod_{k=1}^{m} \hat{j}(y_k) Z(\phi) \;, \tag{B.9}$$

where

$$Z(\phi) = \int D\Phi \mathrm{e}^{\mathrm{i}S(\Phi) - \mathrm{i}V(\Phi+\phi)} \tag{B.10}$$

is the new generating functional.

To calculate the nontrivial elements of the S-matrix we must put the external particles on the mass shell. Formally, this procedure means amputation of the external legs of G^{c}_{nm} and further multiplication on the free particle wave functions. As a result, the amplitude of n- into m-particles transition a_{nm} in the momentum representation has the form

$$a_{nm}(q,p) = (-i)^{n+m} \prod_{k=1}^{n} \hat{\phi}(q_k) \prod_{k=1}^{m} \hat{\phi}^*(p_k) Z(\phi) . \tag{B.11}$$

Here we introduce the particle distraction operator

$$\hat{\phi}(q) = \int \mathrm{d}x \mathrm{e}^{-\mathrm{i}qx} \hat{\phi}(x), \quad \hat{\phi}(x) = \frac{\delta}{\delta\phi(x)} . \tag{B.12}$$

Supposing that the momentum of particles is insufficient for us the probability of n- into m-particles transition is defined by the integral

$$r_{nm} = \frac{1}{n!m!} \int \mathrm{d}\Omega_n(q)\, \mathrm{d}\Omega_m(p) \delta^{(4)}\left(\sum_{k=1}^{n} q_k - \sum_{k=1}^{m} p_k \right) |a_{nm}|^2 , \tag{B.13}$$

where

$$\mathrm{d}\Omega_n(q) = \prod_{k=1}^{n} \mathrm{d}\Omega(q_k) = \prod_{k=1}^{n} \frac{\mathrm{d}^3 q_k}{(2\pi)^3 2\varepsilon(q_k)} , \tag{B.14}$$

is the Lorentz-invariant phase space element. We assume that the energy–momentum conservation δ-function was extracted from the amplitude.

Note that r_{nm} is the divergent quantity. To avoid this problem with trivial divergence, connected integration over reference frame, let us divide the energy–momentum fixing δ-function into two parts:

$$\delta^{(4)}(\sum q_k - \sum p_k) = \int \mathrm{d}^4 P \delta^{(4)}(P - \sum q_k) \delta^{(4)}(P - \sum p_k) \tag{B.15}$$

and consider a new quantity

$$R(P) = \sum_{n,m} \frac{1}{n!m!} \int \mathrm{d}\Omega_n(q)\, \mathrm{d}\Omega_m(p) \delta^{(4)}\left(P - \sum_{k=1}^{n} q_k \right) \delta^{(4)}\left(P - \sum_{k=1}^{n} p_k \right) |a_{nm}|^2 \tag{B.16}$$

defined on the energy–momentum shell (2.6). Here we suppose that the number of particles is not fixed. It is not too hard to see that, up to a phase space volume

$$R = \int \mathrm{d}^4 P R(P) \tag{B.17}$$

is the imaginary part of the amplitude $\langle \mathrm{vac} | \mathrm{vac} \rangle$. Therefore, computing $r(P)$ the standard renormalization procedure can be applied and the new divergences will not arise in our formalism.

The Fourier transformation of δ-functions in (B.16) allows one to write $R(P)$ in the form

$$R(P) = \int \frac{\mathrm{d}^4\alpha_1}{(2\pi)^4} \frac{\mathrm{d}^4\alpha_2}{(2\pi)^4} \mathrm{e}^{\mathrm{i}P(\alpha_1 + \alpha_2)} \rho(\alpha) , \tag{B.18}$$

where

$$\rho(\alpha) = \sum_{n,m} \frac{1}{n!m!} \int \prod_{k=1}^{m} \{ d\Omega(q_k) e^{-i\alpha_1 q_k} \} \prod_{k=1}^{n} \{ d\Omega(p_k) e^{-i\alpha_2 p_k} \} |a_{nm}|^2 . \tag{B.19}$$

Introducing the Fourier-transformed probability $\rho(\alpha)$ we assume that the phase-space volume is not fixed exactly, i.e. it is proposed that the 4-vector P is fixed with some accuracy if α_i are fixed. The energy and momentum in our approach are still locally conserved quantities since an amplitude a_{nm} is translationally invariant. So, we can perform the transformation

$$\alpha_1 \sum q_k = (\alpha_1 - \sigma_1) \sum q_k + \sigma_1 \sum q_k \rightarrow (\alpha_1 - \sigma_1) \sum q_k + \sigma_1 P , \tag{B.20}$$

since 4-momenta are conserved. The choice of σ_1 fixes the reference frame. This degree of freedom of the theory was considered in [11].

Inserting (B.11) into (B.19) we find that

$$\rho(\alpha) = \exp\left\{ -i \int dx\, dx' (\hat{\phi}_+(x) D_{+-}(x - x', \alpha_2) \hat{\phi}_-(x') - \hat{\phi}_-(x) D_{-+}(x - x', \alpha_1) \hat{\phi}_+(x')) \right\} Z(\phi_+) Z^*(-\phi_-) , \tag{B.21}$$

where D_{+-} and D_{-+} are the positive and negative frequency correlation functions, respectively:

$$D_{+-}(x - x', \alpha) = -i \int d\Omega_1(q) e^{iq(x - x' - \alpha)} \tag{B.22}$$

describes the process of particles creation at the time x_0 and its absorption at x'_0, $x_0 > x'_0$, and α is the CM 4-coordinate. Function

$$D_{-+}(x - x', \alpha) = i \int d\Omega_1(q) e^{-iq(x - x' + \alpha)} \tag{B.23}$$

describes the opposite process, $x_0 < x'_0$. These functions obey the homogeneous equations

$$(\partial^2 + m^2)_x G_{+-} = (\partial^2 + m^2)_x G_{-+} = 0 , \tag{B.24}$$

since the propagation of mass-shell particles is described.

We suppose that $Z(\phi)$ may be computed perturbatively. For this purpose the following transformations will be used ($\hat{X} \equiv \delta/\delta X$ at $X = 0$):

$$\begin{aligned} e^{-iV(\phi)} &= e^{-i\int dx \hat{j}(x) \hat{\phi}'(x)} e^{i\int dx j(x)\phi(x)} e^{-iV(\phi')} \\ &= e^{\int dx \phi(x) \hat{\phi}'(x)} e^{-iV(\phi')} = e^{-iV(-i\hat{j})} e^{i\int dx j(x)\phi(x)} , \end{aligned} \tag{B.25}$$

where $\hat{j}$ was defined in (B.2) and $\hat{\phi}$ in (B.12). At the end of the calculations, the auxiliary variables j, ϕ' can be taken equal to zero. Using the first equality in (B.25) we find that

$$Z(\phi) = e^{-i\int dx \hat{j}(x) \hat{\Phi}(x)} e^{-iV(\Phi + \phi)} e^{-i/2 \int dx\, dx' j(x) D_{++}(x - x') j(x')} , \tag{B.26}$$

where D_{++} is the causal Green function:

$$(\partial^2 + m^2)_x G_{++}(x - y) = \delta(x - y) . \tag{B.27}$$

Inserting (B.26) into (B.21) after simple manipulations with differential operators, see (B.25), we may find the expression

$$\rho(\alpha) = \mathrm{e}^{-\mathrm{i}V(-\mathrm{i}\hat{j}_+)+\mathrm{i}V(-\mathrm{i}\hat{j}_-)}\exp\left\{\frac{\mathrm{i}}{2}\int \mathrm{d}x\,\mathrm{d}x'\right.$$
$$(j_+(x)D_{+-}(x-x',\alpha_1)j_-(x') - j_-(x)D_{-+}(x-x',\alpha_2)j_+(x')$$
$$\left. - j_+(x)D_{++}(x-x')j_+(x') + j_-(x)D_{--}(x-x')j_-(x'))\right\}, \tag{B.28}$$

where

$$D_{--} = (D_{++})^* \tag{B.29}$$

is the anticausal Green function.

Considering the system with a large number of particles, we can simplify the calculations choosing the CM frame $P = (P_0 = E, \mathbf{0})$. It is useful also [41,33] to rotate the contours of integration over

$$\alpha_{0,k}\colon \alpha_{0,k} = -\mathrm{i}\beta_k,\ \mathrm{Im}\,\beta_k = 0,\ k = 1,2\ .$$

For the result, omitting the unnecessary constant, we will consider $\rho = \rho(\beta)$.

External particles play a double role in the S-matrix approach: their interactions create and annihilate the system under consideration and, on the other hand, they are probes through which the measurement of a system is performed. Since β_k are the conjugate to the particles energies quantities we will interpret them as the inverse temperatures in the initial (β_1) and final (β_2) states of interacting fields. They are 'good' parameters if and only if the energy correlations are relaxed.

B.1. Kubo–Martin–Schwinger boundary condition

The simplest (minimal) choice of $\Phi(\sigma_\infty) \neq 0$ assumes that the system under consideration is surrounded by black-body radiation. This interpretation restores Niemi–Semenoff's formulation of the real-time finite temperature field theory [28].

Indeed, as follows from (B.21), the generating functional $\rho(\alpha)$ is defined by corresponding generating functional

$$\rho_0(\phi_\pm) = Z(\phi_+)Z^*(-\phi_-) = \int D\Phi_+ D\Phi_-\, \mathrm{e}^{\mathrm{i}S_0(\Phi_+)-\mathrm{i}S_0(\Phi_-)}\mathrm{e}^{-\mathrm{i}V(\Phi_+ + \phi_+)+\mathrm{i}V(\Phi_- - \phi_-)}\,, \tag{B.30}$$

see (B.21). The fields (ϕ_+, Φ_+) and (ϕ_-, Φ_-) were defined on the time contours C_+ and C_-.

As was mentioned above, see (2.11), the path integral (B.30) describes the closed-path motion in the space of fields Φ. We want to use this fact and introduce a more general boundary condition which also guarantees the cancelation of the surface terms in the perturbation framework. We will introduce the equality

$$\int_{\sigma_\infty} \mathrm{d}\sigma_\mu \Phi_+ \partial^\mu \Phi_+ = \int_{\sigma_\infty} \mathrm{d}\sigma_\mu \Phi_- \partial^\mu \Phi_-\ . \tag{B.31}$$

The solution of Eq. (B.31) requires that the fields Φ_+ and Φ_- (and their first derivatives $\partial_\mu \Phi_\pm$) coincide on the boundary hypersurface σ_∞

$$\Phi_\pm(\sigma_\infty) = \Phi(\sigma_\infty) \neq 0 \; , \tag{B.32}$$

where, by definition, $\Phi(\sigma_\infty)$ is the arbitrary "turning-point" field.

In the absence of the surface terms, the existence of a nontrivial field $\Phi(\sigma_\infty)$ has the influence only on the structure of the Green functions

$$\begin{aligned} G_{++} &= \langle T\Phi_+\Phi_+\rangle, \quad G_{+-} = \langle \Phi_+\Phi_-\rangle \; , \\ G_{-+} &= \langle \Phi_-\Phi_+\rangle, \quad G_{--} = \langle \tilde{T}\Phi_-\Phi_-\rangle \; , \end{aligned} \tag{B.33}$$

where $\tilde{T}$ is the antitemporal time ordering operator. These Green functions must obey the equations:

$$\begin{aligned} &(\partial^2 + m^2)_x G_{+-}(x-y) = (\partial^2 + m^2)_x G_{-+}(x-y) = 0 \; , \\ &(\partial^2 + m^2)_x G_{++}(x-y) = (\partial^2 + m^2)_x^* G_{--}(x-y) = \delta(x-y) \end{aligned} \tag{B.34}$$

and the general solutions of these equations

$$\begin{aligned} &G_{ii} = D_{ii} + g_{ii} \; , \\ &G_{ij} = g_{ij}, \quad i \neq j \; , \end{aligned} \tag{B.35}$$

contain the arbitrary terms g_{ij} which are the solutions of homogenous equations

$$(\partial^2 + m^2)_x g_{ij}(x-y) = 0, \quad i,j = +,- \; . \tag{B.36}$$

The general solutions of these equations (they are distinguished by the choice of the time contours $C_\pm$)

$$g_{ij}(x-x') = \int d\Omega_1(q) e^{iq(x-x')} n_{ij}(q) \tag{B.37}$$

are defined through the functions n_{ij} which are the functionals of the 'turning-point' field $\Phi(\sigma_\infty)$: if $\Phi(\sigma_\infty) = 0$ we must have $n_{ij} = 0$.

Our aim is to define n_{ij}. We can suppose that

$$n_{ij} \sim \langle \Phi(\sigma_\infty) \cdots \Phi(\sigma_\infty)\rangle \; .$$

The simplest supposition gives

$$n_{ij} \sim \langle \Phi_i \Phi_j \rangle \sim \langle \Phi^2(\sigma_\infty)\rangle \; . \tag{B.38}$$

We will find the exact definition of n_{ij} starting from the S-matrix interpretation of the theory.

It was noted previously that the turning-point field $\Phi(\sigma_\infty)$ may be arbitrary. We will suppose that on the remote σ_∞ there are only free, on the mass-shell, particles. Formally, it follows from (B.35)–(B.37). This assumption is natural also in the S-matrix framework [40]. In other respects the choice of boundary condition is arbitrary.

Therefore, we wish to describe the evolution of the system in a background field of mass-shell particles. The restrictions connected with energy–momentum conservation laws will be taken into account and in other respects background particles are free. Then our derivation is the same as in [11]. Here we restrict ourselves to mentioning only the main quantitative points.

Calculating the product $a_{nm}a^*_{nm}$ we describe time-ordered processes of particle creation and absorption described by D_{+-} and D_{-+}. In the presence of the background particles, this time-ordered picture is slurred over because of the possibility to absorb particles before their creation occurs.

The processes of creation and absorption are described in vacuum by the product of operators $\hat{\phi}_+\hat{\phi}_-$ and $\hat{\phi}_-\hat{\phi}_+$. We can derive (see also [11]) the generalizations of (B.21). The presence of the background particles will lead to the following generating functional:

$$R_{cp} = \mathrm{e}^{-\mathrm{i}\boldsymbol{N}(\phi^*_i\phi_j)}R_0(\phi_\pm) , \tag{B.39}$$

where $R_0(\phi_\pm)$ is the generating functional for the vacuum case, see (B.30). The operator

$$\boldsymbol{N}(\phi^*_i\phi_j)$$

describes the external particles environment.

The operator $\hat{\phi}^*_i(q)$ can be considered as the creation and $\hat{\phi}_i(q)$ as the annihilation operator and the product $\hat{\phi}^*_i(q)\hat{\phi}_j(q)$ acts as the activity operator. So, in the expansion of $N(\hat{\phi}^*_i\hat{\phi}_j)$ we can leave only the first nontrivial term

$$\boldsymbol{N}(\phi^*_i\phi_j) = \int \mathrm{d}\Omega(q)\hat{\phi}^*_i(q)n_{ij}\hat{\phi}_j(q) , \tag{B.40}$$

since no special correlation among background particles should be expected. If the external (nondynamical) correlations are present then the higher powers of $\hat{\phi}^*_i\hat{\phi}_j$ will appear in expansion (B.40). Following the interpretation of $\hat{\phi}^*_i\hat{\phi}_j$, we conclude that n_{ij} is the mean multiplicity of background particles.

Computing ρ_{cp} we must conserve the translation invariance of amplitudes in the background field. Then, to take into account the energy–momentum conservation laws one should adjust to each vertex of in-going a_{nm} particles the factor $\mathrm{e}^{-\mathrm{i}\alpha_1 q/2}$ and for each out-going particle we have correspondingly $\mathrm{e}^{-\mathrm{i}\alpha_2 q/2}$.

So, the product $\mathrm{e}^{-\mathrm{i}\alpha_k q/2}\mathrm{e}^{-\mathrm{i}\alpha_j q/2}$ can be interpreted as the probability factor of the one-particle (creation + annihilation) process. The n-particles (*creation* + *annihilation*) process probability is the simple product of these factors if there are no special correlations among background particles. This interpretation is evident in the CM frame $\alpha_k = (-\mathrm{i}\beta_k, \mathbf{0})$.

After these preliminaries, it is not too hard to find that in the CM frame we have

$$n_{++}(q_0) = n_{--}(q_0) = \frac{\sum_{n=0}^{\infty} n\mathrm{e}^{-(\beta_1+\beta_2)/2|q_0|n}}{\sum_{n=0}^{\infty}\mathrm{e}^{-(\beta_1+\beta_2)/2|q_0|n}}$$

$$= \frac{1}{\mathrm{e}^{(\beta_1+\beta_2)/2|q_0|}-1} = \tilde{n}\left(|q_0|\frac{\beta_1+\beta_2}{2}\right) . \tag{B.41}$$

Computing n_{ij} for $i \neq j$ we must take into account the presence of one more particle

$$n_{+-}(q_0) = \theta(q_0)\frac{\sum_{n=1}^{\infty} n\mathrm{e}^{-(\beta_1+\beta_1)/2\, q_0 n}}{\sum_{n=1}^{\infty}\mathrm{e}^{-(\beta_1+\beta_1)/2\, q_0 n}} + \Theta(-q_0)\frac{\sum_{n=0}^{\infty} n\mathrm{e}^{(\beta_1+\beta_1)/2\, q_0 n}}{\sum_{n=0}^{\infty}\mathrm{e}^{(\beta_1+\beta_1)/2\, q_0 n}}$$
$$= \Theta(q_0)(1+\tilde{n}(q_0\beta_1)) + \Theta(-q_0)\tilde{n}(-q_0\beta_1) \tag{B.42}$$

and

$$n_{-+}(q_0) = \Theta(q_0)\tilde{n}(q_0\beta_2) + \Theta(-q_0)(1+\tilde{n}(-q_0\beta_2))\,. \tag{B.43}$$

Using (B.41)–(B.43), and the definition (B.35) we find the Green functions:

$$G_{i,j}(x-x',(\beta)) = \int\frac{\mathrm{d}^4 q}{(2\pi)^4}\mathrm{e}^{\mathrm{i}q(x-x')}\tilde{G}_{ij}(q,(\beta)) \tag{B.44}$$

where

$$\mathrm{i}\tilde{G}_i j(q,(\beta)) = \begin{pmatrix} \dfrac{i}{q^2-m^2+\mathrm{i}\varepsilon} & 0 \\ 0 & -\dfrac{i}{q^2-m^2-\mathrm{i}\varepsilon} \end{pmatrix}$$
$$+\, 2\pi\delta(q^2-m^2)\begin{pmatrix} \tilde{n}\left(\dfrac{\beta_1+\beta_2}{2}|q_0|\right) & \tilde{n}(\beta_2|q_0|)a_+(\beta_2) \\ \tilde{n}(\beta_1|q_0|)a_-(\beta_1) & \tilde{n}\left(\dfrac{\beta_1+\beta_2}{2}|q_0|\right) \end{pmatrix} \tag{B.45}$$

and

$$a_\pm(\beta) = -\mathrm{e}^{\beta/2}(|q_0| \pm q_0)\,. \tag{B.46}$$

The corresponding generating functional has the standard form

$$R_p(j_\pm) = \mathrm{e}^{-\mathrm{i}V(-i\hat{j}_+)+\mathrm{i}V(-\mathrm{i}\hat{j}_-)}\mathrm{e}^{i/2\int \mathrm{d}x\,\mathrm{d}x' j_i(x)G_{ij}(x-x',(\beta))j_j(x')}\,, \tag{B.47}$$

where the summation over repeated indexes is assumed.

Inserting (B.47) in the equation of state (2.8) we can find that $\beta_1 = \beta_2 = \beta(E)$. If $\beta(E)$ is a 'good' parameter then $G_{ij}(x-x';\beta)$ coincides with the Green functions of the real-time finite-temperature field theory and the KMS boundary condition:

$$G_{+-}(t-t') = G_{-+}(t-t'-\mathrm{i}\beta), \quad G_{-+}(t-t') = G_{+-}(t-t'+\mathrm{i}\beta)\,, \tag{B.48}$$

is restored. Eq. (B.48) can be deduced from (B.45) by direct calculations.

Appendix C. Local temperatures

We start this consideration from the assumption that the temperature fluctuations are large scale. We can assume that the temperature is a 'good' parameter in a cell whose dimension is much smaller than the fluctuation scale of temperature. (The 'good' parameter means that the corresponding fluctuations are Gaussian.)

Let us divide the remote hypersurface σ_∞ on an N_c and let us propose that we can measure the energy and momentum of groups of in- and out-going particles in each cell. The 4-dimension of cells cannot be arbitrarily small because of the quantum uncertainty principle.

To describe this situation we decompose the δ-function of the initial state constraint (2.6) on the product of $(N_c + 1)$ δ-functions:

$$\delta^{(4)}\left(P - \sum_{k=1}^{m} q_k\right) = \int \prod_{v=1}^{N_c} \left\{ \mathrm{d}Q_v \delta\left(Q_v - \sum_{k=1}^{m_v} q_{k,v}\right)\right\} \delta^{(4)}\left(P - \sum_{v=1}^{N_c} Q_v\right),$$

where $q_{k,v}$ is the momentum of the kth in-going particle in the vth cell and Q_v is the total 4-momenta of n_v in-going particles in this cell, $v = 1, 2, \ldots, N_c$. Therefore,

$$\sum_{v=1}^{N} \sum_{k=1}^{m_v} q_{k,v} = P .$$

The same decomposition will be used for the second δ-function of outgoing particle constraints. We must take into account the multinomial character of particle decomposition on N groups. This will give the coefficient

$$\frac{n!}{n_1! \cdots n_N!} \delta_{\mathrm{K}}\left(n - \sum_{v=1}^{N} n_v\right) \frac{m!}{m_1! \cdots m_N!} \delta_{\mathrm{K}}\left(m - \sum_{v=1}^{N} m_v\right),$$

where δ_{K} is the Kronecker symbol. The summation over

$$\{n_1, n_2, \ldots, n_{N_c}\} = \{n\}_{N_c}, \quad \{m_1, m_2, \ldots, m_{N_c}\} = \{m\}_{N_c}$$

is assumed.

As a result, the quantity

$$R_{N_c}(P, Q) = \sum_{\{n,m\}_{N_c}} \int |a_{nm}|^2 \prod_{v=1}^{N_c} \left\{ \frac{\mathrm{d}\Omega_{m_v}(q_k)}{m_v!} \delta^{(4)}\left(Q_v - \sum_{k=1}^{m_v} q_{k,v}\right) \frac{\mathrm{d}\Omega_{n_v}(p_k)}{n_v!} \delta^{(4)}\left(P_v - \sum_{k=1}^{n_v} p_{k,v}\right)\right\} \tag{C.1}$$

defines the probability to find in the vth cell the fluxes of in-going particles with total 4-momentum Q_v and of out-going particles with the total 4-momentum P_v. The sequence of these two measurements is not fixed.

The Fourier transformation of δ-functions in (C.1) gives

$$R_{N_c}(P, Q) = \int \prod_{k=1}^{N} \frac{\mathrm{d}^4\alpha_{1,v}}{(2\pi)^4} \frac{\mathrm{d}^4\alpha_{2,v}}{(2\pi)^4} \mathrm{e}^{\mathrm{i}\sum_{v=1}^{N} (Q_v \alpha_{1,v} + P_v \alpha_{2,v})} \rho_{N_c}(\alpha),$$

where

$$\rho_{N_c}(\alpha) = \rho_{N_c}(\alpha_{1,1}, \alpha_{1,2} \ldots, \alpha_{1,N_c}; \alpha_{2,1}, \alpha_{2,2}, \ldots, \alpha_{2,N})$$

has the form

$$\rho_{N_c}(\alpha) = \int \prod_{\nu=1}^{N_c} \left\{ \prod_{k=1}^{m_\nu} \frac{d\Omega_{m_\nu}(q)}{m_\nu!} e^{-i\alpha_{1,\nu} q_{k,\nu}} \prod_{k=1}^{n_\nu} \frac{d\Omega_{n_\nu}(p)}{n_\nu!} e^{-i\alpha_{2,\nu} p_{k,\nu}} \right\} |a_{nm}|^2 . \tag{C.2}$$

Inserting

$$a_{nm}(p,q) = (-i)^{n+m} \prod_{k=1}^{m} \hat{\phi}(q_{k,\nu}) \prod_{k=1}^{n} \hat{\phi}^*(p_{k,\nu}) Z(-\phi)$$

into (C.2) we find

$$\rho_{N_c}(\alpha) = \exp\left\{ i \sum_{\nu=1}^{N_c} \int dx\, dx' [\hat{\phi}_+(x) D_{+-}(x-x';\alpha_{2,\nu}) \hat{\phi}_-(x') - \hat{\phi}_-(x) D_{-+}(x-x';\alpha_{1,\nu}) \hat{\phi}_+(x')] \right\} \rho_0(\phi) , \tag{C.3}$$

where $D_{+-}(x-x';\alpha)$, and $D_{-+}(x-x';\alpha)$ are the positive and negative frequency correlation functions.

We must integrate over sets $\{Q\}_{N_c}$ and $\{P\}_{N_c}$ if the distribution of momenta over cells is not fixed. As a result,

$$R(P) = \int D^4\alpha_1(P) D^4\alpha_2(P) \rho_{N_c}(\alpha) , \tag{C.4}$$

where the differential measure

$$D^4\alpha(P) = \prod_{\nu=1}^{N_c} \frac{d^4\alpha_\nu}{(2\pi)^4} K(P, \{\alpha\}_{N_c})$$

takes into account the energy–momentum conservation laws

$$K(P, \{\alpha\}_{N_c}) = \int \prod_{\nu=1}^{N} d^4Q_\nu e^{i\sum_{\nu=1}^{N_c} \alpha_\nu Q_\nu} \delta^{(4)}\left(P - \sum_{\nu=1}^{N_c} Q_\nu \right) .$$

Explicit integration gives that

$$K(P, \{\alpha\}_{N_c}) \sim \prod_{\nu=1}^{N_c} \delta^{(3)}(\alpha - \alpha_\nu) ,$$

where $\boldsymbol{\alpha}$ is 3-vector of the CM frame. Choosing CM frame, $\alpha = (-i\beta, \mathbf{0})$,

$$K(E, \{\beta\}_{N_c}) = \int_0^\infty \prod_{\nu=1}^{N_c} dE_\nu e^{\sum_{\nu=1}^{N_c} \beta_\nu E_\nu} \delta\left(E - \sum_{\nu=1}^{N_c} E_\nu \right) .$$

In this frame

$$\rho_{N_c}(P) = \int D\beta_1(E) D\beta_2(E) \rho_{N_c}(\beta) ,$$

where

$$D\beta(E) = \prod_{\nu=1}^{N_c} \frac{d\beta_\nu}{2\pi i} K(E, \{\beta\}_{N_c})$$

and $\rho_{N_c}(\beta)$ was defined in (C.3) with $\alpha_{k,\nu} = (-i\beta_{k,\nu}, \mathbf{0}), \mathrm{Re}\,\beta_{k,\nu} > 0$, $k = 1, 2$.

We will calculate integrals over β_k using the stationary phase method. The equations for the most probable values of β_k:

$$-\frac{\partial}{\partial\beta_{k,\nu}} \ln K(E, \{\beta\}_{N_c}) = \frac{\partial}{\partial\beta_{k,\nu}} \ln \rho_{N_c}(\beta), \quad k = 1, 2 , \tag{C.5}$$

always have unique positive solutions $\beta^c_{k,\nu}(E)$. We propose that the fluctuations of β_k near $\beta^c_{k,\nu}$ are small, i.e. are Gaussian. This is the basis of the local-equilibrium hypothesis [97]. In this case $1/\beta^c_{1,\nu}$ is the temperature in the initial state in the measurement cell ν and $1/\beta^c_{2,\nu}$ is the temperature of the final state in the νth measurement cell.

The last formulation (C.4) implies that the 4-momenta $\{Q\}_{N_c}$ and $\{P\}_{N_c}$ cannot be measured. It is possible to consider another formulation also. For instance, we can suppose that the initial set $\{Q\}_{N_c}$ is fixed (measured) but $\{P\}_{N_c}$ is not. In this case we will have a mixed experiment: $\beta^c_{1,\nu}$ is defined by the equation

$$E_\nu = -\frac{\partial}{\partial\beta_{1,\nu}} \ln \rho_{N_c}$$

and $\beta^c_{2,\nu}$ is defined by the second equation in (C.5).

Considering the continuum limit, $N_c \to \infty$, the dimension of the cells tends to zero. In this case we are forced by quantum uncertainty principle to assume that the 4-momenta sets $\{Q\}$ and $\{P\}$ are not fixed. This formulation becomes pure thermodynamical: we must assume that just $\{\beta_1\}$ and $\{\beta_2\}$ are measurable quantities. For instance, we can fix $\{\beta_1\}$ and try to find $\{\beta_2\}$ as a function of the total energy E and the functional of $\{\beta_1\}$. In this case, Eqs. (C.5) become the functional equations.

In the considered microcanonical description, the finiteness of temperature does not touch the quantization mechanism. Indeed, one can see from (C.3) that all thermodynamical information is confined in the operator exponent

$$e^{N(\phi_i^* \phi_j)} = \prod_\nu \prod_{i \neq j} e^{i\int \hat{\phi}_i D_{ij} \hat{\phi}_j} ,$$

the expansion of which describes the environment, and the 'mechanical' perturbations are described by the functional $\rho_0(\phi)$. This factorization was achieved by the introduction of the auxiliary field ϕ and is independent of the choice of boundary conditions, i.e. unaffected by the choice of the systems environment.

C.1. Wigner functions

We will adopt the Wigner functions formalism in the Carruthers–Zachariazen formulation [39]. For the sake of generality, the m into n particles transition will be considered. This will allow the inclusion of the heavy ion–ion collisions.

In the previous section, the generating functional $\rho_{N_c}(\beta)$ was calculated by means of dividing the 'measuring device' on the remote hypersurface σ_∞ into N_c cells

$$\rho_{N_c}(\alpha) = e^{-iN(\phi;\beta,z)}\rho_0(\phi) , \tag{C.6}$$

where

$$N(\phi;\beta,z) = \left\{ \sum_{v=1}^{N_c} \int dx\, dx'(\hat{\phi}_+(x)D_{+-}(x-x';\beta_{2,v},z_2)\hat{\phi}_-(x') - \hat{\phi}_-(x)D_{-+}(x-x';\beta_{1,v},z_1)\hat{\phi}_+(x')) \right\} \tag{C.7}$$

is the particle number operator. The frequency correlation functions D_{+-} and D_{+-} are defined by equalities

$$D_{+-}(x-x';\beta_{2,v},z_2) = -i\int d\Omega_1(q)e^{iq_{2,v}(x-x')}e^{-\beta_{2,v}\varepsilon(q_{2,v})}z_2(q_{2,v}) , \tag{C.8}$$

$$D_{-+}(x-x',\beta_{1,v},z_1) = i\int d\Omega_1(q)e^{-iq_{1,v}(x-x')}e^{-\beta_{1,v}\varepsilon(q_{1,v})}z_1(q_{1,v}) . \tag{C.9}$$

It was assumed that the dimension of the device cells tends to zero ($N_c \to \infty$). Now we wish to specify the cells coordinates. As a result we will get to the Wigner function formalism.

Let us introduce Wigner variables [98]

$$x - x' = r, \quad x + x' = 2y: \quad x = y + r/2, \quad x' = y - r/2 . \tag{C.10}$$

Then

$$N(\phi;\beta,z) = -i\sum_{v=1}^{N_c} \int d\Omega(q)\, dr(\hat{\phi}_+(y+r/2)\hat{\phi}_-(y-r/2)z_2(q_{2,v})e^{iq_{2,v}r}e^{-\beta_{2,v}\varepsilon(q_{2,v})} + \hat{\phi}_-(y+r/2)\hat{\phi}_+(y-r/2)z_1(q_{1,v})e^{-iq_{1,v}r}e^{-\beta_{1,v}\varepsilon(q_{1,v})})\, dy . \tag{C.11}$$

The Boltzmann factor, $e^{-\beta_{i,v}\varepsilon(q_{i,v})}$, can be interpreted as the probability to find a particle with the energy $\varepsilon(q_{i,v})$ in the final ($i = 2$) or initial ($i = 1$) state. The total probability, i.e. the process of creation and further absorption of n particles, is defined by multiplication of these factors. Besides, $e^{iq_{2,v}r}$ is the out-going particle momentum measured in the vth cell.

Generally, it is impossible to adjust the 4-index of cell v with coordinate y. For this reason the summation over v and the integration over r are performed in (C.11) independently. But let us assume that the 4-dimension of the cell L is higher than the scale of the characteristic quantum fluctuations L_q,

$$L \gg L_q . \tag{C.12}$$

One can divide the four-dimensional y space into the L-dimensional cells. Then, because of (C.12), the quantum fluctuations cannot take away particles from this cell. Then we can adjust the index of the measurements cell with the index of the y space cell.

As a result,

$$\begin{aligned}N(\phi;\beta,z)= & -\mathrm{i}\int \mathrm{d}y\int \mathrm{d}\Omega_i(q)\,\mathrm{d}r(\hat{\phi}_+(y+r/2)\hat{\phi}_-(y-r/2)z_2(q_2,y)\,\mathrm{e}^{\mathrm{i}q_2 r}\mathrm{e}^{-\beta_2(y)\varepsilon(q_2)}\\ & +\hat{\phi}_-(y+r/2)\hat{\phi}_+(y-r/2)z_1(q_1,y))\,\mathrm{e}^{-\mathrm{i}q_1 r}\mathrm{e}^{-\beta_1(y)\varepsilon(q_1)}\ ,\end{aligned} \tag{C.13}$$

where

$$\int \mathrm{d}y=\sum_v \int_{C(v)} \mathrm{d}y \tag{C.14}$$

and $C(v)$ is the dimension L of the y space cell with index v. Notice that the momentum q did not carry the index v (or the index y of the space cell).

Our formalism allows the introduction of more general 'closed-path' boundary conditions. The presence of external black-body radiation will only reorganize the differential operator $\exp\{\hat{N}(\phi_i^*\phi_j)\}$ and a new generating functional ρ_{cp} has the same form

$$\rho_{cp}(\beta,z)=\mathrm{e}^{-\mathrm{i}N(\phi;\beta,z)}\rho_0(\phi)\ .$$

The calculation of operator $\hat{N}(\phi_i^*\phi_j)$ is strictly the same as in Appendix B. Introducing the cells we will find that

$$\hat{N}(\phi_i^*\phi_j)=\int \mathrm{d}r\,\mathrm{d}y\hat{\phi}_i(r+y/2)\tilde{n}_{ij}(y)\hat{\phi}_j(r-y/2)\ ,$$

where the occupation number $\tilde{n}_{ij}$ carries the cell index y:

$$\tilde{n}_{ij}(r,y)=\int \mathrm{d}\Omega_1(q)\mathrm{e}^{\mathrm{i}qr}n_{ij}(y,q)$$

and ($q_0=\varepsilon(q)$)

$$n_{++}(y,q_0)=n_{--}(y,q_0)=\tilde{n}(y,(\beta_1+\beta_2)|q_0|/2)=\frac{1}{\mathrm{e}^{(\beta_1+\beta_2)(y)|q_0|/2}-1}\ ,$$

$$n_{+-}(y,q_0)=\Theta(q_0)(1+\tilde{n}(y,\beta_2 q_0))+\Theta(-q_0)\tilde{n}(y,-\beta_1 q_0)\ ,$$

$$n_{-+}(y,q_0)=n_{+-}(y,-q_0)\ .$$

For simplicity the CM system was used. Other calculations are the same as the constant temperature case.

Appendix D. Multiperipheral kinematics

First of all [21], two light-like 4-momenta

$$p_{1,2}=P_{1,2}-P_{2,1}m^2/s$$

are introduced. Here $P_{1,2}$ are momenta of colliding particles. The final-state particles momenta have the following representation:

$$p'_1 = \alpha'_1 p_2 + \beta'_1 p_1 + p'_{1\perp}, \quad p'_2 = \alpha'_2 p_2 + \beta'_2 p_1 + p'_{2\perp} ,$$
$$k_i = \alpha_i p_2 + \beta_i p_1 + k_{i\perp} . \tag{D.1}$$

Sudakov's parameters, α, β, are not independent. The mass-shell conditions and the energy–momentum conservation laws give

$$s\alpha'_1\beta'_1 = m^2 + (p'_{1\perp})^2 = E^2_{1\perp}, \quad s\alpha'_2\beta'_2 = E^2_{1\perp}, \quad s\alpha_i\beta_i = E^2_{i\perp} ,$$
$$\alpha'_1 + \alpha'_2 + \sum\alpha_i = 1, \quad \beta'_1 + \beta'_2 + \sum\beta_i = 1 , \tag{D.2}$$

where $E_{i\perp}$ is the transverse energy.

We have for the multiperipheral kinematics

$$1 \approx \beta'_1 \gg \beta_1 \gg \cdots \gg \beta_n \gg \beta'_2 \sim \frac{m^2}{s} ,$$
$$\frac{m^2}{s} \ll \alpha'_1 \ll \alpha_1 \ll \cdots \ll \alpha_n \ll \alpha'_2 \sim 1 \tag{D.3}$$

and the transverse momenta are restricted:

$$|p'_{i\perp}| \sim |k_{i\perp}| \sim m . \tag{D.4}$$

It corresponds to small production angles in the considered CM frame

$$\theta_i = \frac{|k_{i\perp}|}{\sqrt{s\beta_i}}, \quad |\beta_i| \gg |\alpha_i| , \tag{D.5}$$

if the particle moves along $\boldsymbol{P}_1$, and a similar expression exists for particles moving in the opposite direction, where $|\beta_i| \ll |\alpha_i|$. In the 'central region' of the CM frame $|\beta_i| \sim |\alpha_i| \sim (E_{i\perp}/E) \ll 1$ the angles of produced particles are large and energies are small. It should be underlined that all this excludes the (mini)jets formation.

The final-state particles phase space volume element is

$$d\sigma_{2\to 2+n} = \frac{(2\alpha_s)^{2+n}}{16\pi^{2n}} C_V^n \frac{d^2q_1}{q_1^2 + m^2} \frac{d^2q_2}{(q_1 - q_2)^2 + \lambda^2} \cdots \frac{d^2q_{n+1}}{(q_n - q_{n+1})^2 + \lambda^2} \frac{1}{q_{n+1}^2 + \lambda^2}$$
$$\times \frac{d\alpha_1}{\alpha_1} \Theta(\alpha_2 - \alpha_1) \cdots \frac{d\alpha_n}{\alpha_n} \prod_{i=1}^{n+1} \left(\frac{s_i}{s_0}\right)^{2\alpha(q_i^2)} = \frac{1}{q_{n+1}^2 + \lambda^2} dZ_n , \tag{D.6}$$

where $C_V = 3$ and the 4-momentum of produced particle

$$k_i = (\alpha_i - \alpha_{i+1})p_2 + (\beta_i - \beta_{i+1})p_1 + (q_i - q_{i+1})_\perp = -\alpha_{i+1}p_2 + \beta_i p_1 + (q_i - q_{i+1})_\perp .$$

The square of pairs invariant mass

$$s_1 = (p'_1 + k_1)^2 = s|\alpha_2|, \quad s_{n+1} = (k_n + p'_2)^2 = \frac{E^2_{n\perp}}{\alpha_n}, \quad s_i = E^2_{(i-1)\perp}\frac{\alpha_{i+1}}{\alpha_{i-1}} .$$

The energy conservation law takes the form

$$s_1 s_2 \cdots s_n = sE^2_{1\perp} \cdots E^2_{n\perp} .$$

The trajectory of reggeized gluon is

$$\alpha(q^2) = \frac{q^2\alpha_s}{2\pi^2}\int \frac{\mathrm{d}^2k}{(k^2+\lambda^2)((q-k)^2+\lambda^2)} ,$$

where λ is the gluon 'mass'. If this virtuality is large, $\lambda \gg m$, then the gluon decays creating a pQCD jet, but the constraint on the multiperipheral kinematics prevents this possibility.

D.1. Deep inelastic reactions

For the pure deep inelastic case, when one of the initial hadrons which is scattered at the angle θ has the energy E' in the cms of beams whereas another which is scattered at small angle and the large transfer momentum $Q = 4EE'\sin^2(\theta/2) \gg m^2$, is distributed to the same number of the emitted particles due to evolution mechanism we have [91] (θ is small)

$$\mathrm{d}\sigma_n^{\mathrm{DIS}} = \frac{4\alpha^2E'^2}{Q^4M}\mathrm{d}D_n\,\mathrm{d}E'\mathrm{d}\cos\theta ,$$

$$\mathrm{d}D_n = \left(\frac{\alpha_s}{4\pi}\right)^n \int_{m^2}^{Q^2}\frac{\mathrm{d}k_n^2}{k_n^2}\int_{m^2}^{k_n^2}\frac{\mathrm{d}k_{n-1}^2}{k_{n-1}^2}\cdots\int_{m^2}^{k_2}\frac{\mathrm{d}k_1^2}{k_1^2}\int_x^1\mathrm{d}\beta_n\Theta_n^{(1)}\int_{\beta_n}^1\mathrm{d}\beta_{n-1}\Theta_{n-1}^{(1)}\cdots$$

$$\times\int_{\beta_2}^1\mathrm{d}\beta_1\Theta_1^{(1)}P\left(\frac{\beta_n}{\beta_{n-1}}\right)\cdots P(\beta_1), \quad P(z) = 2\frac{1+z^2}{1-z} , \tag{D.7}$$

where $\Theta^{(i)} = \Theta(\theta_{i+1} - \theta_i)$ and the emission angle $\theta_i = |k_i|/(E\max(\alpha_i.\beta_i))$.

D.2. Large angle production

For the large-angle particle production process the differential cross section (as well as the total one) decreases with CM energy $\sqrt{s}$. Let us consider for definiteness the process of annihilation of electron–positron pairs to n photons [99]:

$$\mathrm{d}\sigma_n^{DL} = \frac{2\pi\alpha^2}{s}\mathrm{d}F_n ,$$

$$\mathrm{d}F_n = \left(\frac{\alpha}{2\pi}\right)^n\prod_{i=1}^n \mathrm{d}x_i\,\mathrm{d}y_i\Theta(x_i - y_i)\Theta(y_i - y_{i-1})\Theta(y_i)\Theta(x_i)$$

$$\times\Theta(\rho - x_n)\Theta(\rho - y_n), \quad y_i = \ln\frac{1}{\beta_i}, \quad \rho = \ln\frac{s}{m^2} , \tag{D.8}$$

Similar formulae can be written for subprocess of quark-antiquark annihilation into n large-angle moving gluons.

At the end, one can consider the following possibilities:

(a) Pomeron regime (P);
(b) Evolution regime (DIS);
(c) Double logarithmic regime (DL);
(d) DIS + P regime;
(e) P + DL + P regime.

The description of every regime may be performed in terms of effective ladder-type Feynman diagrams. This can be done using the blocks dZ_n, dD_n and dF_n.

Appendix E. Reggeon diagram technique for generating function

We will consider, see (3.11)

$$\mathscr{P}(q,\omega;z) = \int_0^\infty d\xi e^{-\omega\xi} P(q,\xi;z) = \frac{1}{\omega + \alpha_0' q^2 + \psi_0(z)}, \quad \xi = \ln(s/m^2) . \tag{E.1}$$

as the 'propagator of the cut Pomeron'. It will be assumed also that

$$\psi_0(z) = -\Delta + (1-z)n_0, \quad n_0 > 0 .$$

So, the resonance short-range correlations will be ignored in this definition or propagator. It was assumed also that the 'bare' slope α' is z independent.

It should be underlined that the 'propagator' (E.1) is written phenomenologically. It absorbs the assumptions that (i) the diffraction cone shrinks with energy and (ii) the inclusive cross sections are universal, see (3.2).

The set of principal rules concerning multiperipheral kinematics of Feynman diagrams is given in Appendix D. The reggeon calculus supposes that the virtuality of each line of the Feynman diagram is restricted. This ignores 'hard jets', later known as the pQCD jets.

Then the ν Pomeron exchange eikonal diagram has only $(\nu+1)$ ways of being cut. If the cut line goes through μ Pomerons, then the corresponding contributions are:

$$\Phi_\nu^\mu(\omega,q) = \int d\Omega_\nu (M_\nu^\mu(q_1,\ldots,q_\nu))^2 \mathscr{Y}_\nu^\mu \prod_{l=1}^{\mu} \mathscr{P}(q_i,\omega_i;z) \prod_{i=\mu+1}^{\nu} \mathscr{P}(q_I,\omega_i;z=1) , \tag{E.2}$$

where $M_\nu^\mu(q_1,\ldots,q_\nu)$ is the 'vertex function', the combinatorial coefficient is

$$\mathscr{Y}_\nu^\mu = \frac{(-1)^{(\nu-\mu)} 2^\nu \nu!}{\mu!(\nu-\mu)!}$$

and the phase space element is

$$d\Omega_\nu = \prod_{i=1}^{\nu} \frac{d\omega_l\, d^2 q_l}{(2\pi)^3 i} \delta\left(\omega - \sum_{l=1}^{\nu} \omega_l\right) \delta^2\left(q - \sum_{i=1}^{\nu} q_l\right) . \tag{E.3}$$

As usual, contribution (E.2) leads to the following mean multiplicity of produced particles:

$$\bar{n}(s)^{\mu}_{\nu} = \frac{\partial}{\partial z}\ln \int d\Omega_{\nu}(s/m^2)^{\omega}\Phi^{\mu}_{\nu}(\omega, q=0)\bigg|_{z=1} \sim \mu\bar{n}(s)\ . \tag{E.4}$$

Therefore,

$$\mu \sim \bar{n}(s) \tag{E.5}$$

is essential in the VHM region, where $n \sim \bar{n}(s)^2$ is assumed.

The impact parameter representation:

$$\varphi^{\mu}_{\nu}(s,q) = \int d\Omega \left(\frac{s}{m^2}\right)^{\omega} \frac{d^2 q}{(2\pi)^2} e^{i\boldsymbol{q}\boldsymbol{b}} \Phi^{\mu}_{\nu}(\omega, q) \tag{E.6}$$

would be useful also. The contribution (E.2) describes interactions with impact parameter

$$\langle \boldsymbol{b}^2 \rangle \simeq 4\alpha' \ln(s/m^2)/\nu\ . \tag{E.7}$$

Notice that $\langle \boldsymbol{b}^2 \rangle$ is the number of cut pomerons independent of μ. But, remembering that $\mu \geq \nu$ and that the Regge model is only able to describe large-distance interactions, $m^2\langle \boldsymbol{b}^2 \rangle \geq 1$, one can conclude that the Regge pole description is valid only for

$$n \leq \bar{n}(s)^2\ . \tag{E.8}$$

This is why the VHM region is defined by $\bar{n}(s)^2$.

Appendix F. Pomeron with $\Delta > 0$

Then the cut Pomeron propagator in the impact parameter representation

$$\tilde{g}(\boldsymbol{b}, \xi; z) = g(\boldsymbol{b}, \xi) e^{(z-1)\bar{n}(s)}\ , \tag{F.1}$$

where

$$g(\boldsymbol{b}, \xi) = \frac{1}{2\alpha'\xi} e^{\Delta\xi} e^{-\boldsymbol{b}^2/4\alpha'\xi} \tag{F.2}$$

is the uncut Pomeron profile function. Using this definition one can find that the contribution of the eikonal diagrams gives a contribution

$$\mathscr{F}_0(\boldsymbol{b}, \xi; z) = (1 - e^{-\lambda^2 g(\boldsymbol{b},\xi)}) - \frac{1}{2}(1 - e^{2\lambda^2 g(\boldsymbol{b},\xi)(e^{(z-1)\bar{n}(s)} - 1)})\ , \tag{F.3}$$

where λ is a constant.

The first bracket is essential for $\boldsymbol{b}^2 \leq 4\alpha'\Delta\xi^2$. So, with exponential accuracy, the first term is equal to

$$\Theta(4\alpha'\Delta\xi^2 - \boldsymbol{b}^2)\ .$$

Let us now consider the second bracket. For $z < 1$

$$b^2 \le 4\alpha' \Delta \xi^2 \left\{ 1 + \frac{1}{\xi} \ln(1 - e^{(z-1)\bar{n}(s)}) \right\} = 4\alpha' \Delta \xi^2 \gamma(\xi, z) \tag{F.4}$$

are essential. It is not hard to see that $\gamma(\xi, z)$ decreases if $z \to 1$ and $\gamma(\xi, z)$ is equal to zero for $z = 1$. In this case, by the definition of the generating function, the integral over $\boldsymbol{b}$ of $\mathscr{F}_0(\boldsymbol{b}, \xi; z = 1)$ defines the contribution to the total cross section. So, the model predicts the production of particles in the ring

$$4\alpha' \xi^2 \gamma(\xi, z) \le \boldsymbol{b}^2 \le 4\alpha' \xi^2 \Delta \ . \tag{F.5}$$

if $z < 1$, i.e. if $n < \bar{n}(s)$. Notice also that $\gamma(\xi, z = 0) = \Delta + O(e^{-\xi})$. Then,

$$\mathscr{F}_0(\boldsymbol{b}, \xi; z = 0) = \tfrac{1}{2} \mathscr{F}_0(\boldsymbol{b}, \xi; z = 1) \ . \tag{F.6}$$

So, the elastic part of the total cross section is half of the total cross section. This means, using the optical analogy, that the scattering on the absolutely black disk is well described.

The last conclusion means that the interaction radii should increase with n in the VHM region. Indeed, as follows from (F.3),

$$\mathscr{F}_0(\boldsymbol{b}, \xi; z) \simeq \frac{1}{2}(e^{2\lambda^2 g(\boldsymbol{b}, \xi)(e^{(z-1)\bar{n}(s)} - 1)} - 1) \ , \tag{F.7}$$

at $z > 1$ and, therefore,

$$0 \le \boldsymbol{b}^2 \le B^2 = 4\alpha' \xi (\Delta \xi + (z - 1)\bar{n}(s)) \tag{F.8}$$

are essential.

Appendix G. Dual resonance model of VHM events

Our purpose is to investigate the role of the exponential spectrum (3.19) in the asymptotic region over multiplicity n. In this case one can validate heavy resonance creation and such a formulation of the problem has definite advantages.

(i) If creation of heavy resonances at $n \to \infty$ is expected, then one can neglect the dependence on the resonance momentum $\boldsymbol{q}_i$. So, the 'low-temperature' expansion is valid in the VHM region.
(ii) Having the big parameter n, one can construct the perturbations expanding over $1/n$.
(iii) We will be able to show at the end the range of applicability of these assumptions.

For this purpose, the following formal phenomena will be used. The grand partition function

$$T(z, s) = \sum_n z^n \sigma_n(s), \quad T(1, s) = \sigma_{\mathrm{tot}}(s), \quad n \le \sqrt{s}/m_0 \equiv n_{\max}(s) \tag{G.1}$$

will be introduced, see (2.35). Then the inverse Mellin transformation, see (2.37)

$$\sigma_n(s) = \frac{1}{2\pi i} \int \frac{dz}{z^{n+1}} T(z, s) \ . \tag{G.2}$$

will be performed expanding it in the vicinity of the solution $z_c > 0$ of the equation of state, see (2.38):

$$n = z\frac{\partial}{\partial z}\ln T(z,s) \; . \tag{G.3}$$

It is assumed, and this should be confirmed at the end, that the fluctuations in the vicinity of z_c are Gaussian.

It is natural at first glance to consider $z_c = z_c(n,s)$ as an increasing function of n. Indeed, this immediately follows from the positivity of $\sigma_n(s)$ and the finiteness of $n_{max}(s)$ at finite s. But one can consider the 'thermodynamical limit', see Section 2.3.1, or the limit $m_0 \to 0$. Theoretically, the last one is right because of the PCAC hypotheses and nothing should happen if the pion mass $m_0 \to 0$. In this sense, $T(z,s)$ may be considered as the whole function of z. Then, $z_c = z_c(n,s)$ would be an increasing function of n if and only if $T(z,s)$ is a regular function at $z = 1$.

The proof of this statement is as follows. We should conclude, as follows from Eq. (G.3), that

$$z_c(n,s) \to z_s \quad \text{at } n \to \infty \quad \text{and} \quad \text{at } s = \text{const} \; , \tag{G.4}$$

i.e. the singularity point z_s *attracts* z_c in asymptotics over n. If $z_s = 1$, then $(z_c - z_s) \to +0$, when n tends to infinity [50]. The concrete realization of this possibility is shown in Section 3.3. But if $z_s > 1$, then $(z_c - z_s) \to -0$ in VHM region, see Sections 3.1, 3.5.

One may use the estimation, see also (2.39):

$$-\frac{1}{n}\ln\frac{\sigma_n(s)}{\sigma_{tot}(s)} = \ln z_c(n,s) + O(1/n) \; , \tag{G.5}$$

where z_c is the *smallest* solution of (G.3). It should be underlined that this estimation is *independent* of the character of singularity, i.e. the position z_s is only important with $O(1/n)$ accuracy.

G.1. Partition function

Introducing the 'grand partition function' (G.1) the 'two-level' description means that

$$\ln\frac{T(z,\beta)}{\sigma_{tot}(s)} = \sum_k \frac{1}{k!}\int \prod_{i=1}^{k} \{\mathrm{d}\Omega_k(q)\,\mathrm{d}m_i\,\xi(q_i,z)\mathrm{e}^{-\beta\varepsilon_i}\} N_k(q_1,q_2,\ldots,q_k;\beta) \equiv -\beta\mathscr{F}(z,s) \; , \tag{G.6}$$

where $\varepsilon(q_i) = (q_i^2 + m_i^2)^{1/2}$. This is our group decomposition. The quantity $\xi(q,z)$ may be considered as the local activity. So,

$$\left.\frac{\delta T}{\delta\xi(q,z)}\right|_{\xi=1} \sim \sigma_{tot} B_1(q) \; . \tag{G.7}$$

If the resonance decay forms a group of particles with total 4-momentum q, then $B_1(q)$ is the mean number of such groups. The second derivative gives

$$\left.\frac{\delta^2 T}{\delta\xi(q_1,z)\delta\xi(q_2,z)}\right|_{\xi=1} \sim \sigma_{tot}\{B_2(q_1,q_2) - B_1(q_1)B_1(q_2)\} \equiv \sigma_{tot} K_2(q_1,q_2) \; , \tag{G.8}$$

where $K_2(q_1, q_2)$ is the two groups correlation function, and so on. One can consider B_k as Mayer's group coefficients, see Section 2.3.3.

The Lagrange multiplier β was introduced in (G.6) to each resonance: the Boltzmann exponent $\exp\{-\beta\varepsilon\}$ takes into account the energy conservation law $\sum_i \varepsilon_i = E$, where E is the total energy of colliding particles, $2E = \sqrt{s}$ in the CM frame. This conservation law means that β is defined by the equation

$$\sqrt{s} = \frac{\partial}{\partial\beta} \ln T(z, \beta) . \tag{G.9}$$

So, to define the state one should solve two equations of state (G.3) and (G.9).

The solution β_c of Eq. (G.9) has the meaning of inverse temperature of the gas of resonances if and only if the fluctuations in the vicinity of β_c are Gaussian, see Section 2.2.2.

On the second level, we should describe the resonance decay into hadrons. Using (3.24) we can write in the vicinity of $z = 1$:

$$\xi(q, z) = \sum_n z^n \sigma_N^{\mathrm{R}}(q) = g^{\mathrm{R}}\left(\frac{m_0}{m}\right) \mathrm{e}^{(z-1)\bar{n}(m)} . \tag{G.10}$$

The assumptions B and D, see (3.21), were used here

So,

$$-\beta\mathscr{F}(z, s) = \sum_k \int \prod_{i=1}^{k} \{\mathrm{d}m_i \xi(m_i, z)\} \tilde{B}_k(m; \beta) , \tag{G.11}$$

where $m = (m_1, m_2, \ldots, m_k)\xi$ was defined in (G.10) and

$$\tilde{B}_k(m; \beta) = \int \prod_{i=1}^{k} \{\mathrm{d}\Omega_k(q) \mathrm{e}^{-\beta\varepsilon_i(q_i)}\} B_k(m; q) . \tag{G.12}$$

Assuming now that $|\boldsymbol{q}_i| \ll m$ are essential,

$$\tilde{B}_k(m; \beta) \simeq B_k(m) \prod_{i=1}^{k} \left\{ \sqrt{\frac{2m_i}{\beta^3}} \mathrm{e}^{-\beta m_i} \right\} . \tag{G.13}$$

Following the duality assumption, one may write

$$B_k(m) = \bar{B}_k(m) \prod_{i=1}^{k} \{m_i^{\gamma} \mathrm{e}^{\beta_0 m_i}\} \tag{G.14}$$

and $\bar{B}_k(m)$ is a slowly varying function of $m = (m_1, m_2, \ldots, m_k)$:

$$\bar{N}_k(m) \simeq b_k .$$

As a result, the low-temperature expansion is as follows:

$$-\beta\mathscr{F}(z, s) = \sum_k \frac{2^{k/2} m_0^k (g^{\mathrm{R}})^k b_k}{\beta^{3k/2}} \left\{ \int_{m_0}^{\infty} \mathrm{d}m\, m^{\gamma + 3/2} \mathrm{e}^{(z-1)\bar{n}^{\mathrm{R}}(m) - (\beta - \beta_0)m} \right\}^k . \tag{G.15}$$

We should assume that $(\beta-\beta_0)\geq 0$. In this sense one may consider $1/\beta_0$ as the limiting temperature and the above-mentioned constraint means that the resonance energies should be high enough.

G.2. Thermodynamical parameters

Remembering that the position of the singularity over z is essential, let us assume that the resonance interactions cannot renormalize it, i.e. that the sum (G.15) is convergent. Then, leaving the first term in the sum (G.15),

$$-\beta\mathscr{F}(z,s)=\frac{m_0 g^{\mathrm{R}} C_1}{\beta^{3/2}}\int_{m_0}^{\infty} \mathrm{d}m(m/m_0)^{\gamma+3/2}\,\mathrm{e}^{(z-1)\bar{n}^{\mathrm{R}}(m)-(\beta-\beta_0)m}\,. \tag{G.16}$$

We expect that this assumption is satisfied if

$$\int_{m_0}^{\infty}\mathrm{d}m m^{\gamma+3/2}\,\mathrm{e}^{(z-1)\bar{n}^{\mathrm{R}}(m)-(\beta-\beta_0)m}\gg\frac{2^{1/2}m_0(g^{\mathrm{R}})b_2}{b_1\beta^{3/2}}\left\{\int_{m_0}^{\infty}\mathrm{d}m m^{\gamma+3/2}\,\mathrm{e}^{(z-1)\bar{n}^{\mathrm{R}}(m)-(\beta-\beta_0)m}\right\}^2 \tag{G.17}$$

for

$$n\to\infty\,,\ s\to\infty\,,\quad \frac{nm_0}{\sqrt{s}}\equiv\frac{n}{n_{\max}}\ll 1\,. \tag{G.18}$$

So, we would solve our equations of state with the following free energy:

$$-\beta\mathscr{F}(z,s)=\frac{\alpha}{\beta^{3/2}}\int_{m_0}^{\infty}\mathrm{d}\left(\frac{m}{m_0}\right)\left(\frac{m}{m_0}\right)^{\gamma'-1}\mathrm{e}^{-\Delta(m/m_0)}\,, \tag{G.19}$$

where, using (3.20),

$$\gamma'=\gamma+2(z-1)\bar{n}_0^{\mathrm{R}}+5/2=2(z-1)\bar{n}_0^{\mathrm{R}},\ \Delta=m_0(\beta-\beta_0)\geq 0,\ \alpha=\mathrm{const}\,. \tag{G.20}$$

We have in terms of these new variables the following equation for z:

$$n=z\frac{2\alpha\bar{n}_0^{\mathrm{R}}}{\beta^{3/2}}\frac{\partial}{\partial\gamma'}\frac{\Gamma(\gamma',\Delta)}{\Delta^{\gamma'}}\,. \tag{G.21}$$

The equation for β takes the form

$$n_{\max}=\frac{\alpha m_0}{\beta^{3/2}}\frac{\Gamma(\gamma'+1,\Delta)}{\Delta^{\gamma'+1}}\,, \tag{G.22}$$

where $n_{\max}=(\sqrt{s}/m_0)$ and $\Gamma(\Delta,\gamma')$ is the incomplete Γ-function

$$\Gamma(\gamma',\Delta)=\int_{\Delta}^{\infty}\mathrm{d}x x^{\gamma'-1}\mathrm{e}^{-x}\,.$$

G.3. Asymptotic solutions

Following physical intuition, one should expect the cooling of the system when $n \to \infty$, for fixed $\sqrt{s}$, and heating when $n_{\max} \to \infty$, for fixed n. But, as was mentioned above, since the solution of Eq. (G.22) β_c is defined by the value of the total energy, one should expect that β_c decreases in both cases. So, the solution

$$\Delta_c \geq 0, \frac{\partial \Delta_c}{\partial n} < 0 \text{ at } n \to \infty, \frac{\partial \Delta_c}{\partial s} < 0 \text{ at } s \to \infty \tag{G.23}$$

is natural for our consideration.

The physical meaning of z is activity. It defines at $\beta = \text{const}$ the work needed for the creation of one particle. Then, if the system is stable and $T(z,s)$ may be singular at $z > 1$ only,

$$\frac{\partial z_c}{\partial n} > 0 \text{ at } n \to \infty, \frac{\partial z_c}{\partial s} < 0 \text{ at } s \to \infty \ . \tag{G.24}$$

One should assume solving Eqs. (G.21) and (G.22) that

$$z_c \Delta^{\gamma'_c + 1} \frac{\partial}{\partial \gamma'_c} \frac{\Gamma(\gamma'_c, \Delta_c)}{\Delta_c^{\gamma'_c}} \ll \Gamma(\gamma'_c + 1, \Delta_c) \ . \tag{G.25}$$

This condition contains the physical requirement that $n \ll n_{\max}$. In the opposite case, the finiteness of the phase space for $m_0 \neq 0$ should be taken into account.

As was mentioned above, the singularity z_s attracts z_c at $n \to \infty$. For this reason one may consider the following solutions:

A. $z_s = \infty : z_c \gg \Delta$, $\Delta \ll 1$.

In this case

$$\Delta^{-\gamma'} \Gamma(\gamma', \Delta) \sim e^{\gamma' \ln(\gamma'/\Delta)} \ . \tag{G.26}$$

This estimation gives the following equations:

$$n = C_1 \gamma' \ln(\gamma'/\Delta) e^{\gamma' \ln(\gamma'/\Delta)}, \quad \frac{n}{n_{\max}} = C_2 \Delta \gamma' \ln\left(\frac{\gamma'}{\Delta}\right) \ll 1 \ , \tag{G.27}$$

where $C_i = O(1)$ are the unimportant constants. The inequality is a consequence of (G.25).

These equations have the following solutions:

$$\Delta_c \simeq \frac{n}{n_{\max} \ln n} \ll 1, \quad \gamma'_c \sim \ln n \gg 1 \ . \tag{G.28}$$

Using them one can see from (G.5) that it gives

$$\sigma_n < O(e^{-n}) \ . \tag{G.29}$$

B. $z_s = +1 : z_c \to 1$, $\Delta_c \ll 1$.

One should estimate $\Gamma(\gamma', \Delta)$ near the singularity at $z = 1$ and in the vicinity of $\Delta = 0$ to consider the consequence of this solution. Expanding $\Gamma(\gamma', \Delta)$ over Δ at $\gamma' \to 0$,

$$\Gamma(\gamma', \Delta) = \Gamma(\gamma') - \Delta^{\gamma'} \mathrm{e}^{-\Delta} + \mathrm{O}(\Delta^{\gamma'+1}) \simeq \frac{1}{\gamma'} + \mathrm{O}(1) . \tag{G.30}$$

This gives the following equations for γ':

$$n = C'_1 \frac{\gamma' \ln(1/\Delta) - 1}{\gamma'} \mathrm{e}^{\gamma' \ln(1/\Delta)} . \tag{G.31}$$

The equation for Δ has the form

$$n_{\max} = C'_2 \mathrm{e}^{(\gamma'+1)\ln(1/\Delta)} , \tag{G.32}$$

where $C'_i = \mathrm{O}(1)$ are unimportant constants.

At

$$0 < \gamma' \ln(1/\Delta) - 1 \ll 1, \text{ i.e. at } \ln(1/\Delta) \ll n \ll \ln^2(1/\Delta) , \tag{G.33}$$

we find

$$\gamma'_{\mathrm{c}} \sim \frac{1}{\ln(1/\Delta_{\mathrm{c}})} . \tag{G.34}$$

Inserting this solution into (G.32)

$$\Delta_{\mathrm{c}} \sim \frac{1}{n_{\max}} . \tag{G.35}$$

It is remarkable that Δ_{c} in the leading approximation is n independent. By this reason γ'_{c} becomes n independent also

$$\gamma'_{\mathrm{c}} \sim \frac{1}{\ln(n_{\max})} ; \; z_{\mathrm{c}} = 1 + \frac{1}{\bar{n}_0^{\mathrm{R}} \ln(n_{\max})} . \tag{G.36}$$

This means that

$$\sigma_n = \mathrm{O}(\mathrm{e}^{-n}) \tag{G.37}$$

and obeys the KNO scaling with mean multiplicity $\bar{n} = \bar{n}_0^{\mathrm{R}} \ln(n_{\max})$.

Appendix H. Correlation functions in DIS kinematics

Considering particle creation in the DIS processes, one should distinguish correlation of particles in the (mini)jets and the correlations between (mini)jets. We will start from the description of the jet correlations. One should introduce the inclusive cross section for the ν jets creation

$$\Phi_\nu^{(r)_\nu}(k_1, k_2, \ldots, k_\nu; q^2, x) ,$$

where k_i, $i=1,2,\dots,n$ are the jets 4-momentum in the DIS kinetics, $-q^2 \gg \lambda^2$. Having Φ_ν we can find the correlation functions

$$N_\nu^{(r)}(k_1,k_2,\dots,k_\nu;q^2,x)\ ,$$

where $(r)=r_1,\dots,r_\nu$ and $r_i=(q,g)$ defines the sort of created color particle. It is useful to introduce the generating functional

$$F^{ab}(q^2,x;w)=\sum_n \int \mathrm{d}\Omega_n(k)\prod_{i=1}^{n} w^{r_i}(k_i)|a_n^{ab}(k_1,k_2,\dots,k_n;q^2,x)|^2\ , \tag{H.1}$$

where a_n^{ab} is the amplitude, $\mathrm{d}\Omega_n(k)$ is the phase space volume and $w^{r_i}(k_i)$ are the arbitrary functions. It is evident that

$$F^{ab}(q^2,x;w)|_{w=1}=D^{ab}(q^2,x)\ . \tag{H.2}$$

The inclusive cross sections

$$\Phi_\nu^{(r)}(k_1,k_2,\dots,k_\nu;q^2,x)=\prod_{i=1}^{\nu}\frac{\delta}{\delta w^{r_i}(k_i)}F^{ab}(q^2,x;w)|_{w=1}\ . \tag{H.3}$$

The correlation function

$$N_\nu^{(r)}(k;q^2,x)=\prod_{i=1}^{\nu}\frac{\delta}{\delta w^{r_i}(k_i)}\ln F^{ab}(q^2,x;w)|_{w=1}\ . \tag{H.4}$$

We can find the partial structure functions $D^{ab}(q^2,x;n)$, where n is the number of produced (time-like) gluons, using their definitions.

It will be useful to introduce the Laplace transform over the variable $\ln(1/x)$:

$$F^{ab}(q^2,x;w)=\int_{\mathrm{Re}\,j<0}\frac{\mathrm{d}j}{2\pi\mathrm{i}}\left(\frac{1}{x}\right)^{j} f^{ab}(q^2,j;w)\ . \tag{H.5}$$

The expansion parameter of our problem $\alpha_s\ln(-q^2/\lambda^2)\sim 1$. For this reason one should take into account all possible cuts of the ladder diagrams. So, calculating $D^{ab}(q^2,x)$ in the LLA all possible cuts of the skeleton ladder diagrams are defined by the factor [78]:

$$\frac{1}{\pi}\{\Gamma_r^{ab}G_r\Gamma_r^{ab}\}\ , \tag{H.6}$$

i.e. the cut line may not only get through the exact Green function $G_r(k_i^2)$ but through the exact vertex functions $\Gamma_r^{ab}(q_i,q_{i+1},k_i)$ also (q_i^2,q_{i+1}^2 are negative). We have in the LLA (see Appendix D)

$$\lambda^2 \ll -q_i^2 \ll -q_{i+1}^2 \ll -q^2$$

and

$$x\le x_{i+1}\le x_i\le 1\ .$$

Following our approximation, see the previous section, we could not distinguish the way in which the cut line goes through the Born amplitude

$$a_r^{ab} = \{(\Gamma_r^{ab})^2 G_r\} \; .$$

We will simply associate $w^r \operatorname{Im} a_r^{ab}$ to each rung of the ladder.

Considering the asymptotics over n, the time-like partons virtuality $k_i \simeq -q_i^2/y_i$ should be maximal. Here y_i is the fraction of the longitudinal momentum of the jet. Then, slightly limiting the jets phase space,

$$\ln k_i^2 = \ln |q_{i+1}|^2 (1 + \mathrm{O}(\ln(1/x)/|q_{i+1}|^2)) \; . \tag{H.7}$$

As a result, introducing $\tau_i = \ln(q_i^2/\Lambda^2)$, where $\alpha_s(q^2) = 1/\beta\tau$, $\beta = (11N/3) - (2n_f/3)$ in the LLA variable, we can find the following set of equations:

$$\tau \frac{\partial}{\partial \tau} f_{ab}(q^2, j; w) = \sum_{c,r} \varphi_{ac}^r(j) w^r(\tau) f_{ab}(q^2, x; w) \; , \tag{H.8}$$

where

$$\varphi_{ac}^r(j) = \varphi_{ac}(j) = \int_0^1 \frac{\mathrm{d}x}{x} x^j P_{ac}(x) \tag{H.9}$$

and $P_{ac}(x)$ is the regular kernel of the Bethe–Salpeter equation [78]. At $w = 1$ this equation is the ordinary one for $D^{ab}(q^2, x)$.

We will search the correlation functions from Eq. (H.8) in terms of the Laplace transform

$$n_{ab}^{(r)_\nu}(k_1, k_2, \ldots, k_\nu; q^2, j) = n_{ab}^{(r)_\nu}(k; q^2, j) \; .$$

Let us write

$$f_{ab}(q^2, j; w) = d_{ab}(q^2, j) \exp\left\{ \sum_\nu \frac{1}{\nu!} \int \prod_{i=1}^{\nu} \left(\frac{\mathrm{d}\tau_i}{\tau_i} (w^{r_i}(\tau_i) - 1) \right) n_{ab}^{(r)_\nu}(k; q^2, j) \right\} . \tag{H.10}$$

Inserting (H.10) into (H.8) and expanding over $(w-1)$ we find the sequence of coupled equations.

Omitting the cumbersome calculations, we write in the LLA that

$$\phi_{ab}^{(r)_\nu}(\tau_1, \tau_2, \ldots, \tau_\nu; q^2, j) = d_{ac_1}(j, \tau_1) \varphi_{c_1 c_2}^{r_1}(j) d_{c_2 c_3}(j, \tau_2) \cdots \varphi_{c_\nu c_{\nu+1}}^{r_\nu}(j) d_{c_{\nu+1} b}(j, \tau_{\nu+1}) \; . \tag{H.11}$$

One should take into account the conservation laws:

$$\tau_1 \cdot \tau_2 \cdots \tau_{\nu+1} = \tau, \quad \tau_1 < \tau_2 < \cdots < \tau_{\nu+1} < \tau \; . \tag{H.12}$$

Computing the Laplace transform of this expression we find

$$\Phi_{ab}^{(r)_\nu}(\tau_1, \tau_2, \ldots, \tau_\nu; q^2, x) \; .$$

The kernel $d_{ab}(j, \tau)$ was introduced in (H.11). Let us write it in the form

$$d_{ab}(j, \tau) = \sum_{\sigma = \pm} \sigma \frac{d_{ab}(j)}{\nu_+ - \nu_-} \tau^{\nu_\sigma(j)} \; , \tag{H.13}$$

where

$$d^{\sigma}_{qq} = \nu_{\rm s} - \varphi_{gg}, \quad d^{\sigma}_{qg} = \nu_{\rm s} - \varphi_{qq}, \quad d^{\sigma}_{gg} = \varphi_{gq}, \quad d^{\sigma}_{gq} = \varphi_{qg} \tag{H.14}$$

and

$$\nu_{\sigma} = \tfrac{1}{2}\{\varphi_{qq} + \varphi_{gg} + \sigma[(\varphi_{qq} - \varphi_{gg})^2 - 4n_f\varphi_{qg}\varphi_{gq}]^{1/2}\} . \tag{H.15}$$

If $x \ll 1$, then $(j-1) \ll 1$ are essential. In this case [78],

$$\varphi_{gg} \sim \varphi_{gq} \gg \varphi_{qg} \sim \varphi_{qq} = \mathrm{O}(1) . \tag{H.16}$$

This means that the gluon jets dominate and

$$n^{g}_{gg} = \varphi_{gg} + \mathrm{O}(1) . \tag{H.17}$$

One can find the following estimation of the two-jet correlation function:

$$n^{r_1 r_2}_{ab}(\tau_1, \tau_2; \jmath, \tau) = \mathrm{O}(\max\{(\tau_1/\tau)^{\varphi_{gg}}, (\tau_2/\tau)^{\varphi_{gg}}, (\tau_1/\tau_2)^{\varphi_{gg}}\}) . \tag{H.18}$$

This correlation function is small since in the LLA $\tau_1 < \tau_2 < \tau$. This means that the jet correlation becomes high if and only if the masses of the correlated jets are comparable. But this condition shrinks the range of integration over τ and for this reason one may neglect the 'short-range' correlations among jets. Therefore, as follows from (H.10),

$$f_{ab}(q^2, j; w) = d_{gg}(\tau, j)\exp\left\{\varphi_{gg}\int_{\tau_0}^{\tau}\frac{d\tau'}{\tau'}w^g(\tau')\right\} . \tag{H.19}$$

We will use this expression to find the multiplicity distribution in the DIS domain.

H.1. Generating function

To describe particle production, one should replace:

$$w^r \,\mathrm{Im}\, a^{ab}_r \to w^r_n \,\mathrm{Im}\, a^{ab}_r ,$$

where w^r_n is the *probability* of n particle production,

$$\sum_n w^r_n = 1 . \tag{H.20}$$

Having ν jets, one should take into account the conservation condition $n_1 + n_2 + \cdots + n_\nu = n$. For this reason, the generating functions formalism is useful. As a result, one can find that if we take (H.19)

$$w^g = w^g(\tau, z), \quad w^g(\tau, z)|_{z=1} = 1 , \tag{H.21}$$

then $f_{ab}(q^2, j; w)$ defined by (H.19) is the generating functional of the multiplicity distribution in the 'j representation'. In this expression $w^g(\tau, z)$ is the generating function of the multiplicity distribution in the jet of mass $|k| = \lambda e^{\tau/2}$.

As a result, see (H.5),

$$F^{ab}(q^2,x;w) \propto \int_{\mathrm{Re}\, j<0} \frac{\mathrm{d}j}{2\pi\mathrm{i}} (1/x)^j \mathrm{e}^{\varphi_{gg}\,\omega(\tau,z)} , \tag{H.22}$$

where

$$\omega(\tau,z) = \int_{\tau_0}^{\tau} \frac{\mathrm{d}\tau'}{\tau'} w^g(\tau',z) . \tag{H.23}$$

Noting the normalization condition (H.21),

$$\omega(\tau,z=1) = \ln\tau . \tag{H.24}$$

The integral (H.22) may be calculated by the steepest descent method. It is not hard to see that

$$j \simeq j_{\mathrm{c}} = 1 + \{4N\omega(\tau,z)/\ln(1/x)\}^{1/2} \tag{H.25}$$

is essential. Notice that $j-1 \ll 1$ should be essential but we find, instead of the constraint (3.2), that

$$\omega(\tau,z) \ll \ln(1/x) . \tag{H.26}$$

In the frame of this constraint,

$$F^{ab}(q^2,x;w) \propto \exp\{4\sqrt{N\omega(\tau,z)\ln 1/x)}\} . \tag{H.27}$$

Generally speaking, there exist such values of z that $j_{\mathrm{c}} - 1 \sim 1$. This is possible if $\omega(\tau,z)$ is a regular function of z at $z=1$. Then z_{c} should be an increasing function of n and consequently $\omega(\tau,z_{\mathrm{c}})$ would be an increasing function of n. Therefore, one may expect that in the VHM domain $j_{\mathrm{c}} - 1 \sim 1$.

Then $j \simeq 1 + \omega(\tau,z)/\ln(1/x)$ would be essential in the integral (H.22). This leads to the following estimation:

$$F^{ab}(q^2,x;w) \propto \mathrm{e}^{-\omega(\tau,z)} .$$

But this is impossible since $F^{ab}(q^2,x;w)$ should be an increasing function of z. This shows that the estimation (H.27) has a finite range of validity.

Solution of this problem with unitarity is evident. One should take into account correlations among jets considering the expansion (H.10). Indeed, smallness of $n_{ab}^{(r)_\nu}$ may be compensated by large values of $\prod_i^{\nu} w^{r_i}(\tau_i,z)$ in the VHM domain.

Appendix I. Solution of the jets evolution equation

One may neglect quark jets in the VHM region since the gluons mean multiplicity $\bar{n}_g > \bar{n}_q$-quarks multiplicity [81,85] and in the VHM region the leftmost singularities are important. Then we can write [84]

$$\frac{\partial}{\partial\tau} T_j(\tau,z) = \frac{12}{11} T_j(\tau,z) \int_{\tau_0}^{\tau} \mathrm{d}\tau' (T_j(\tau',z) - 1) , \tag{I.1}$$

where $\tau = \ln(q^2/\lambda^2)$ and $T_j(\tau, z)$ is the generating function of the distribution over the number of gluons w_n(tau):

$$T_j(\tau, z) = \sum_n z^n w_n(\tau), \quad T_j(\tau, z = 1) = 1 . \tag{I.2}$$

We search a solution in the VHM region, where

$$n \gg n_j \propto \exp\{\sqrt{a\tau}\}, \quad a = \tfrac{12}{11} . \tag{I.3}$$

Let us consider the following solution:

$$w_n = \left(\frac{n}{\bar{n}_j}\right)^{\gamma} \mathrm{e}^{-\alpha n/\bar{n}_j} . \tag{I.4}$$

It is useful to introduce

$$\alpha_k(\tau) = \frac{1}{k!} \sum_{n=1}^{\infty} n^{k-1} w_n(\tau) \tag{I.5}$$

for this solution. Inserting (I.4) into this expression,

$$\alpha_k(\tau) = \bar{n}^k(\tau)\beta_k , \tag{I.6}$$

where β_k (i) should be positive and (ii) τ independent.

These conditions are satisfied for the following values of k. Indeed, at $k \gg 1$:

$$\beta_k = \frac{1}{k!} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{n}{\bar{n}_j}\right)^{k+\gamma} \mathrm{e}^{-\alpha n/\bar{n}_j} \simeq \alpha^{-(k+\gamma)} \frac{\Gamma(k+\gamma)}{\Gamma(k+1)} . \tag{I.7}$$

The generating function T_j has the following form in terms of α_k:

$$T_j(\tau, z) = \sum_{n=1}^{\infty} (\ln z)^k \alpha_k(\tau) . \tag{I.8}$$

Inserting (I.8) into (I.1) and assuming that β_k is a τ-independent quantity, we find the following recurrent equation for β_k:

$$\beta_k = \frac{4}{k} \sum_{k_1=1}^{k} \frac{1}{k_1} \beta_{k_1} \beta_{k-k_1} - 2\frac{\sqrt{a\tau}}{k} \beta_k . \tag{I.9}$$

Therefore, if

$$k \gg \sqrt{a\tau} , \tag{I.10}$$

then we can neglect the last term on the right-hand side of (I.9) and in this case β_k are positive and τ independent. Noting that in (I.7) $n \sim k\bar{n}_j$ are essential inequality (I.10) means that the solution (I.4) is correct if

$$n \gg \bar{n}_j \ln \bar{n}_j , \tag{I.11}$$

i.e., only for this value of n w_n has the form (I.4) and the corresponding generating functional has singularity at

$$z_s = 1 + \frac{\alpha}{\bar{n}_j} . \tag{I.12}$$

Appendix J. Condensation and type of asymptotics over multiplicity

It is important for the VHM experiment to have an upper restriction on the asymptotics. We wish to show that σ_n decreases faster than any power of $1/n$:

$$\sigma_n < O(1/n) . \tag{J.1}$$

To prove this estimation, one should know the type of singularity at $z = 1$.

One can imagine that the points, where the external particles are created, form the system. Here we assume that this system is in equilibrium, i.e. in this system, there are no macroscopical flows of energy, particles, charges and so on.

The lattice gas approximation is used to describe such a system. This description is quite general and does not depend on details. Motion of the gas particles leads to the necessity to sum over all distributions of the particles on cells. For simplicity, we will assume that only one particle can occupy the cell.

So, we will introduce the occupation number $\sigma_i = \pm 1$ in the ith cell: $\sigma_i = +1$ means that we have no particle in the cell and $\sigma_i = -1$ means that a particle exists in a cell. Assuming that the system is in equilibrium, we may use the ergodic hypothesis and sum over all 'spin' configurations of σ_i, with the restriction: $\sigma_i^2 = 1$. It is evident that this restriction introduces the interactions [100].

The corresponding partition function in temperature representation [52]

$$\rho(\beta, H) = \int D\sigma \mathrm{e}^{-S\lambda(\sigma)} , \tag{J.2}$$

where integration is performed over $|\sigma(x)| \leq \infty$ and, considering the continuum limit, $D\sigma = \prod_x \mathrm{d}\sigma(x)$. The action

$$S_\lambda(\sigma) = \int \mathrm{d}x \left\{ \frac{1}{2}(\nabla\sigma)^2 - \omega\sigma^2 + g\sigma^4 - \lambda\sigma \right\} , \tag{J.3}$$

where

$$\omega \sim \left(1 - \frac{\beta_{\mathrm{cr}}}{\beta}\right), \quad g \sim \frac{\beta_{\mathrm{cr}}}{\beta}, \quad \lambda \sim \left(\frac{\beta_{\mathrm{cr}}}{\beta}\right)^{1/2} \beta H . \tag{J.4}$$

and $1/\beta_{\mathrm{cr}}$ is the critical temperature.

J.1. Unstable vacuum

We start this consideration from the case $\omega > 0$, i.e. assuming that $\beta > \beta_{\mathrm{cr}}$. In this case the ground state is degenerate if $H = 0$. The extra term $\sim \sigma H$ in (J.3) can be interpreted as the interaction with external magnetic field H. This term regulates the number of 'down' spins with $\sigma = -1$ and is related to the activity

$$z^{1/2} = e^{\beta H} , \tag{J.5}$$

i.e. H coincides with the chemical potential.

The potential

$$v(\sigma) = -\omega\sigma^2 + g\sigma^4, \quad \omega > 0 , \tag{J.6}$$

has two minima at

$$\sigma_\pm = \pm\sqrt{\omega/2g} .$$

If the dimension $d > 1$, no tunnelling phenomena exist. But choosing $H < 0$ the system in the correct minimum (it corresponds to the state without particles) becomes unstable. The system tunnels into the state with an absolute minimum of energy.

The partition function $\rho(\beta, z)$ becomes singular at $H = 0$ because of this instability. The square root branch point gives

$$\mathrm{Im}\,\rho(b, z) = \frac{a_1(\beta)}{H^4} e^{-a_2(\beta)/H^2}, \quad a_i > 0 . \tag{J.7}$$

Note, $\mathrm{Im}\,\rho(b, z) = 0$ at $H = 0$. Deforming the contour in the Mellin integral over z on the branch line,

$$\rho_n(\beta) = \frac{1}{\pi}\int_1^\infty \frac{dz}{z^{n+1}} \frac{8a_1\beta^4}{\ln^4 z} e^{-4a_2\beta^2/\ln^2 z} . \tag{J.8}$$

In this integral

$$z_c \propto \exp\left\{\frac{8a_2\beta^2}{n}\right\}^{1/3} \tag{J.9}$$

is essential. This leads to the following estimation:

$$\rho_n \propto e^{-3(a_2\beta^2)^{1/3} n^{2/3}} < O(1/n) . \tag{J.10}$$

It is useful to note at the end of this section that

(i) The value of ρ_n is defined by $\mathrm{Im}\,\rho(b, z)$ and the metastable states, the decay of which gives a contribution to $\mathrm{Re}\,\rho(b, z)$, are not important.

(ii) It follows from (J.9) that in the VHM domain

$$H \sim H_c \sim \ln z_c \sim (1/n)^{1/3} \to 0 . \tag{J.11}$$

So, the calculations are performed for the 'weak' external field case, when the degeneracy is weakly broken. It is evident that the lifetime of the unstable (without particles) state is large in this case and for this reason the semiclassical approximation is correct. This is an important consequence of (3.1).

J.2. Stable vacuum

Let us consider now $\omega < 0$, i.e. $\beta < \beta_{cr}$. The potential (J.6) has only one minimum at $\sigma = 0$ in this case. The inclusion of an external field shifts the minimum to the point $\sigma_c = \sigma_c(H)$. In this case the expansion in the vicinity of σ_c should be useful. As a result,

$$\rho(\beta, z) = \exp\left\{\int dx \lambda \sigma_c - W(\sigma_c)\right\}, \tag{J.12}$$

where $W(\sigma_c)$ can be expanded over σ_c:

$$W(\sigma_c) = \sum_l \frac{1}{l} \int \prod_k \{dx_k \sigma_c(x_k; H)\} \tilde{b}_l(x_1, \ldots, x_l) . \tag{J.13}$$

In this expression, $\tilde{b}_l(x_1, \ldots, x_l)$ is the one-particle irreducible Green function, i.e. $\tilde{b}_l$ is the virial coefficient. Then σ_c can be considered as the effective activity of the correlated l-particle group.

The sum in (J.13) should be convergent and, therefore, $|s_c| \to \infty$ if $|H| \to \infty$. But in this case the virial decomposition is equivalent to the expansion over the inverse density of particles [25]. In the VHM region it is high and the mean field approximation becomes correct. As a result,

$$\sigma_c \simeq -\left(\frac{|\lambda|}{4g}\right)^{1/3} : |s_c| \to \infty \quad \text{if } |\lambda| \to \infty \tag{J.14}$$

and

$$\rho(\beta, z) \propto e^{(3|\lambda|^{4/3})/((4g)^{1/3})} \left\{12g\left(\frac{|\lambda|}{4g}\right)^{2/3}\right\}^{-1/2} . \tag{J.15}$$

We can use this expression to calculate ρ_n. In this case

$$z_c \propto e^{4gn^3} \to \infty \quad \text{at } n \to \infty , \tag{J.16}$$

is essential and in the VHM domain

$$\rho_n \propto e^{-4gn^4} < O(e^{-n}) . \tag{J.17}$$

This result is an evident consequence of vacuum stability. It should be noted once more that the conditions (3.1) considerably simplify calculations.

Appendix K. New multiple production formalism and integrable systems

K.1. S-matrix unitarity constraints

To explain our idea, let us consider the spectral representation of the one-particle amplitude:

$$A_1(x_1, x_2; E) = \frac{\Psi_n^*(x_2)\Psi_n(x_1)}{E - E_n - i\varepsilon}, \quad \varepsilon \to +0 . \tag{K.1}$$

It describes the transition of a particle with energy E from point x_1 to x_2. According to our general idea, see introduction to Section 2.1, we will calculate

$$R_1(E) = \int dx_1\, dx_2\, A_1(x_1, x_2; E) A_1^*(x_1, x_2; E) . \tag{K.2}$$

The integration over the end points x_1 and x_2 is performed only for the sake of simplicity.

Inserting (K.1) into (K.2) and using ortho-normalizability of the wave functions $\Psi_n(x)$ we find that

$$\varepsilon R_1(E) = \varepsilon \sum_n \left| \frac{1}{E - E_n - i\varepsilon} \right|^2 = \frac{1}{2i} \sum_n \left\{ \frac{1}{E - E_n - i\varepsilon} - \frac{1}{E - E_n + i\varepsilon} \right\}$$

$$= \mathrm{Im} \sum_n \frac{1}{E - E_n - i\varepsilon} = \pi \sum_n \delta(E - E_n) . \tag{K.3}$$

On the other hand, the closed-path amplitude, offered for calculation in [101],

$$C_1(E) = \sum_n \int dx \frac{\Psi_n^*(x) \Psi_n(x)}{E - E_n - i\varepsilon} = \sum_n \frac{1}{E - E_n - i\varepsilon}$$

$$= \sum_n \left\{ \mathscr{P} \frac{1}{E - E_n} + i\pi\delta(E - E_n) \right\} = \sum_n \mathscr{P} \frac{1}{E - E_n} + i\varepsilon R_1(E) . \tag{K.4}$$

So, we wish to calculate only the imaginary part of the closed-path contribution

$$\varepsilon R(E) = \mathrm{Im}\, C_1(E) .$$

Notice the extra factor ε on the left-hand side.

The reason for this choice is evident: the real part of $C_1(E)$ is equal to zero at $E = E_n$, i.e. did not contribute to the measurable. To calculate the bound states energy spectrum, it is enough to know only the imaginary part of the closed-path amplitude.

This property is not accidental. It is known as the optical theorem and is the consequence of the total probability conservation principles. The formal realization of this is the unitarity condition for the $\boldsymbol{S}$-matrix: $\boldsymbol{S}\boldsymbol{S}^+ = \boldsymbol{I}$. In terms of the amplitudes $\boldsymbol{A}$, $\boldsymbol{S} = \boldsymbol{I} + i\boldsymbol{A}$, the unitarity condition presents an infinite set of nonlinear operator equalities:

$$i\boldsymbol{A}\boldsymbol{A}^* = \boldsymbol{A} - \boldsymbol{A}^* . \tag{K.5}$$

Notice that expressing the amplitude by the path integral one can see that the left-hand side of this equality offers the double integral and, at the same time, the right-hand side is the linear combination of integrals. Thus, the continuum contributions into the amplitudes should be canceled to provide the conservation of total probability. In this sense it is a necessary condition.

Indeed, to see the integral form of our approach, let us use the proper-time representation:

$$A_1(x_1, x_2; E) = \sum_n \Psi_n(x_1)\Psi_n^*(x_2) i \int_0^\infty \mathrm{d}T \mathrm{e}^{\mathrm{i}(E - E_n + \mathrm{i}\varepsilon)T} \tag{K.6}$$

and insert it into (K.2):

$$R_1(E) = \sum_n \int_0^\infty \mathrm{d}T_+ \, \mathrm{d}T_- \, \mathrm{e}^{-(T_+ + T_-)\varepsilon} \mathrm{e}^{\mathrm{i}(E - E_n)(T_+ - T_-)} \,. \tag{K.7}$$

We will introduce new time variables instead of $T_\pm$:

$$T_\pm = T \pm \tau \,, \tag{K.8}$$

where, as follows from the Jacobian of transformation, $|\tau| \leq T$, $0 \leq T \leq \infty$. But we can put $|\tau| \leq \infty$ since $T \sim 1/\varepsilon \to \infty$ is essential in the integral over T. As a result,

$$\rho_1(E) = 2\pi \sum_n \int_0^\infty \mathrm{d}T \mathrm{e}^{-2\varepsilon T} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau}{\pi} \mathrm{e}^{2\mathrm{i}(E - E_n)\tau} \,. \tag{K.9}$$

In the last integral, the continuum of contributions with $E \neq E_n$ are canceled. Note that the product of amplitudes AA^* was 'linearized' after the introduction of 'virtual' time [103] $\tau = (T_+ - T_-)/2$.

We wish to calculate the density matrix $\rho(\beta, z)$ including the consequence of the unitarity condition cancelation of unnecessary contributions. Here we demonstrate the result and the intermediate steps we will formulate, without proof, as the statements offered in [16,15], where the formalities are described.

K.2. Dirac measure

The statement, see [15] and references cited therein,

S1. *The unitarity condition unambiguously determines contributions in the path integrals for* ρ looks like a tautology since $\mathrm{e}^{\mathrm{i}S(x)}$, where $S(x)$ is the action, is the unitary operator which shifts a system along the trajectory.[11] So, it seems evident that the unitarity condition is already included in the path integrals.

The rule as the path integrals should be calculated is well known, see e.g. [102]. Nevertheless, the general path-integral solution contains unnecessary degrees of freedom (unobservable states with $E \neq E_n$ in the above example). We would define the path integrals in such a way that the condition of absence of unnecessary contributions in the final (measurable) result be loaded from the very beginning. Just in this sense, the unitarity looks like the necessary and sufficient condition unambiguously determining the complete set of contributions.

[11] It is well known that this unitary transformation is the analogy of the tangent transformations of classical mechanics [103].

S2. *The m- into n-particles transition (unnormalized) probability R_{nm} would have on the Dirac measure the following symmetrical form*

$$R_{nm}(p_1,\dots,p_n,q_1,\dots,q_m) = \left\langle \prod_{k=1}^{m} |\Gamma(q_k;u)|^2 \prod_{k=1}^{n} |\Gamma(p_k;u)|^2 \right\rangle_u$$

$$= \mathrm{e}^{-\mathrm{i}\boldsymbol{K}(j,e)} \int DM(u) \mathrm{e}^{\mathrm{i}S_O(u) - \mathrm{i}U(u,e)} \prod_{k=1}^{m} |\Gamma(q_k;u)|^2 \prod_{k=1}^{n} |\Gamma(p_k;u)|^2 \equiv \hat{\mathcal{O}}(u) \prod_{k=1}^{m} |\Gamma(q_k;u)|^2 \prod_{k=1}^{n} |\Gamma(p_k;u)|^2 . \tag{K.10}$$

Here $p(q)$ are the in(out)-going particle momenta. It should be underlined that this representation is strict and is valid for arbitrary Lagrange theory of arbitrary dimensions. The eikonal approximation for inelastic amplitudes was considered in [104].

The operator $\hat{\mathcal{O}}$ contains three elements, the Dirac measure DM, the functional $U(x,e)$ and the operator $\boldsymbol{K}(j,e)$.

The expansion over the operator

$$\boldsymbol{K}(j,e) = \frac{1}{2}\mathrm{Re}\int_{C_+} \mathrm{d}x\,\mathrm{d}t \frac{\delta}{\delta j(x,t)} \frac{\delta}{\delta e(x,t)} \equiv \frac{1}{2}\mathrm{Re}\int_{C_+} \mathrm{d}x\,\mathrm{d}t \hat{j}(x,t)\hat{e}(x,t) \tag{K.11}$$

generates the perturbation series. We will assume that this series exists (at least in Borel sense).

The functionals $U(u,e)$ and $S_O(u)$ are defined by the equalities

$$S_O(u) = (S_0(u+e) - S_0(u-e)) + 2\,\mathrm{Re}\int_{C_+} \mathrm{d}x\,\mathrm{d}t e(x,t)(\partial^2 + m^2)u(x,t) , \tag{K.12}$$

$$U(u,e) = V(u+e) - V(u-e) - 2\,\mathrm{Re}\int_{C_+} \mathrm{d}x\,\mathrm{d}t e(x,t) v'(u) , \tag{K.13}$$

where $S_0(u)$ is the free part of the Lagrangian and $V(u)$ describes interactions. The quantity $S_O(u)$ is not equal to zero if u have nontrivial topological charge (see also [105]).

According to S1, considering motion in the phase space (u,p) the measure $\mathrm{DM}(u,p)$ has the Dirac form

$$\mathrm{DM}(u,p) = \prod_{x,t} \mathrm{d}u(x,t)\,\mathrm{d}p(x,t) \delta\left(\dot{u} - \frac{\delta H_j(u,p)}{\delta p} \right) \delta\left(\dot{p} + \frac{\delta H_j(u,p)}{\delta u} \right) \tag{K.14}$$

with the total Hamiltonian

$$H_j(u,p) = \int \mathrm{d}x \left\{ \frac{1}{2}p^2 + \frac{1}{2}(\nabla u)^2 + v(u) - ju \right\} . \tag{K.15}$$

This last one includes the energy ju of quantum fluctuations.

The measure (K.14) contains the following information:

a. Only *strict* solutions of the equations

$$\dot{u} - \frac{\delta H_j(u,p)}{\delta p} = 0, \quad \dot{p} + \frac{\delta H_j(u,p)}{\delta u} = 0 \tag{K.16}$$

with $j = 0$ should be taken into account. This 'rigidness' of the formalism means the absence of pseudo-solutions (similar to multi-instanton, or multi-kink) contribution.
b. ρ_{nm} is described by the *sum* of all solutions of Eq. (K.16), independent of their 'nearness' in the functional space;
c. ρ_{nm} did not contain the interference terms from various topologically nonequivalent contributions. This displays the orthogonality of the corresponding Hilbert spaces;
d. The measure (K.14) includes $j(x)$ as the external adiabatic source. Its fluctuation disturbs the solutions of Eq. (K.16) and vice versa since the measure (K.14) is strict;
e. In the frame of the adiabaticity condition, the field disturbed by $j(x)$ belongs to the same manifold (topology class) as the classical field defined by (K.16) [105].
f. The Dirac measure is derived for *real-time* processes only, i.e. (K.14) is not valid for tunneling ones. For this reason, the above conclusions should be taken carefully.
g. It can be shown that theory on the measure (K.14) restores ordinary (canonical) perturbation theory.

The parameter $\Gamma(q;u)$ is connected directly with *external* particle energy, momentum, spin, polarization, charge, etc., and is sensitive to the symmetry properties of the interacting fields system.[12] For the sake of simplicity, $u(x)$ is the real scalar field. The generalization would be evident.

As a consequence of (K.14), $\Gamma(q;u)$ is the function of the external particle momentum q and is a *linear* functional of $u(x)$:

$$\Gamma(q;u) = -\int \mathrm{d}x e^{iqx} \frac{\delta S_0(u)}{\delta u(x)} = \int \mathrm{d}x e^{iqx}(\partial^2 + m^2)u(x), \quad q^2 = m^2 , \tag{K.17}$$

for the mass m field. This parameter presents the momentum distribution of the interacting field $u(x)$ on the remote hypersurface σ_∞ if $u(x)$ is the regular function. Notice, the operator $(\partial^2 + m^2)$ cancels the mass-shell states of $u(x)$.

The construction (K.17) means, because of the Klein–Gordon operator and the external states being mass-shell by definition [40], that the solution $\rho_{nm} = 0$ is possible for a particular topology (compactness and analytic properties) of *quantum* field $u(x)$. So, $\Gamma(q;u)$ carries the following remarkable properties:

- it directly defines the observables,
- it is defined by the topology of $u(x)$,
- it is the linear functional of the actions symmetry group element $u(x)$.

Notice, the space-time topology of $u(x,t)$ becomes important in calculating integral (3.2) by parts. This procedure is available if and only if $u(x,t)$ is the regular function. But the *quantum* fields are always singular. Therefore, the solution $\Gamma(q;u) = 0$ is valid if and only if the quasiclassical approximation is exact. Just this situation is realized in the soliton sector of sin-Gordon model.

[12] The following trivial analogy with ferromagnetic may be useful. So, the external magnetic field $\mathscr{H} \sim \bar{\mu}$, if $\bar{\mu}$ is the magnetics order parameter, and the phase transition means that $\bar{\mu} \neq 0$. $\Gamma(q,u)$ has just the same meaning as $\mathscr{H}$.

Despite evident ambiguity $\Gamma(q;u)$ carries the definite properties of the order parameter since the opposite solution $\rho_{nm}=0$ can only be the dynamical display of an *unbroken* symmetry,[13] i.e. of the nontrivial topology of interacting fields, as the consequence of unbroken symmetry.

If (K.16) have nontrivial solution $u_c(x,t)$, then this 'extended objects' quantization problem [107] arises. We solve it by introducing convenient dynamical variables [15]. The main formal difficulty, see e.g. [108], of this program consists of transformation of the path-integral measure which was solved in [105].[14]

Then

S3. *The measure* (K.14) *admits the transformation*

$$u_c\colon (u,p)\to(\xi,\eta)\in W=G/G_c\ . \tag{K.18}$$

and the transformed measure has the form

$$\mathrm{DM}(u,p)=\prod_{x,tC}\mathrm{d}\xi(t)\,\mathrm{d}\eta(t)\delta\left(\dot{\xi}-\frac{\delta h_j(\xi,\eta)}{\delta\eta}\right)\delta\left(\dot{\eta}+\frac{\delta h_j(\xi,\eta)}{\delta\xi}\right), \tag{K.19}$$

where $h_j(\xi,\eta)=H_j(u_c,p_c)$ is the transformed Hamiltonian.

It is evident that (ξ,η) are parameters of integration of Eqs. (K.16) and they form the factor space $W=G/G_c$. For instance, if one particle dynamics is considered, then one may choose $\xi=x(0)$ and $\eta=p(0)$. One may consider also the following possibility:

$$\xi=\int^x\frac{\mathrm{d}u}{\sqrt{2(\eta-v(u))}}$$

and

$$\eta=p^2/2+v(x)\ .$$

In these terms $h_j=\eta-j(t)u_c(\xi,\eta)$ and new Hamilton equations have the form

$$\dot{\xi}=1-j\frac{\partial u_c(\xi,\eta)}{\partial\eta},\quad \dot{\eta}=j\frac{\partial u_c(\xi,\eta)}{\partial\xi}\ . \tag{K.20}$$

So, we have at $j=0$: $\xi=t+t_0$ and $\eta=\eta_0$. For this reason

S4. *The* (*action*, *angle*)-*type variables are most useful.*

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[13] The S-matrix was introduced 'phenomenologically', see also the example considered in [106,95], postulating the LSZ reduction formulae, see Eq. (B.1). So, the formal constraints, e.g. the Haag theorem, would not be taken into account on the chosen level of accuracy.

[14] A number of problems of quantum mechanics were solved using also the 'time sliced' method [109]. This approach presents the path integral as the finite product of well-defined ordinary integrals and, therefore, allows to perform arbitrary space and space-time transformations. But the transformed 'effective' Lagrangian gains an additional term $\sim\hbar^2$. The latter crucially depends on the way in which the 'slicing' was performed. This phenomenon considerably complicates calculations and the general solution of this problem is unknown to us. It is evident that this method is especially effective if the quantum corrections $\sim\hbar$ play no role. Such models are well known. For instance, the Coulomb model in quantum mechanics, the sine-Gordon model in field theory, where the bound-state energies are exactly quasiclassical.

According to (K.18) there exists transformation of the perturbation generating operator:

S5. *The operator* $\boldsymbol{K}$ *has the following transformed form*

$$2\boldsymbol{K} = \int \mathrm{d}t\{\hat{j}_\xi \cdot \hat{e}_\xi + \hat{j}_\eta \cdot \hat{e}_\eta\} , \tag{K.21}$$

in the factor space, where j_X, e_X, $X = \xi, \eta$, *are new auxiliary variables.*

As a result of mapping of the perturbation generating operator $\boldsymbol{K}$ on the manifold W the equations of motion became linearized:

$$\mathrm{DM} = \prod_t \delta\left(\dot{\xi} - \frac{\delta h(\eta)}{\delta \eta} - j_\xi\right)\delta(\dot{\eta} - j_\eta) . \tag{K.22}$$

Then

S6. *If Feynman's* iε*-prescription is adopted, then the Green function of Eq.* (*K*.22)

$$g(t - t') = \Theta(t - t') . \tag{K.23}$$

Later on we will consider the soliton sector of the sin-Gordon model. In this case ξ_i is the coordinate and η_i is the momentum of the ith soliton and N is the number of solitons.

Expansion of $\exp\{\mathrm{K}(je)\}$ gives the 'strong coupling' perturbation series. Its analysis shows that

S7. *Action of the integro-differential operator* $\hat{\mathcal{O}}$ *leads to the following representation*:

$$R_{nm}(p, q) = \int_W \left\{\mathrm{d}\xi(0) \cdot \frac{\partial}{\partial \xi(0)} R^{\xi}_{nm}(p, q) + \mathrm{d}\eta(0) \cdot \frac{\partial}{\partial \xi(0)} R^{\eta}_{nm}(p, q)\right\} . \tag{K.24}$$

This means that the contributions into $R_{nm}(p, q)$ are accumulated strictly on the boundary 'bifurcation manifold' ∂W [110], i.e. depends directly on the topology of W.

K.3. Multiple production in sin-Gordon model

Let us consider now the completely integrable sin-Gordon model. For the sake of simplicity the integral

$$R_2(q) = \mathrm{e}^{-\mathrm{i}\hat{K}(j,e)} \int \mathrm{DM}(u, p)|\Gamma(q; u)|^2 \mathrm{e}^{\mathrm{i}s_0(u) - \mathrm{i}U(u,e)} , \tag{K.25}$$

where $\Gamma(q; u)$ was defined in (3.2), will be calculated.

The effective potential of the sin-Gordon model

$$U(u_N; e_c) = -\frac{2m^2}{\lambda^2} \int \mathrm{d}x\, \mathrm{d}t \sin \lambda u_N (\sin \lambda e - \lambda e) \tag{K.26}$$

with

$$e_c = e_\xi \cdot \frac{\partial u_c}{\partial \eta} - e_\eta \cdot \frac{\partial u_c}{\partial \xi} . \tag{K.27}$$

Performing the shifts in (K.22):

$$\xi_i(t) \to \xi_i(t) + \int dt' g(t-t') j_{\xi,i}(t') \equiv \xi_i(t) + \xi'_i(t) ,$$

$$\eta_i(t) \to \eta_i(t) + \int dt' g(t-t') j_{\eta,i}(t') \equiv \eta_i(t) + \eta'_i(t) , \qquad \text{(K.28)}$$

we can get the Green function $g(t-t')$ into the operator exponent:

$$\boldsymbol{K}(ej) = \frac{1}{2}\int dt\, dt' \Theta(t-t')\{\hat{\xi}'(t)\cdot \hat{e}_\xi(t') + \hat{\eta}'(t)\cdot \hat{e}_\eta(t')\} . \qquad \text{(K.29)}$$

Note the Lorentz noncovariantness of our perturbation theory with Green function (K.23).

As a result

$$D^N M(\xi,\eta) = \prod_{i=1}^{N}\prod_t d\xi_i(t)\, d\eta_i(t) \delta(\dot{\xi}_i - \omega(\eta+\eta'))\delta(\dot{\eta}_i), \quad \omega(\eta) = \frac{\partial h}{\partial \eta} \qquad \text{(K.30)}$$

with

$$u_N = u_N(x; \xi + \xi', \eta + \eta') . \qquad \text{(K.31)}$$

Using the definition

$$\int Dx \delta(\dot{x}) = \int dx(0) = \int dx_0 ,$$

the functional integrals on the measure (K.30) are reduced to the ordinary integrals over initial data $(\xi,\eta)_0$. These integrals define zero modes volume. Notice that the zero-modes measure was defined without the Faddeev–Popov *anzats*.

We divide the calculations into two parts. First of all, we consider the quasiclassical approximation and then we will show that this approximation is exact.

This strategy is necessary since it seems to be important to show the role of quantum corrections noting that for all physically acceptable field theories $R_{nm} = 0$ in the quasiclassical approximation.

The N-soliton solution u_N depends upon $2N$ parameters. Half of them, N, can be considered as the position of solitons and the other N as the solitons momentum. Generally, at $|t| \to \infty$ the u_N solution decomposed on the single solitons u_s and on the double-soliton bound states u_b [111]:

$$u_N(x,t) = \sum_{j=1}^{n_1} u_{s,j}(x,t) + \sum_{k=1}^{n_2} u_{b,k}(x,t) + O(e^{-|t|}) .$$

Note that this asymptotic is achieved if $\xi_i \to \infty$ or/and $\eta_i \to \infty$. This latter defines the bifurcation line of our model. So, the one soliton u_s and two-soliton bound state u_b would be the main elements of our formalism. Its (ξ,η) parametrizations have the form

$$u_s(x;\xi,\eta) = -\frac{4}{\lambda}\arctan\{\exp(mx\cosh\beta\eta - \xi)\}, \quad \beta = \frac{\lambda^2}{8} \qquad \text{(K.32)}$$

and

$$u_{\mathrm{b}}(x;\xi,\eta) = -\frac{4}{\lambda}\arctan\left\{\tan\frac{\beta\eta_2}{2}\frac{mx\sinh\beta\eta_1/2\cos\beta\eta_2/2-\xi_2}{mx\cosh\beta\eta_1/2\sin\beta\eta_2/2-\xi_1}\right\}. \tag{K.33}$$

Performing the last integration we find

$$R_2(q) = \sum_N \int \prod_{i=1}^N \{\mathrm{d}\xi_0\,\mathrm{d}\eta_0\}_i \mathrm{e}^{-\mathrm{i}\hat{K}} \mathrm{e}^{\mathrm{i}S_0(u_N)} \mathrm{e}^{-\mathrm{i}U(u_N;e_\xi,e_\eta)} |\Gamma(q;u_N)|^2 \tag{K.34}$$

where

$$u_N = u_N(\eta_0+\eta',\xi_0+\omega(t)+\xi') \tag{K.35}$$

and

$$\omega(t) = \int \mathrm{d}t'\theta(t-t')\omega(\eta_0+\eta')(t'). \tag{K.36}$$

In the quasiclassical approximation $\xi'=\eta'=0$ we have

$$u_N = u_N(x;\eta_0,\xi_0+\omega(\eta_0)t). \tag{K.37}$$

Notice that the surface term

$$\int \mathrm{d}x^\mu \partial_\mu(\mathrm{e}^{\mathrm{i}qx}u_N) = 0. \tag{K.38}$$

Then

$$\int \mathrm{d}^2x\mathrm{e}^{\mathrm{i}qx}(\partial^2+m^2)u_N(x,t) = -(q^2-m^2)\int \mathrm{d}^2x\mathrm{e}^{\mathrm{i}qx}u_N(x,t) = 0, \tag{K.39}$$

since q^2 belongs to the mass shell by definition. The condition (K.38) is satisfied for all $q_\mu \neq 0$ since u_N belongs to Schwarz space (the periodic boundary condition for $u(x,t)$ does not alter this conclusion). Therefore, in the quasiclassical approximation $R_2 = 0$.

Expanding the operator exponent in (K.34) we find that action of operators $\hat{\xi}'$, $\hat{\eta}'$ creates terms

$$\sim \int \mathrm{d}^2x\mathrm{e}^{\mathrm{i}qx}\theta(t-t')(\partial^2+m^2)u_N(x,t) \neq 0. \tag{K.40}$$

So, generally, if the quantum corrections are included, $R_2 \neq 0$.

Now we will show that the quasiclassical approximation is exact in the soliton sector of the sin-Gordon model. The structure of the perturbation theory is readily seen in the 'normal-product' form

$$R_2(q) = \sum_N \int \prod_{i=1}^N \{\mathrm{d}\xi_0\,\mathrm{d}\eta_0\}_i : \mathrm{e}^{-\mathrm{i}U(u_N;\hat{j}/2i)} \mathrm{e}^{\mathrm{i}s_0(u_N)} |\Gamma(q;u_N)|^2, \tag{K.41}$$

where

$$\hat{j} = \hat{j}_\xi \cdot \frac{\partial u_N}{\partial \eta} - \hat{j}_\eta \cdot \frac{\partial u_N}{\partial \xi} = \Omega \hat{j}_X \frac{\partial u_N}{\partial X} \tag{K.42}$$

and

$$\hat{j}_X = \int dt' \Theta(t - t') \hat{X}(t') \tag{K.43}$$

with the $2N$-dimensional vector $X = (\xi, \eta)$. In Eq. (K.42) Ω is the ordinary symplectic matrix.

The colons in (K.41) mean that the operator $\hat{j}$ should stay to the left of all functions in the perturbation theory expansion over it. The structure (K.42) shows that each order over $\hat{j}_{X_i}$ is proportional at least to the first order derivative of u_N over the variable conjugate to X_i.

The expansion of (K.41) over $\hat{j}_X$ can be written [105] in the form (omitting the quasiclassical approximation):

$$R_2(q) = \sum_N \int \prod_{i=1}^{N} \{d\xi_0 \, d\eta_0\}_i \left\{ \sum_{i=1}^{2N} \frac{\partial}{\partial X_{0i}} P_{X_i}(u_N) \right\}, \tag{K.44}$$

where $P_{X_i}(u_N)$ is the infinite sum of 'time-ordered' polynomials (see [105]) over u_N and its derivatives. The explicit form of $P_{X_i}(u_N)$ is complicated since the interaction potential is non-polynomial. But it is enough to know, see (K.42), that

$$P_{X_i}(u_N) \sim \Omega_{ij} \frac{\partial u_N}{\partial X_{0j}} . \tag{K.45}$$

Therefore,

$$R_2(q) = 0 \tag{K.46}$$

since (i) each term in (K.44) is the total derivative, (ii) we have (K.45) and (iii) u_N belongs to Schwarz space.

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