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Quantization of solitons in coset space

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The perturbation theory around the soliton fields of the sin-Gordon model is developed in the coset space. It is shown by explicit calculations that all corrections to the topological soliton contribution are canceled exactly. © *2001 American Institute of Physics.* [DOI: 10.1063/1.1337613]

I. INTRODUCTION

The problem of quantization of the extended objects was formulated mainly in the middle of the 1970s, see the review paper[1] and references cited therein. One starts from the classical Lagrange equation:

$$\frac{\delta S(u)}{\delta u(x,t)}=0, \tag{1}$$

where, for simplicity, $u(x,t)$ is the real scalar field.[2] If this equation has nontrivial solution $u_c(x,t)$ then the problem of its quantization will arise. One of the first attempts to construct the perturbation theory was based on the WKB expansion in the vicinity of u_c.[3,4]

The Born–Oppenheimer method was adopted also.[5,6] First of all, to construct the quantum mechanics, the structure of Hilbert space $\mathcal{H}$ is postulated. So, it is assumed that the Fock column consists from the vacuum state $|0\rangle$ and from the multiple meson states $|p_1,p_2,\ldots,p_n\rangle$, $n\geq 1$. The ordinary perturbation theory operates just with this meson sector only. The *anzats* $|P_1,P_2,\ldots,P_l\rangle$[5] for the l-soliton state, $l\geq 1$, is introduced also.

It is postulated that the quantum excitations in the soliton sector are described by the excitation of the meson field.[5] Therefore, to construct the perturbation theory, there should also be the mixed states:

$$|P_1,\ldots,P_l;p_1,\ldots,p_n\rangle,\quad l\geq 1,\ n\geq 1, \tag{2}$$

but, at the same time,

$$\langle P_1,\ldots,P_l;p_1,\ldots,p_n|p_1,\ldots,p_{n'}\rangle\equiv 0,\ l\geq 1,\ n+n'\geq 0, \tag{3}$$

i.e., it is assumed that the solitons are the absolutely stable field configurations.[1]

The present paper in a definite sense completes the picture in Refs. 5 and 6, The $(1+1)$-dimensional exactly integrable sin-Gordon model will be considered to illustrate our result. We will investigate the multiple production of mesons by soliton and the truth of (3) will be shown at the end of explicit calculations. In other words, it will be shown that the postulate in

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Refs. 5 and 6 concerning orthogonality of the meson $\mathcal{H}_m$ and soliton $\mathcal{H}_s$ Hilbert spaces can be proved. We will see that this conclusion follows from exactness of the semiclassical approximation for the sin-Gordon model.

It should be noted that the exactness of the semiclassical approximation in the topological soliton sector of the sin-Gordon model is not beyond the realm.[3] It is well known also that the integrable Coulomb problem is exactly semiclassical. We have the same for the quantum rigid rotator,[7] which is the isomorphic to the Poshle–Teller model. The general discussion of the exactness of the semiclassical approximation from a geometrical point of view was given in Ref. 8.

It will be crucial for us in many respects to follow the WKB ideology. So, we will consider the meson production amplitudes

$$a_{nm}(p,q)=\langle p_1,\ldots,p_n|q_1,\ldots,q_m\rangle_c, \quad n,m=1,2,\ldots. \tag{4}$$

The index c means that the calculations are performed in the soliton sector and p_i and q_i are the meson momenta. By definition,

$$p_i^2=q_i^2=m^2, \tag{5}$$

since the quantum uncertainty principle leads to the impossibility of mass-shell observation of the field.[9] The ordinary reduction formalism will be used to calculate a_{nm}. This means that we will construct the *phenomenological* S-matrix of the meson interaction through the soliton fields, i.e., we will start from the assumption that the states (2) exist, and it will be shown at the end of the calculations that such S-matrix is trivial:

$$a_{nm}(p,q)\equiv 0, \quad n+m>0. \tag{6}$$

The formalism allows to prove (6). For this purpose we will build the perturbation theory expansion over $1/g$, where g is the interaction constant.[10] This perturbation theory is dual to the theory described in Ref. 1, over g, i.e., one cannot decompose the definite order over g contribution in terms of the $1/g$ expansion, and vice versa. So, only the summary results of both expansion may be compared.

Following to WKB ideology, to find the corrections to the semiclassical approximation in the vicinity of the extremum $u_c(x,t)$, one should find the solution of the equation for the Green function:

$$(\partial^2+v''(u_c))G(x,t;x',t')=\delta(x-x')\delta(t-t'),$$

where $v''(u)$ is the second derivative of the potential function $v(u)$. This Green function describes propagation of a particle in the time dependent inhomogeneous and anisotropic external field $u_c(x,t)$. Generally, this problem has no closed solution. So, for instance, the attempt to solve the problem using the momentum decomposition[11] leads to the hardly handling double-parametric perturbation theory. To avoid this problem we will build a new perturbation theory over $1/g$.

Imagining particle coordinates as the elements of the Lee group, the classical particle motion may be described mapping the trajectory on group manifold. Roughly speaking, this means that the group combination law creates the particles classical trajectory.[12]

Moreover, this program was realized for description of the particle quantum motion.[13] It was shown for essentially nonlinear Lagrangian $L=\frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu$ that the semiclassical approximation is exact on the (semi)simple Lee group manifold. But this slender solution of quantum problems is destructed in presence of the interaction potential $v(x)=O(x^n)$, $n>2$, since the last one breaks the isotropy and homogeneity of the Lee group manifolds.[10] The developed perturbation theory will describe the quantum perturbations breaking isotropy and homogeneity of the group manifold.

Developed formalism contains the following steps.[10,14] (i) We will introduce the manifold W_G of trajectories u_c, solving the Eq. (1). The manifold W_G will be labeled by the local coordinates

(ξ, η), i.e., we will consider $u_c = u_c(x;\xi,\eta)$ since u_c should belong to W_G *completely*. (ii) The numbers (ξ, η) are interpreted as the generalized coordinates of the particle. Then $u_c(x;\xi,\eta)$ will define the external potential for it. The quantum motion of the particle may be described noting that W_G is the homogeneous and isotropic manifold, since this case is a rather quantum mechanical problem in the flat space.

It was shown in Ref. 14 that the WKB model,[5] where the field excitations in vicinity of u_c are decomposed over the meson states, and our model quantum mechanics of the particle in the external potential defined by u_c, are isomorphic. In other words, we know that the quantum trajectory of the particle covers the phase space $(\xi, \eta) \in W_G$ densely. But it should be noted also that the model described in Ref. 5 presents the expansion over the interaction constant λ and our perturbation theory describes expansion over the $(1/\lambda)$.

In the classical limit (labeled by the index 0) the motion of our particle must be free,[14] i.e., its velocity should be a constant,

$$\dot{\xi}_0 = \text{const}, \quad \dot{\eta}_0 = 0. \tag{7}$$

This may be achieved expressing the set $\{\eta\}$ through the set of generators of the subgroup broken by u_c.[15] It is evident, such choice of the particles coordinate gives the same effect as in the above discussed transformation to the homogeneous and isotropic (semi)simple Lee group manifold,[10] see also Ref. 16. Moreover, we will see that even in the case of nontrivial potential function, one can get to the free particles motion, rescaling the quantum sources.[10,14]

Thus, the necessary invariant subspace W_G would be chosen equal to the coset space G/G_c:

$$W_G = G/G_c, \tag{8}$$

where G is the symmetry group and $G_c \subset G$ is the classical solutions u_c symmetry group. The problem of quantization of the coset space have a reach history, see, e.g., Ref. 17. As described in Refs. 10 and 14, the formalism presents one possible realization of the coset spaces quantization scheme.

The last one means that we will realize the transformation generated by the classical trajectory:[14]

$$u_c: (u,p)(x,t) \rightarrow (\xi, \eta). \tag{9}$$

Such construction of perturbation theory in the W_G space requires the additional effort noting that the dimension of the original phase space $(u,p) \in T^* V$ is infinite. Therefore, Eq. (9) assumes the infinite reduction since the dimension of coset space W_G is finite.[18] The crucial reduction scheme was formulated in Ref. 14.

In other words, quantizing the sin-Gordon soliton fields, the space coordinate would be an irrelevant variable. This is the well-known fact, e.g., Ref. 4, and it leads to the Lorentz noncovariant perturbation theory. It is the consequence of the solitary profile of considered field configurations and its absolute stability, i.e., of conservation of the topological charge. The necessary information concerning this question will be given in Sec. III.

Having the complete theory, one can analyze the perturbations. The crucial point of the new perturbation theory is the statement[10] that the quantum corrections are accumulated strictly on the boundaries ∂W_G (bifurcation manifolds[19,20,15]) of the W_G space. Therefore, if

$$\partial u_c \cap \partial W_G = \emptyset, \tag{10}$$

then the problem is exactly semiclassical. On other hand, Eq. (10) means conservation of the topological charge: ∂u_c is the flow induced by the quantum perturbations in W_G and if (10) is not satisfied, then one should exist a flow into the forbidden, separated by the bifurcation boundary, domain with other topological charge. So, Eq. (10) is the topological charge conservation.

On the other hand, Eq. (10) leads to (6) since particle production is the pure quantum effect. This will be shown in Sec. IV.

The paper is organized as follows. In Sec. II we will (i) formulate the necessary boundary conditions to derive the LSZ reduction formula, (ii) find the explicit expression for a_{nm}, (iii) formulate the mapping into the coset space W_G. In Sec. III we (i) consider the sin-Gordon model, (ii) discuss the coset space boundary condition, (iii) remind the structure of the new perturbation theory,[14] (iv) describe meson multiple production to show (6).

II. DENSITY MATRIX ON THE DIRAC MEASURE

The main point of this section is the attempt to generalize the ordinary for field theory boundary condition

$$u(x\in\sigma_\infty)=0,$$

where σ_∞ is the remote hypersurface. This boundary condition is used to remove the surface term, and it is necessary to formulate the reduction formalism. We would like to introduce the new boundary condition to have a possibility to include the nonvanishing on σ_∞ field configurations and, at the same time, throw off the surface term.

The $(n+m)$-point Green functions G_{nm} are introduced through the generating functional Z_j:[21]

$$G_{nm}(x,y)=(-i)^{n+m}\prod_{k=1}^{n}\hat{j}(x_k)\prod_{k=1}^{m}\hat{j}(y_k)Z_j, \tag{11}$$

where $\hat{j}(x)=\delta/\delta j(x)$ and the generating functional

$$Z_j=\int Du\, e^{iS_j(u)}. \tag{12}$$

The action

$$S_j(u)=S(u)-V(u)+\int dx\, dt\, j(x,t)u(x,t), \tag{13}$$

where

$$S(u)=\int dx\, dt(\tfrac{1}{2}(\partial u)^2-m^2u^2),\quad m^2\geqslant 0, \tag{14}$$

is the free part and $V(u)$ describes the interactions. At the end of the calculations one should put $j=0$.

To provide convergence, the integral (12) will be defined on the Mills complex time contour C_+.[22] For example,

$$C_\pm:\quad t\to t+i\varepsilon,\varepsilon\to+0,\ -\infty\leqslant t\leqslant+\infty \tag{15}$$

and after all calculations, one should return the time contour on the real axis putting $\varepsilon=0$.

In a 'meson' sector the integration in (12) is performed over all field configurations with standard vacuum boundary condition:

$$\int d^2x\partial_\mu(u\partial^\mu u)=\int_{\sigma_\infty} d\sigma_\mu\, u\partial^\mu u=0. \tag{16}$$

It follows from this conditions that

$$u(x\in\sigma_\infty)=0,\quad pa_\mu u(x\in\sigma_\infty)=0. \tag{17}$$

It excludes a contribution from the surface term, since it assumes that field disappeared on the remote hypersurface σ_∞. Considering the soliton sector this boundary condition requires the modification since there is in the $(x-t)$ space such direction along which the soliton field does not disappear. The integral (12) would have a formal meaning until this boundary condition will not be specified.

Let us introduce now the field φ through the equation

$$-\frac{\delta S(\varphi)}{\delta\varphi(x,t)}=j(x,t). \tag{18}$$

It is assumed that we can formulate such boundary condition that the surface term may be neglected calculating the variational derivative in (18). Then we perform the ordinary shift $u\to u+\varphi$ in integral (12). Considering φ as the probe field created by the source:

$$\varphi(x)=\int d^2x'\, G_0(x-x')j(x'),\quad (\partial^2+m^2)G_0(x-x')=\delta(x-x'), \tag{19}$$

the connected Green function G^c_{nm} will only be interesting for us,

$$G^c_{nm}(x,y)=(-i)^{n+m}\prod_{k=1}^{n}\hat{j}(x_k)\prod_{k=1}^{m}\hat{j}(y_k)Z(\varphi), \tag{20}$$

where

$$Z(\varphi)=\int Du\, e^{iS(u)-iV(u+\varphi)} \tag{21}$$

is the new generating functional.

To calculate the nontrivial elements of the S matrix we must put the external particles on the mass shell. Formally this procedure means amputation of the external legs of G^c_{nm} and further multiplication on the free particle wave functions. In result the amplitude of n into m particle transition a_{nm} in the momentum representation has the form

$$a_{nm}(q,p)=(-i)^{n+m}\prod_{k=1}^{n}\hat{\varphi}(q_k)\prod_{k=1}^{m}\hat{\varphi}^*(p_k)Z(\varphi). \tag{22}$$

Here the particles creation operator

$$\hat{\phi}^*(q)=\int d^2x\, e^{iqx}\hat{\phi}(x),\quad \hat{\phi}(x)=\frac{\delta}{\delta\phi(x)}. \tag{23}$$

was introduced. The Eq. (22) is the ordinary LSZ reduction formulas. But one should remember that the boundary condition (16) should be generalized to have permission for inclusion of the soliton contributions calculating $Z(\varphi)$.

Describing the particles multiple production it is enough to consider the generating functional

$$\rho(\alpha,z)=\exp\left\{-\int d\Omega_1(p)(\hat{\varphi}^*_+(p)\hat{\varphi}_-(p)e^{i\alpha_+p}z_+(p)\right.$$
$$\left.+\hat{\varphi}^*_-(p)\hat{\varphi}_+(p)e^{i\alpha_-p}z_-(p))\right\}Z(\varphi_+)Z^*(\varphi_-), \tag{24}$$

where

$$d\Omega_n(p)=\prod_{k=1}^{n}\frac{d^1p_k}{(2\pi)2\epsilon(p_k)}=\prod_{k=1}^{n}d\Omega_1(p_k),\quad \epsilon(p)=(p^2+m^2)^{1/2}.$$

Let us calculate

$$\int\frac{d^2\alpha_+}{(2\pi)^2}e^{-iP\alpha_+}\frac{d^2\alpha_-}{(2\pi)^2}e^{-iP\alpha_-}\prod_{k=1}^{n}\frac{\delta}{\delta z_+(p_k)}\prod_{k=1}^{m}\frac{\delta}{\delta z_-(q_k)}\rho(\alpha,z)|_{z_+=z_-=0}.$$

Inserting here the definition (24), one can find that this expression gives

$$\delta\left(P-\sum_{k=1}^{n}p_k\right)\delta\left(P-\sum_{k=1}^{m}q_k\right)|a_{nm}(p,q)|^2,$$

where the δ functions are the result of integration over $\alpha_\pm$. So, the factors $e^{i\alpha_\pm P}$ in (24) permit to introduce the energy-momentum shell and the δ function defines the restriction on the shell. Both restrictions

$$P=\sum_{k=1}^{m}q_k,\quad P=\sum_{k=1}^{n}p_k$$

are compatible since the amplitude a_{nm} is translationally invariant. The integration over P gives energy-momentum conservation law.

Notice now that $\rho(\alpha,z)$ is defined through the generating functional

$$\rho_0(\varphi)=Z(\varphi_+)Z^*(-\varphi_-)=\int Du_+\,Du_-\,e^{iS_+(u_+)-iS_-(u_-)}e^{-iV_+(u_++\varphi_+)+iV_-(u_--\varphi_-)}.\tag{25}$$

Then, we can consider the closed-path boundary condition

$$\int_{\sigma_\infty}d\sigma_\mu\,u_+\partial^\mu u_+=\int_{\sigma_\infty}d\sigma_\mu\,u_-\partial^\mu u_-,\tag{26}$$

instead of (16) and (17). The natural solution of this boundary condition is

$$u_+(x\in\sigma_\infty)=u_-(x\in\sigma_\infty)=u(x\in\sigma_\infty).\tag{27}$$

It provides cancellation of the surface term on the remote hypersurface σ_∞ independently on the value of the field $u(x\in\sigma_\infty)$.

Considering the system with the large number of particles, we can simplify calculations choosing the center-of-mass (c.m.) frame $P=(P_0=E,\vec{0})$. It is useful also[23] to rotate the contours of integration over

$$\alpha_{0,k}:\ \alpha_{0,k}=-i\beta_k,\ \operatorname{Im}\beta_k=0,\quad k=1,2.$$

Then $\rho(\beta,z)$ have a meaning of the density matrix, where β would have, in the some definite case,[24] meaning of the inverse temperature and z is the activity.[25]

It was shown in Ref. 14 that the unitarity condition unambiguously determines contributions in the path integrals for ρ. Exist the statement:

S1. The density matrix $\rho(\alpha,z)$ has the following representation:

$$\rho(\alpha,z)=e^{-i\hat{K}(je)}\int DM(u)e^{iS_O(u)-iU(u,e)}e^{N(\alpha,z;u)}\equiv\mathcal{O}(u)e^{N(\alpha,z;u)}.\tag{28}$$

It should be underlined that this representation is strict and is valid for arbitrary Lagrange theory of arbitrary dimensions. The derivation of (28) is given in the Appendix.

Expansion over the operator

$$\hat{K}(je)=\frac{1}{2}\mathrm{Re}\int_{C_+}\mathrm{d}x\,\mathrm{d}t\frac{\delta}{\delta j(x,t)}\frac{\delta}{\delta e(x,t)}\equiv\frac{1}{2}\mathrm{Re}\int_{C_+}\mathrm{d}x\,\mathrm{d}t\,\hat{j}(x,t)\hat{e}(x,t) \tag{29}$$

generates the perturbation series. We will assume that this series exist (at least in the Borel sense). The variational derivatives in (29) are defined as follows:

$$\frac{\delta\phi(x,t\in C_i)}{\delta\phi(x',t'\in C_j)}=\delta_{ij}\delta(x-x')\delta(t-t')\quad i,j=+,-,$$

where C_i is the Mills time contour. The auxiliary variables (j,e) must be taken equal to zero at the very end of the calculations.

The functionals $U(u,e)$ and $S_O(u)$ are defined by the equalities

$$S_O(u)=(S(u+e)-S(u-e))+2\,\mathrm{Re}\int_{C_+}\mathrm{d}x\,\mathrm{d}t\,e(x,t)(\partial^2+m^2)u(x,t), \tag{30}$$

$$U(u,e)=V(u+e)-V(u-e)-2\,\mathrm{Re}\int_{C_+}\mathrm{d}x\,\mathrm{d}t\,e(x,t)v'(u), \tag{31}$$

where $S(u)$ is the free part of the Lagrangian and $V(u)$ describes interactions. The phase $S_O(u)$ is not equal to zero if u have the nontrivial topological charge.[14] We will discuss carefully this question later.

The measure $DM(u,p)$ has the form

$$DM(u)=\prod_{x,t}\mathrm{d}u(x,t)\delta\left(\frac{\delta(S(u)-V(u))}{\delta u(x,t)}+j(x,t)\right). \tag{32}$$

The functional δ function in the measure means that the necessary and sufficient set of contributions in the integral over $u(x,t)$ is defined by the classical equation

$$-\frac{\delta(S(u)-V(u))}{\delta u(x,t)}=j(x,t), \tag{33}$$

disturbed by the quantum source $j(x,t)$.

For further calculation another representation will be useful. If we insert into the integral (28)

$$1=\int\prod_{x,t}\mathrm{d}p(x,t)\delta(p(x,t)-\dot{u}(x,t))$$

then the measure DM takes the form

$$DM(u,p)=\prod_{x,t}\mathrm{d}u(x,t)\mathrm{d}p(x,t)\delta\left(\dot{u}(x,t)-\frac{\delta H_j(u,p)}{\delta p(x,t)}\right)\delta\left(\dot{p}(x,t)+\frac{\delta H_j(u,p)}{\delta u(x,t)}\right) \tag{34}$$

with the total Hamiltonian

$$H_j(u,p)=\int\mathrm{d}x\left\{\frac{1}{2}p^2+\frac{1}{2}(\nabla u)^2+v(u)-ju\right\}. \tag{35}$$

The last one includes the energy ju of quantum fluctuations. The measure (34) describes motion in the symplectic space $(u,p)\in V$. But it should be underlined that the used expansion is not the Lagrange transformation. So, generally, it is quite possible, considering x as the index of the space shell, that not all of $p(x,t)$ are the independent variables. For this reason the measure (34) has mostly a Lagrange meaning.

The measure (34) contains the following information.[10,14]

(a) *Only the strict solutions of equations*

$$\dot{u}-\frac{\delta H_j(u,p)}{\delta p}=0, \quad \dot{p}+\frac{\delta H_j(u;p)}{\delta u}=0 \tag{36}$$

at $j=0$ should be taken into account. This rigidness means absence in the formalism of the pseudo-solution (similar to multi-instanton, or multikink) contributions.

(b) *$\rho(\alpha,z)$ is described by the sum of all solutions of Eq. (36), independently from their nearness in the functional space.*

(c) *The field disturbed by $j(x)$ belongs to the same manifold (topology class) as the classical field defined by (36).*[10]

(d) *The consequence of properties b. and c. is the selection rule: quantum dynamics is realized in the coset space of highest dimension.*[10] This, excluding from consideration the pure meson sector.

The particle density

$$N(\alpha,z;u)=N_+(\alpha_+,z_+;u)+N_-(\alpha_-,z_-;u), \tag{37}$$

where

$$N_\pm(\alpha_\pm,z_\pm;u)=\int d\Omega_1(q)e^{i\alpha_\pm q}z_\pm(q)|\Gamma(q;u)|^2. \tag{38}$$

The vertex $\Gamma(q;u)$ is the function of the external particle momentum q and is the linear functional of $u(x)$:

$$\Gamma(q;u)=-\int dx\, e^{iqx}\frac{\delta S(u)}{\delta u(x)}=\int dx\, e^{iqx}(\partial^2+m^2)u(x), \quad q^2=m^2, \tag{39}$$

for the mass m field. This parameter presents the momentum distribution of the interacting field $u(x)$ on the remote hypersurface σ_∞ *if* $u(x)$ is the regular function. Notice, the operator cancels the mass-shell states of $u(x)$.

Generally $\Gamma(q;u)$ is connected directly with *external* particles properties and sensitive to the symmetry of the interacting fields system.[26]

The construction (39) means, because of the operator (∂^2+m^2) and remembering that the external states should be mass shell by definition,[9] the solution $\rho(\alpha,z)=0$ is actually possible for particular topology (compactness and analytic properties) of *quantum* field $u(x)$. So, $\Gamma(q;u)$ carry remarkable properties: (i) it directly defines the observables, (ii) is defined by the topology of $u(x)$. Notice that the space-time topology of $u(x,t)$ becomes important calculating integral (39) by parts. This procedure is available if $u(x,t)$ is the regular function. But the *quantum* fields are always singular. Therefore, the solution $\Gamma(q;u)=0$ is valid iff the semiclassical approximation is exact, i.e., the particle production is the pure quantum effect. Just this situation is realized in the soliton sector of the sin-Gordon model.

Let G be the symmetry of the problem and let G_c be the symmetry of the solution u_c. Then

S2. *The measure (34) admits the transformation:*

$$u_c:(u,p)\rightarrow(\xi,\eta)\in W=G/G_c \tag{40}$$

and transformed measure has the form

$$DM(u,p)=\prod_{x,t C} d\xi(t)d\eta(t)\delta\left(\dot\xi-\frac{\delta h_j(\xi,\eta)}{\delta\eta}\right)\delta\left(\dot\eta+\frac{\delta h_j(\xi,\eta)}{\delta\xi}\right), \tag{41}$$

where $h_j(\xi,\eta)=H_j(u_c,p_c)$ is the transformed Hamiltonian

$$h_j(\xi,\eta;t)=h(\eta)-\int dx\, j(x,t)u_c(x;\xi,\eta) \tag{42}$$

and $u_c(x;\xi,\eta)$ is the soliton solution parametrized by (ξ,η).

The proof of Eq. (41) is the same as for the Coulomb problem considered in Ref. 14. But the case of the $(1+1)$-dimensional model needs the additional explanations. First of all, one must introduce the functional

$$\Delta(u,p)=\int\prod_t d^N\xi(t)d^N\eta(t)\prod_{x,t}\delta(u(x,t)-u_c(x;\xi,\eta))\delta(p(x,t)-p_c(x;\xi,\eta)). \tag{43}$$

The equalities

$$u(x,t)=u_c(x;\xi,\eta),\quad p(x,t)=p_c(x;\xi,\eta) \tag{44}$$

assume that for given $u(x,t)$ and $p(x,t)$ one can hide the t dependence into the N functions $\xi=\xi(t)$ and $\eta=\eta(t)$. It is assumed that this procedure can be done for arbitrary x. In other respects, functions $u(x,t)$ and $p(x,t)$, and therefore, $u_c(x;\xi,\eta)$ and $p(x;\xi,\eta)$, are arbitrary.

For more confidence, one may divide the space onto the N cells and to each $(u,p)_x$ we may adjust $(\xi,\eta)_x$. It is possible that (ξ,η) are x independent. In this degenerate case $\Delta\sim(\delta(0))^k$, where $k\leq N$ is the degree of the degeneracy. We will omit the index x considering $(\xi,\eta)_x$ as the vector of the necessary dimension.

If (ξ,η) are the solutions of (44), then

$$\Delta(u,p)=\int\prod_t d\xi'(t)d\eta'(t)\delta(u_c^\xi\xi'+u_c^\eta\eta')\delta(p_c^\xi\xi'+p_c^\eta\eta')=\Delta_c(\xi,\eta)\neq 0, \tag{45}$$

where, for instance, $u_c^X=\partial u_c(x;\xi,\eta)/\partial X$, $X=\xi,\eta$. Notice the importance of the last condition. If it is fulfilled, then one may insert into (28), with measure (41),

$$1=\frac{\Delta(u,p)}{\Delta_c(\xi,\eta)} \tag{46}$$

and integrate over $u(x,t)$ and $p(x,t)$. Notice that the possible infinite factor $(\delta(0))^k$ would be canceled in the ratio (46).

The Jacobian of transformation

$$J=\int\frac{Du\,Dp}{\Delta_c(\xi,\eta)}\prod_{x,t}\delta\left(\dot u-\frac{\delta H_j(u,p)}{\delta p}\right)\delta\left(\dot p+\frac{\delta H_j(u,p)}{\delta u}\right)$$
$$\times\,\delta(u(x,t)-u_c(x;\xi,\eta))\delta(p(x,t)-p_c(x;\xi,\eta)), \tag{47}$$

is proportional to functional δ-functions again. To have the transformation, we should use the last two δ functions. Notice, if the first two δ functions are used to calculate J, then the last two δ functions realize the constraints. In result,

$$J=\frac{1}{\Delta_c(\xi,\eta)}\prod_{x,t}\delta\left(\dot u_c-\frac{\delta H_j(u_c,p_c)}{\delta p_c}\right)\delta\left(\dot p_c+\frac{\delta H_j(u_c,p_c)}{\delta u_c}\right). \tag{48}$$

It should be underlined that u_c and p_c are *arbitrary* functions of ξ and η, i.e., on this stage we make the transformation of arbitrary functions $u(x,t)$ and $p(x,t)$ on the new arbitrary functions $u_c(x;\xi,\eta)$ and $p_c(x;\xi,\eta)$, where, generally speaking, $\xi=\xi(x,t)$ and $\eta=\eta(x,t)$. Then Δ_c is the corresponding determinant.

The expression (48) can be rewritten identically to the form

$$J=\frac{1}{\Delta_c(\xi,\eta)}\int\prod_{x,t}d\xi'(t)d\eta'(t)\delta\left(\xi'-\left(\dot{\xi}-\frac{\delta h_j(\xi,\eta;t)}{\delta\eta}\right)\right)\delta\left(\eta'-\left(\dot{\eta}+\frac{\delta h_j(\xi,\eta;t)}{\delta\xi}\right)\right)$$
$$\times\delta\left(u_c^{\xi}\xi'+u_c^{\eta}\eta'+\{u_c,h_j\}-\frac{\delta H_j}{\delta p_c(x,t)}\right)\delta\left(p_c^{\xi}\xi'+p_c^{\eta}\eta'-\{p_c,h_j\}+\frac{\delta H_j}{\delta u_c(x,t)}\right), \tag{49}$$

where {,} is the Poisson bracket.

Let us assume now that the auxiliary function $h_j(\xi,\eta;t)$ is chosen so that the equalities

$$\{u_c,h_j\}=\frac{\delta H_j}{\delta p_c(x,t)},\quad \{p_c,h_j\}=-\frac{\delta H_j}{\delta u_c(x,t)} \tag{50}$$

are satisfied identically. Then, taking into account the condition (45), one can find

$$J=\delta\left(\dot{\xi}-\frac{\delta h_j(\xi,\eta;t)}{\delta\eta}\right)\delta\left(\dot{\eta}+\frac{\delta h_j(\xi,\eta;t)}{\delta\xi}\right). \tag{51}$$

This ends the transformation. Notice that the determinant Δ_c was canceled identically.

The transformation specify by the Eqs. (50) the function h_j. It assumes that one can find such functions $u_c=u_c(x;\xi,\eta)$ and $p_c=p_c(x;\xi,\eta)$, with property (45), that (50) has unique solution $h_j(\xi,\eta;t)$.

Let us convert the problem assuming that just h_j is known. It is natural to assume that

$$h_j(\xi,\eta;t)=H_j(u_c,p_c), \tag{52}$$

then u_c and p_c are defined by Eqs. (50) and

$$\dot{\xi}=\frac{\delta h_j(\xi,\eta;t)}{\delta\eta},\quad \dot{\eta}=-\frac{\delta h_j(\xi,\eta;t)}{\delta\xi}. \tag{53}$$

It is not hard to see that (50) together with (53) are equivalent to incident equations (36). This is seen from the following chain of equalities:

$$\dot{u}_c(x;\xi\eta)=u_c^{\xi}\dot{\xi}+u_c^{\eta}\dot{\eta}=u_c^{\xi}\frac{\partial h_j(\xi,\eta;t)}{\partial\eta}-u_c^{\eta}\frac{\partial h_j(\xi,\eta;t)}{\partial\xi}$$
$$=\{u_c,h_j\}=\frac{\delta H_j}{\delta p_c(x,t)}$$

and the same we have for p_c. Therefore $(u_c;p_c)$ is the classical phase space flow and the space W_G, labeled by (ξ,η), is the coset space G/G_c.

In result, the new measure takes the form (41), i.e., ξ and η should obey the equations (53):

$$\dot{\xi}=\omega(\eta)-\int dx\, j(x,t)\frac{\partial u_N(x;\xi,\eta)}{\partial\eta},\quad \dot{\eta}=\int dx\, j(x,t)\frac{\partial u_N(\xi,\eta)}{\partial\xi}, \tag{54}$$

where $\omega(\eta)\equiv\partial h(\eta)/\partial\eta$. Hence the source of quantum perturbations are proportional to the time-local tangent vectors

$$\int \mathrm{d}x\, \partial u_N(x;\xi,\eta)/\partial \eta, \quad \int \mathrm{d}x\, \partial u_N(x;\xi,\eta)/\partial \xi$$

to the soliton configurations. It suggests the idea in Ref. 14 to split the Lagrange sources

$$j(x,t) \rightarrow (j_\xi, j_\eta)(t).$$

The mechanism of splitting was described in Ref. 10. The resulting operator $\mathcal{O}(u_c)$, defined in (28), has the same structure. But new perturbations of the generating operator

$$\hat{K}(e_\xi, e_\eta; j_\xi, j_\eta) = \tfrac{1}{2}\mathrm{Re}\int_{C_+} \mathrm{d}t\{\hat{j}_\xi(t)\cdot \hat{e}_\xi(t) + \hat{j}_\eta(t)\cdot \hat{e}_\eta(t)\}. \tag{55}$$

The measure takes the form

$$DM(\xi,\eta) = \prod_t \mathrm{d}\xi(t)\mathrm{d}\eta(t)\,\delta(\dot{\xi} - \omega(\eta) - j_\xi(t))\,\delta(\dot{\eta} - j_\eta(t)). \tag{56}$$

The effective potential $U = U(u_c; e_c)$ with

$$e_c(x,t) = e_\xi(t)\cdot \frac{\partial u_N(x;\xi,\eta)}{\partial \eta(t)} - e_\eta(t)\cdot \frac{\partial u_N(x;\xi,\eta)}{\partial \xi(t)}. \tag{57}$$

Notice that the space degree of freedom is disappeared from our consideration.

III. MULTIPLE PRODUCTION IN SIN-GORDON MODEL

We would consider the theory with the Lagrangian

$$L = \frac{1}{2}(\partial_\mu u)^2 + \frac{m^2}{\lambda^2}[\cos(\lambda u) - 1]. \tag{58}$$

It is well known that this field model possesses the soliton excitations in the $(1+1)$ dimension.

Formally nothing prevents to linearize partly our problem considering the Lagrangian

$$L = \frac{1}{2}[(\partial_\mu u)^2 - \alpha m^2 u^2] + \frac{m^2}{\lambda^2}\left[\cos(\lambda u) - 1 + \alpha \frac{\lambda^2}{2} u^2\right] \equiv S(u) - v(u). \tag{59}$$

The last term $v(u) = O(u^4)$ describes interactions. The corresponding vertex function is

$$\Gamma(q;u) = \int \mathrm{d}x\, \mathrm{d}t\, e^{iqx}(\partial^2 + m^2)u(x,t), \quad q^2 = m^2. \tag{60}$$

It should be noted here that the division chosen in (59) onto the free and interaction parts did not affect the equation of motion, see (33), and effective potential, see (31), i.e., in this sense α may be chosen arbitrary. But α will arise in the definition of the mass: one should change $m^2 \rightarrow \alpha m^2$ in (60). This means that our S-matrix approach requires additional, external, normalization condition for the mass shell. We will choose $\alpha = 1$ assuming that m is the measured mass of the meson.

We assume that $u(x,t)$ belongs to Schwarz space:

$$u(x,t)|_{|x|=\infty} = 0\left(\mathrm{mod}\frac{2\pi}{\lambda}\right). \tag{61}$$

This means that $u(x,t)$ tends to zero $[\mathrm{mod}(2\pi/\lambda)]$ at $|x| \rightarrow \infty$ faster then any power of $1/|x|$.

The ν-soliton classical Hamiltonian h_ν is the sum

$$h_\nu(\eta)=\int \mathrm{d}r\,\sigma(r)\sqrt{r^2+m^2}+\sum_{i=1}^{\nu} h(\eta_i), \tag{62}$$

where $\sigma(r)$ is the continuous spectrum and $h(\eta)$ is the soliton energy. Notice absence of the energy of soliton interactions.

The ν-soliton solution u_ν depends on the 2ν parameters. Half of the ν can be considered as the position of the solitons and the other ν as the solitons momentum. Generally, at $|t|\to\infty$ the u_ν solution decomposed on the single solitons u_s and on the double soliton bound states u_b:

$$u_\nu(x,t)=\sum_{j=1}^{n_1} u_{s,j}(x,t)+\sum_{k=1}^{n_2} u_{b,k}(x,t)+O(e^{-|t|}). \tag{63}$$

For this reason the one soliton u_s and two-soliton bound state u_b would be the main elements of our formalism. Its (ξ,η) parametrizations, i.e., the solution of Eq. (50), has the form:[27]

$$u_s(x;\xi,\eta)=-\frac{4}{\lambda}\arctan\{\exp(mx\cosh\beta\eta-\xi)\},\quad \beta=\frac{\lambda^2}{8} \tag{64}$$

and

$$u_b(x;\xi,\eta)=-\frac{4}{\lambda}\arctan\left\{\tan\frac{\beta\eta_2}{2}\,\frac{mx\sinh\frac{\beta\eta_1}{2}\cos\frac{\beta\eta_2}{2}-\xi_2}{mx\cosh\frac{\beta\eta_1}{2}\sin\frac{\beta\eta_2}{2}-\xi_1}\right\}. \tag{65}$$

The (ξ,η) parametrization of soliton individual energies $h(\eta)$ takes the form

$$h_s(\eta)=\frac{m}{\beta}\cosh\beta\eta,\quad h_b(\eta)=\frac{2m}{\beta}\cosh\frac{\beta\eta_1}{2}\sin\frac{\beta\eta_2}{2}\geqslant 0.$$

The bound-state energy h_b depends on η_2 amd η_1. First one defines the inner motion of two bounded solitons and the second one defines the bound states center of mass motion. Correspondingly we will call these parameters as the internal and external ones. Note that the inner motion is periodic, see (65).

Following the definition of the Dirac measure one should sum over all solutions of the Lagrange equation, see the property (b). In Sec. II. As follows from the equality:

$$\sum_{\{u_c\}}=\int_{W_G}\mathrm{d}\xi_0\,\mathrm{d}\eta_0\sigma(u;\xi_0,\eta_0),$$

we should define the density $\sigma(u;\xi_0,\eta_0)$ of states in the element of the coset space W_G. The Faddeev–Popov *ansatz* is used for this purpose.[4]

In our approach, performing the transformation into the coset space W_G, we define the density $\sigma(u;\xi_0,\eta_0)$. Indeed, using the definition

$$\int Dx\prod_t \delta(\dot{x})=\int \mathrm{d}x(0)=\int \mathrm{d}x_0$$

the functional integrals with measure (56) are reduced to the ordinary ones over the initial data $(\xi,\eta)_0$.

But it is important here to trace on the following question. One can note that, at first glance, integration over $(\xi,\eta)_0$ may only give $\rho\sim V_0^1$, where V_0 is the zero modes volume, i.e., is a volume of the W_G space. On other hand, as follows from definition of $\rho\sim|a_{nm}|^2$, one may expect that $\rho\sim V_0^2$. This discrepancy should have an explanation.

Remembering definition of ρ as the square of the amplitudes, we should define the contributions on the whole time contour $C=C_+ + C_-$, see (15), to take into account the input condition that the trajectories $u_+(t\in C_+)$ and $u_-(t\in C_-)$ are absolutely independent in the frame of the closed-path boundary condition (27):

$$u_c(x,t\in\partial C_+)=u_c(x,t\in\partial C_-), \tag{66}$$

where $\partial C_\pm$ is the boundary of $C_\pm$. Other directions to the $\sigma_|infty$ are not important here.

Then, if we introduce $(\xi,\eta)(t\in C_\pm)|_0\equiv(\xi_0,\eta_0)_\pm$, one should have in mind that, generally speaking, $(\xi_0,\eta_0)_+\neq(\xi_0,\eta_0)_-$ and the integration over them should be performed independently. This may explain the above discrepancy and one should have $\rho\sim V_0^2$.

It is not hard to see that for our topological solitons the condition (66) leads to the equalities

$$(\xi_0,\eta_0)_+=(\xi_0,\eta_0)_-=(\xi_0,\eta_0). \tag{67}$$

To see this it is enough to insert (64), or (65), into (66) and take into account that at $t\in\partial C_\pm$ the estimation (63) is right.

Solution (67) means that, for arbitrary functional $F(\xi,\eta)$,

$$\int\prod_{t\in C_+ + C_-} d\xi d\eta\,\delta(\dot\xi)\delta(\dot\eta)F(\xi,\eta)=\int d\xi_{0+}d\eta_{0+}\int d\xi_0 d\eta_0 F(\xi_0,\eta_0). \tag{68}$$

Therefore, $\rho\sim V_0^2$. We will put out the integrals over inessential variables ξ_{0+} and η_{0+}.

It should be underlined that (67) is the consequence of the conservation of the topological charge: the solitons by this reason are the stable formation and, therefore, to satisfy the closed path boundary condition, one should have (67).

Performing the shifts

$$\xi_i(t)\to\xi_i(t)+\int dt'\, g(t-t')j_{\xi,i}(t')\equiv\xi_i(t)+\xi_i'(t),$$

$$\eta_i(t)\to\eta_i(t)+\int dt'\, g(t-t')j_{\eta,i}(t')\equiv\eta_i(t)+\eta_i'(t),$$

we can get the Green function $g(t-t')$ into the operator exponent

$$\hat K(ej)=\frac{1}{2}\int dt\,dt'\,\Theta(t-t')\{\hat\xi'(t')\cdot\hat e_\xi(t)+\hat\eta'(t')\cdot\hat e_\eta(t)\}, \tag{69}$$

since the Green function $g(t-t')$ of the transformed theory is the step function[10]

$$g(t-t')=\Theta(t-t'). \tag{70}$$

Such Green function allows to shift $C_\pm$ on the real-time axis. This, noting (67), excludes doubling of the degrees of freedom.

Notice the Lorentz noncovariantness of our perturbation theory with Green function (70).

The measure takes the form

$$D^\nu M(\xi,\eta)=\prod_{i=1}^{\nu}\prod_t d\xi_i(t)d\eta_i(t)\,\delta(\xi_i-\omega(\eta+\eta'))\,\delta(\dot\eta_i). \tag{71}$$

The interactions are described by

$$U(u_\nu;e_c)=-\frac{2m^2}{\lambda^2}\int \mathrm{d}x\,\mathrm{d}t\,\sin\lambda u_\nu(\sin\lambda e_c-\lambda e_c) \tag{72}$$

with

$$u_\nu=u_\nu(x;\xi+\xi',\eta+\eta') \tag{73}$$

and e_c was defined in (57).

The equation

$$\dot{\xi}_i=\omega(\eta_i+\eta_i') \tag{74}$$

is trivially integrable. In the quantum case $\eta_i'\neq 0$ this equation describes motion in the nonhomogeneous and anisotropic manifold. So, the expansion over $(\hat{\xi}',\hat{e}_\xi,\hat{\eta}',\hat{e}_\eta)$ generates the local in time fluctuations of W_G manifold. The weight of these fluctuations is defined by $U(u_\nu;e_c)$.

Expansion of $\exp\{\hat{K}(je)\}$ gives the strong coupling perturbation series. The analyses show that[14]

S3. Action of the integro-differential operator $\hat{O}$ leads to the following representation:

$$\rho(\alpha,z)=\int_{W_G}\left\{\mathrm{d}\xi(o)\cdot\frac{\partial}{\partial\xi(0)}R^{\xi}(\alpha,z)+\mathrm{d}\eta(0)\cdot\frac{\partial}{\partial\eta(0)}R^{\eta}(\alpha,z)\right\}. \tag{75}$$

This means that the contributions into ρ are accumulated strictly on the boundary bifurcation manifold ∂W_G. The proof of this important result was given in Refs. 10 and 14 and we will use it without comments.

We would divide calculations on two parts. First of all, we would consider the semiclassical approximation and then we will show that this approximation is exact.

Performing the last integration we find

$$\rho(\alpha,z)=\int\prod_{i=1}^{\nu}\{\mathrm{d}\xi_0\,\mathrm{d}\eta_0\}_i e^{-i\hat{K}}e^{iS_O(u_\nu)}e^{-iU(u_\nu;e_c)}e^{N(\alpha,z;u_\nu)}, \tag{76}$$

where

$$u_\nu=u_\nu(\eta_0+\eta',\xi_0+\omega(t)+\xi') \tag{77}$$

and

$$\omega(t)=\int \mathrm{d}t'\,\Theta(t-t')\omega(\eta_0+\eta')(t'). \tag{78}$$

In the semiclassical approximation $\xi'=\eta'=0$ we have

$$u_\nu=u_\nu(x;\eta_0,\xi_0+\omega(\eta_0)t). \tag{79}$$

Notice that the surface term

$$\int \mathrm{d}x^\mu\,\partial_\mu(e^{iqx}u_\nu)=0. \tag{80}$$

Then

$$\int d^2x\, e^{iqx}(\partial^2+m^2)u_\nu(x,t) = -(q^2-m^2)\int d^2x\, e^{iqx}u_\nu(x,t)=0, \tag{81}$$

since q^2 belongs to the mass shell by definition. The condition (80) is satisfied for all $q_\mu \neq 0$ since u_ν belongs to the Schwarz space. Therefore, in the semiclassical approximation $R^c(\alpha,z)$ is the trivial function of z: $\partial R^c(\alpha,z)/\partial z=0$.

Expending the operator exponent in (76), we find that action of the operators $\hat{\xi}'$, $\hat{\eta}'$ create the terms

$$\sim \int d^2x\, e^{iqx}\theta(t-t')(\partial^2+m^2)u_\nu(x,t)\neq 0. \tag{82}$$

So, generally $R(\alpha,z)$ is the nontrivial function of z.

Now we will show that the semiclassical approximation is exact in the soliton sector of the sin-Gordon model. The structure of the perturbation theory is readily seen in the normal-product form

$$R(\alpha,z)=\sum_\nu \int \prod_{i=1}^{N} \{d\xi_0 d\eta_0\}_i :e^{-iU(u_\nu;\hat{j}/2i)}e^{iS_O(u_\nu)}e^{N(\alpha,z;u_\nu)}:, \tag{83}$$

where

$$\hat{j}=\hat{j}_\xi\cdot\frac{\partial u_\nu}{\partial\eta}-\hat{j}_\eta\cdot\frac{\partial u_\nu}{\partial\xi}=\Omega\hat{j}_X\frac{\partial u_\nu}{\partial X} \tag{84}$$

and

$$\hat{j}_X=\int dt'\,\Theta(t-t')\hat{X}(t') \tag{85}$$

with the $2N$-dimensional vector $X=(\xi,\eta)$. In Eq. (84) Ω is the ordinary symplectic matrix.

The colons in (83) mean that the operator $\hat{j}$ should stay to the left of all functions. The structure (84) shows that each order over $\hat{j}_{X_i}$ is proportional at least to the first order derivative of u_ν over conjugate to X_i variable.

The expansion of (83) over $\hat{j}_X$ can be written using the form

$$\rho(\alpha,z)=\sum_\nu \int \prod_{i=1}^{\nu} \{d\xi_0\, d\eta_0\}_i \left\{\sum_{i=1}^{2\nu}\frac{\partial}{\partial X_{0i}}P_{X_i}(u_\nu)\right\}, \tag{86}$$

where $P_{X_i}(u_\nu)$ is the infinite sum of the time-ordered polynomial over u_ν and its derivatives.[14] The explicit form of $P_{X_i}(u_\nu)$ is unimportant, it is enough to know, see (84), that

$$P_{X_i}(u_\nu)\sim\Omega_{ij}\frac{\partial u_\nu}{\partial X_{0j}}. \tag{87}$$

Therefore,

$$\frac{\partial}{\partial z}R(\alpha,z)=0 \tag{88}$$

since (i) each term in (86) is the total derivative, (ii) we have (87), and (iii) u_ν belongs to Schwarz space.

IV. CONCLUSION

We would like to conclude this paper noting the role of the coset space G/G_c topology. It was shown that if

(i) $W_G = G/G_c \neq \varnothing$,
(ii) $W_G = T^*V$ is the simplectic manifold,
(iii) ∂u_c is the phase space flow [see (87)],
(iv) $\partial u_c \cap \partial W_G = \varnothing$,

then the semiclassical approximation is exact.

For this reason, being absolutely stable, topological solitons are unable to describe the multiple production processes. This property of the exactly integrable models was formulated also as the absence of stochastization in the integrable systems.[28] The $O(4)\times O(2)$-invariant solution of $O(4,2)$-invariant theories[29] is noticeably more interesting from this point of view.[30]

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APPENDIX: DERIVATION OF EQ. (29)

The generating functional (24) can be written in the form

$$\rho(\beta,z) = e^{-\bar{n}(s)(\beta,z;\varphi)}\rho_0(\varphi), \tag{A1}$$

where the particles number operator

$$\bar{n}(s)(\beta,z;\varphi) = \bar{n}(s)(\beta_+,z_+;\varphi) + \bar{n}(s)^*(\beta_-,z_-;\varphi) \tag{A2}$$

and

$$\bar{n}(s)(\beta_+,z_+;\varphi_+) = \int d\Omega_1(q)\hat{\varphi}_+^*(q)\hat{\varphi}_-(q)e^{-\beta_+\epsilon(q)}z_+(q) \tag{A3}$$

is the produced particle number operator.

The functional ρ_0 was introduced in (25):

$$\rho_0(\varphi) = Z(\varphi_+)Z^*(-\varphi_-)$$
$$= \int Du_+ Du_- e^{iS_+(u_+) - iS_-(u_-)} e^{-iV_+(u_+ + \varphi_+) + iV_-(u_- - \varphi_-)}. \tag{A4}$$

So, the integration over u_+ and u_- is not performed independently: one should take into account the boundary condition (27). We can perform in this integral the linear transformation

$$u_\pm(x) = u(x) \pm \phi(x). \tag{A5}$$

Then the boundary condition (27) leads to the equality

$$\phi(x\in\sigma_\infty) = 0, \tag{A6}$$

leaving $u(x\in\sigma_\infty)$ arbitrary. Last one means that the integration over this turning-point field $u(x \in \sigma_\infty)$ should be performed, see Sec. III.

Let us extract in the exponents (A4) the linear term over $(\phi+\varphi)$:

$$V_+(u+(\phi+\varphi))-V_-(u-(\phi+\varphi))$$
$$=U(u,\phi+\varphi)+2\,\mathrm{Re}\int_{C_+}dx(\phi(x)+\varphi(x))v'(u) \tag{A7}$$

and

$$S_+(u+\varphi)-S_-(u-\varphi)=S_O(u)-2i\,\mathrm{Re}\int_{C_+}dx\,\varphi(x)(\partial_\mu^2+m^2)u(x), \tag{A8}$$

where

$$2\,\mathrm{Re}\int_{C_+}=\int_{C_+}+\int_{C_-}.$$

Notice that generally speaking, $S_O(u)\neq 0$, if the topology of the field $u(x)$ is nontrivial, see Sec. III.

The expansion over $(\phi+\varphi)$ can be written in the form

$$e^{-iU(u,\phi+\varphi)}=e^{(1/2i)\,\mathrm{Re}\int_{C_+}dx\,\hat{j}(x)\hat{\varphi'}(x)}e^{i2\,\mathrm{Re}\int_{C_+}dx\,dt\,j(x)(\phi(x)+\varphi(x))}e^{-iU(u,\varphi')}, \tag{A9}$$

where $\hat{j}(x)$, $\hat{\varphi'}(x)$ are the variational derivatives. The auxiliary variables (j,φ') must be taken equal to zero at the very end of the calculations.

In result,

$$\rho_0(\phi)=e^{(1/2i)\mathrm{Re}\int_{C_+}dx\,\hat{j}(x)\hat{\phi}(x)}\int Du\,e^{is_0(u)}e^{-iU(u,\varphi)}e^{i2\,\mathrm{Re}\int_{C_+}dx(j(x)-v'(u))\phi(x)}$$
$$\times\prod_x\delta(\partial_\mu^2u+m^2u+v'(u)-j), \tag{A10}$$

where the functional δ function was defined by the equality

$$\prod_x\delta(\partial_\mu^2u+m^2u+v'(u)-j)=\int D'\phi\,e^{-2i\,\mathrm{Re}\int_{C_+}dx(\partial_\mu^2u+m^2u+v'(u)-j)\varphi(x)}, \tag{A11}$$

where the prime means that $D'\phi$ does not includes the integration over $\phi(x\in\sigma_\infty)$. This condition is not seen in the functional δ function because of the definition

$$\int\prod_x du(x)\delta(\partial_\mu u(x))=\int du(x_\mu\in\sigma_\infty).$$

Equation (A10) can be rewritten in the equivalent form

$$\rho_0(\phi)=e^{-i\hat{K}(j,\varphi)}\int DM(u)e^{is_0(u)-iU(u,\varphi)}e^{i2\,\mathrm{Re}\int_{C_+}dx\,\phi(x)(\partial_\mu^2+m^2)u(x)} \tag{A12}$$

because of the δ functional measure

$$DM(u)=\prod_x du(x)\delta(\partial_\mu^2u+m^2u+v'(u)-j), \tag{A13}$$

with

$$\hat{K}(j\varphi)=\frac{1}{2}\mathrm{Re}\int_{C_+}dx\hat{j}(x)\hat{\phi}(x). \tag{A14}$$

Notice at the end that the contour C_+ in (A14) cannot be shifted on the real time axis since the Green function of the equation

$$\partial_\mu^2 u + m^2 u + v'(u) = j$$

is singular on the light cone.

The action of operator $N(\beta, z; \hat{\phi})$ maps the interacting fields system on the physical states. Last ones are marked by $z_\pm$ and $\beta_\pm$. The operator exponent is the linear functional over ϕ and this allows easily find (28).

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