Quantisation of Extended Objects

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Abstract

The 'strong-coupling' perturbation theory over the inverce interaction constant $1/g$ near the nontrivial solution of Lagrange equation is formulated. The ordinary 'week-coupling' perturbation theory over g is described also to compare both perturbation theories. The 'strong-coupling' perturbation theory is developed by unitary mapping of the quantum dynamics into the space with local coordinates of (action, angle)-type.

1 Introduction

The extended (soliton-like) objects quantisation problem considered in this paper have more than twenty years old history, see the review papers [1, 2]. But absence of progress in this field evokes anxiety, noting a number of unsolved by this reason important physical problems.

One of possible solutions of this problem was offered in [3]. The aim of this article is to show that the approach described in this paper leads to the strong coupling perturbation theory over inverse interaction constant $1/g$.

Our approach is based on the idea [4] that the measure DM of the functional integral representation for

$$\rho(E) = \int du_1 du_2 | < u_2; E | u_1; E > |^2 \tag{1.1}$$

is δ-like (Diracian):

$$DM(u) = \prod_{x,t} du(x,t) \delta \left(\frac{\delta S(u)}{\delta u(x,t)} - j(x,t) \right), \tag{1.2}$$

where $j(x,t)$ is the random force of quantum excitations. We will consider the symplest one particle quantum problem and the states in (1.1) are described by the boundary values of coordinate u_i and energy E. In Sec.2.1 the physical basis of (1.2) will be discussed and full derivation will be given in Sec.2.2.

The δ-function of (1.2) means that the strict space-time local equality:

$$- \frac{\delta S(u)}{\delta u(x,t)} = j(x,t) \tag{1.3}$$

defines the complete set of necessary contributions. The general properties of theory defined on the δ-like measure is listed in Sec.2.3. Note, (1.3) is not the consequence of Hamiltonian variational principle and (1.2) will be derived in Sec.2.2, proceeding from the conservation of total probability (unitarity condition) [4].

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Eq.(1.3) shows that a transformation of kinetic part of Lagrangian without fail induce the tangent transformation of quantum source $j(x,t)$ [5]. We will use this possibility to describe the quantum dynamics in useful terms. Namely, we will apply the canonical transformation of (1.2) to the collective coordinates. They will have a meaning of (action, angle)-type variables and will form the cotangent foliation $T^*\Omega$ to the incident phase space Ω. So, used in this paper transformation is the ordinary momentum mapping of classical mechanics [6]:

$$J: \ (u,p) \to (\xi,\eta), \tag{1.4}$$

where p is the conjugate to u momentum, see Secs.4.1 and 4.2. The Hamiltonian description will be useful by this reason.

In this paper we restrict ourselves by one dimensional quantum mechanics assuming that

$$v(u;g) = \frac{1}{g}v(g^{1/2}u;1) \equiv \frac{1}{g}v(g^{1/2}u), \tag{1.5}$$

where, for simplicity, $v(u)$ is the potential hole with one minimum at $u=0$ and g is the interaction constant. Then the nontrivial solution u_c of eq.(1.3) would be singular at $g=0$:

$$u_c = O(g^{-1/2}). \tag{1.6}$$

in the lowest order over j.

We will find that the transformed perturbation theory presents the expansion over $\sim 1/g$ and the expansion coefficients are simply calculable iff $T^*\Omega$ is the homogeneous and isotropic space

This result cardinally distinguished from the week coupling perturbation theory developed in [1, 2, 7]. We will derive in Sec.3 this ordinary perturbation series over g using the measure (1.3) to show exactly where we turn from habitual way to formulate new perturbation theory. Strictly speaking, there is not any connections among both perturbation theories and they are dual to each other, see Sec.5.

The paper is organised as follows. In Sec.2 we will find the integral representation for ρ with measure (1.2). In Sec.3 we will describe the week-coupling perturbation theory. In Sec.4 the mapping (1.4) for the quantum system is described to show the decomposition over $1/g$ and the rule of calculation of corresponding coefficient will be given. In concluding Sec.5 we will offer the dynamical interpretation of new perturbation theory.

2 Unitary definition of the functional measure

Starting this section we will try to explain the role of unitarity in definition of functional measure, Sec.2.1. Then we will show as the d'Alembert's variational principle may be *derived* for quantum systems (Sec.2.2) and, at the end, the general properties of theory on the δ-like measure will be offered.

2.1 Formulation of method

To calculate the bound state energies E_n it is enough to consider the trace[8]:

$$R(E) = \sum_n \int du \frac{\psi_n(u)\psi_n^*(u)}{E-E_n-i\varepsilon} = \sum_n \frac{1}{E-E_n-i\varepsilon} = \mathrm{Sp}\ \frac{1}{E-\mathbf{H}-i\varepsilon}, \tag{2.1}$$

where $\mathbf{H}$ is the Hamiltonian operator and $\varepsilon \to +0$ and the wave functions ψ_n ortho-normalizability was used.

The semiclassical approximation leads to

$$R(E) \sim \sum_{k=0}^{\infty} e^{ik(S_1(u_c)-\pi)} = \frac{1}{1+e^{iS_1(u_c)}}, \tag{2.2}$$

where $S_1(u_c)$ is the action on the elementary (one period) closed path trajectory $u_c = u_c(E)$ [8]. The position of poles in (2.2) defines the value of E_n.

Note now that

$$\frac{1}{E-E_n-i\varepsilon} = \mathrm{P}\frac{1}{E-E_n} + i\pi\delta(E-E_n) \text{ at } \varepsilon = 0,$$

i.e. it is not necessary to calculate the real part since it did not contain the masurable value of energy E_n:

$$\mathrm{P}\frac{1}{E-E_n} = 0 \quad E = E_n.$$

By this reason, following to [4], we will calculate much more simple quantity

$$\varepsilon\rho(E) = \varepsilon \sum_{n_1,n_2} \int dx_1 dx_2 \frac{\psi_{n_1}(x_1)\psi^*_{n_1}(x_2)}{E-E_{n_1}-i\varepsilon} \frac{\psi^*_{n_2}(x_1)\psi_{n_2}(x_2)}{E-E_{n_2}+i\varepsilon} =$$

$$= \varepsilon \sum_n \left|\frac{1}{E-E_n-i\varepsilon}\right|^2 = \frac{1}{2i}\sum_n \left\{\frac{1}{E-E_n-i\varepsilon} - \frac{1}{E-E_n+i\varepsilon}\right\} =$$

$$= \pi \sum_n \delta(E-E_n) = \pi \ \mathrm{Sp}\ \delta(E-\mathbf{H}) = \mathrm{Im}\ R(E). \tag{2.3}$$

Therefore, we wish exclude from consideration the unnecessary contributions[2] with $E \neq E_n$. It should be noted that we exclude continuum of contributions contained in $\mathrm{Re}\{1/E-E_n-i\varepsilon\}$ and leave the set of point-like contributions $\mathrm{Im}\{1/E-E_n-i\varepsilon\}$, but with infitite amplitudes.

We can find [4] that

$$\rho(E) \sim \sum_{k=-\infty}^{+\infty} e^{ik(S_1(u_c)-\pi)} = 2\pi \sum_n \delta(S_1(u_c)-(2n+1)\pi) \tag{2.4}$$

So, as in (2.2), the aim of quantum perturbation theory is to define the corrections to the phase $S_1(u_c)$

In terms of integrals the cancellation phenomena, shown in (2.3), looks as follows:

$$\rho(E) = \sum_n \left|\frac{1}{E-E_n-i\varepsilon}\right|^2 = \sum_n \int_0^{\infty} dT_+ dT_- e^{-\varepsilon(T_++T_-)+i(E-E_n)(T_+-T_-)} \tag{2.5}$$

To see the effect of cancellation let us introduce new time variables T and τ:

$$T_\pm = T \pm \tau. \tag{2.6}$$

[2] 'Unnecessary' means for us the unmeasurable quantity in *given* experiment.

The Jacobian of transformation gives: $0 \leq T \leq \infty$ and $-T \leq \tau \leq T$. But in the integral (2.5) $T \sim (1/\varepsilon) \to \infty$ are essential at $\varepsilon \to 0$. By this reason we can put $|\tau| \leq \infty$. In result,

$$\varepsilon\rho(E) = 2\pi\varepsilon \int_0^\infty dT e^{2\varepsilon T} \int_{-\infty}^{+\infty} \frac{d\tau}{\pi} e^{2i(E-E_n)\tau} \tag{2.7}$$

In the last integral all contributions, except for the case $E = E_n$, are cancelled.

Described cancellation is not accidental, or approximate, being the consequence of optical theorem, i.e. is the consequence of unitarity condition. The δ-likeness of measure (1.2) has the same nature as the δ-function in the r.h.s. of (2.3), i.e. the δ-like measure will arise when the absorption part of amplitudes is calculated.

Note, we start from claculation of modulo squire of amplitudes since we know the path integral reprisantation for them. Then, using the unitarity condition we find the correct measure for imaginary part of the amplitude.

2.2 Functional δ-like measure

We will use following path-integral representation for amplitude

$$a(u_1, u_2; E) = i \int_0^\infty dT e^{iET} \int_{u(0)=u_1}^{u(T)=u_2} Du e^{iS_{C_+(T)}(u)}, \; Du = \prod_{t \in C_+(T)} \frac{du(t)}{\sqrt{2\pi}}, \tag{2.8}$$

to calculate

$$\rho(E) = \int du_1 du_2 \left| a(u_1, u_2; E) \right|^2 . \tag{2.9}$$

The action $S_{C_+(T)}(u)$ is defined on the Mills complex time contour [9]:

$$C_\pm(T): \; t \to t \pm i\varepsilon, \; \varepsilon \to +0, \; 0 \leq t \leq T \tag{2.10}$$

Inserting $a(u_1, u_2; E)$ into (2.9) we find:

$$\rho(E) = \int_0^\infty dT_+ dT_- e^{iE(T_+ - T_-)} \int_{u_+(0)=u_-(0)}^{u_+(T_+)=u_-(T_-)} Du_+ Du_- e^{iS_{C_+(T_+)}(u_+) - iS_{C_-(T_-)}(u_-)} \tag{2.11}$$

Note crucial for us the 'closed-path' boundary conditions:

$$u_+(0) = u_-(0), \; u_+(T_+) = u_-(T_-). \tag{2.12}$$

We will introduce new variables T and τ, see (2.6). The integral over τ will be calculated perturbatively. In zero order over τ we would have from (2.12):

$$u_+(0) = u_-(0), \; u_+(T) = u_-(T). \tag{2.13}$$

It should be underlined that this is unique solution of the boundary condition (2.12) which did not contradict to the quantum uncertainty principle (other solutions of (2.12) would involve constraints for time derivatives of coordinate).

If we introduce now new coordinates u and x:

$$u_\pm(t) = u(t) \pm x(t), \tag{2.14}$$

then (2.13) gives:

$$x(0) = x(T) = 0 \tag{2.15}$$

and $u(0)$ and $u(T)$ are arbitrary. We will see that this 'minimal' boundary condition is sufficient to define the integrals over τ and u.

Let us expand the closed path action

$$S_{cl}(u \pm x; T \pm \tau) \equiv (S_{C_+(T+\tau)}(u + x) - S_{C_-(T-\tau)}(u - x))$$

over τ:

$$S_{cl}(u \pm x; T \pm \tau) = S_{cl}(u \pm x; T) - 2\tau H_T(u) - 2\tilde{H}_T(u; \tau), \tag{2.16}$$

where the Hamiltonian at the time moment T

$$H_T(u) = -\frac{\partial}{\partial T} S_{C_+(T)}(u). \tag{2.17}$$

is x independent because of (2.15). The remainder term $\tilde{H}_T(u; \tau)$ contains higher powers over τ:

$$\tilde{H}_T(u; \tau) = \sum_{n=1}^{\infty} \frac{\tau^{2n+1}}{(2n+1)!} \frac{d^{2n}}{dT^{2n}} H_T(u).$$

Therefore, the conditions (2.15) factorize τ and $x(t)$ dependence: the x dependence is contained in the τ independent quantity $S_{cl}(u \pm x; T)$ only. So, we may construct the perturbation theory over τ and x independently.

Let us consider now expansion over x:

$$S_{cl}(u \pm x; T) = S_{P(T)}(u) - 2\mathrm{Re} \int_{C_+(T)} dt x(t) \frac{\delta S_{C_+}(u)}{\delta u(t)} - 2\tilde{V}_T(u, x), \tag{2.18}$$

where the first term in this decomposition is:

$$S_{P(T)}(u) = (S_{C_+(T)}(u) - S_{C_-(T)}(u)). \tag{2.19}$$

If the motion is periodic then $S_{P(T)}(u)$ is not equal to zero even on the real time axis [4]. In semiclassical approximation

$$S_P(T)(u_c) = kS_1(u_c), k = 0, 1, ...,$$

and is T independent. The reason of this conclusion is explained in Sec.3.3. As usual,

$$2\mathrm{Re} \int_{C_+} dt = \int_{C_+} dt + \int_{C_-} dt \tag{2.20}$$

since for arbitrary analytic function $f(t \in C_+) = f^*(t \in C_-)$.

Following formal trick will be useful. We can write:

$$e^{-2i\tilde{H}_T(u;\tau)} = \sum_n \frac{\tau^n}{n!} K_n(u; T),$$

where

$$K_n(u;T) = \frac{d^n}{d\tau_1^n} e^{-2iH_T(u;\tau_1)}|_{\tau_1=0} \equiv \hat{\tau}_1^n e^{-2i\tilde{H}_T(u;\tau_1)}.$$

On other hand,

$$(2i\tau)^n = \frac{d^n}{d\varepsilon^n} e^{2i\varepsilon\tau}|_{\varepsilon=0} \equiv \hat{\varepsilon}^n e^{2i\varepsilon\tau}.$$

Therefore,

$$e^{-2i\tilde{H}_T(u;\tau)} = \sum_n \frac{(\hat{\tau}_1\hat{\varepsilon}/2i)^n}{n!} e^{2i\varepsilon\tau} e^{-2i\tilde{H}_T(u;\tau_1)} = e^{-i\hat{\tau}_1\hat{\varepsilon}/2} e^{2i\varepsilon\tau} e^{-2i\tilde{H}_T(u;\tau_1)}. \tag{2.21}$$

The expansion of the operator $e^{-i\hat{\tau}_1\hat{\varepsilon}/2}$ will generate corresponding perturbation series.

The same operator can be introduced for expansion over the local quantity x:

$$e^{-2i\tilde{V}_T(u,x)} = e^{-\frac{i}{2}\mathrm{Re}\int_{C_+} dt\hat{j}(t)\hat{x}_1(t)} e^{2i\mathrm{Re}\int_{C_+} dtj(t)x(t)} e^{-2i\tilde{V}_T(u,x_1)}. \tag{2.22}$$

Note, the eqs.(2.21), (2.22) linearise the arguments of corresponding exponents. Then, using (2.16), (2.18) and (2.21), (2.22) we find that

$$\rho(E) = 2\pi \int_0^\infty dT e^{-i\mathbf{K}(\varepsilon\tau,\mathbf{jx})} \int DM(u)\delta(E+\varepsilon - H_T(u)) e^{iS_{P(T)}(u)} e^{-2i\tilde{H}_T(u;\tau)-2i\tilde{V}_T(u,x)}, \tag{2.23}$$

where expansion over the operator

$$\mathbf{K} = \frac{1}{2}\left(\hat{\tau}\hat{\varepsilon} + \mathrm{Re}\int_{C_+} dt\hat{j}(t)\hat{x}(t)\right) \tag{2.24}$$

gives the perturbation series. At the very end of calculations all auxiliary variables τ, ε, j and x should be taken equal to zero.

The measure in (2.23) is defined as follows:

$$DM(u) = \prod_t du\delta\left(-\frac{\delta S(u)}{\delta u} - j\right) = \prod_t du\delta(\ddot{u} + v'(u) - j) \tag{2.25}$$

and the δ-function is defined by the equality:

$$\prod_t \delta(\ddot{u} + v'(u) - j) = \int_{x(0)=0}^{x(T)=0} \prod_t \frac{dx}{\pi} e^{2i\mathrm{Re}\int dtx(\ddot{u}+v'(u)-j)}. \tag{2.26}$$

Argument of this δ-function did not contain the boundary values $u(0)$ and $u(T)$. But this is not important since to solve the second order equation (2.28) two constant of integration is necessary.

The exponent in (2.26) is equal to the sum: $\mathrm{Re}x\ \mathrm{Re}(\ddot{u} + v'(u) - j) + \mathrm{Im}x\ \mathrm{Im}(\ddot{u} + v'(u) - j)$, being defined on the complex time contour. This means that

$$\prod_t \delta(\ddot{u} + v'(u) - j) = \prod_{t\in C} \delta(\mathrm{Re}\{\ddot{u} + v'(u) - j\})\delta(i\mathrm{Im}\{\ddot{u} + v'(u) - j\}), \tag{2.27}$$

where $C = C_+ + C_-$. So, the measure (2.25) defines both the real and imaginary part of contributions.

By definition, $(\ddot{u}+v'(u)-j)$ is the total force, then the product $(\ddot{u}+v'(u)-j)x$ is the virtual work. In classical mechanics this work should be equal to zero, since the classical motion is time reversible (d'Alembert). Then, noting that virtual deviation is arbitrary, one finds the local condition:

$$\ddot{u} + v'(u) - j = 0. \tag{2.28}$$

when the motion is time reversible.

In quantum case the virtual work is not equal to zero (quantum corrections shift the energy levels), but the integral over $x(t)$ gives the same result (2.28). We can conclude that the unitarity condition of quantum mechanics allows to *derive* the d'Alembert's variational principle of classics mechanics [4], see also [10].

2.3 Properties of theory with δ-like measure

The eq.(2.28) should be solved expanding over $j(t)$:

$$u_j(t) = u_c(t) + \int dt' G(t,t';u_c)j(t') + (\text{higher powers of } j) \tag{2.29}$$

where u_c is the solution of homogeneous equation:

$$\ddot{u} + v'(u) = 0 \tag{2.30}$$

and $G(t,t';u_c)$ is the Green function:

$$(\partial_t^2 + v''(u_c))G(t,t';u_c) = \delta(t-t'). \tag{2.31}$$

The eq.(2.30) have in our case the trivial constant solution

$$u_0 : \ \dot{u}_0 = 0, \ v'(u_0) = 0 \tag{2.32}$$

and nontrivial one

$$u_c = u_c(t) : \ \dot{u}_c(t) \neq 0, \ \ddot{u}_c + v'(u_c) \equiv 0. \tag{2.33}$$

Because of definition of the δ-function and since there is not any special restriction on the contributions both one should be taken into account:

$$\rho(E) = \rho_0(E) + \rho_c(E). \tag{2.34}$$

This means that one should sum over all possible topological classes of trajectory, if the single class is unable to cover all phase space. Each class of trajectories belongs to restricted domain of phase space: $\Omega = W^0 \times W^c$ in our case. Each sub-domain W^i is restricted by the bifurcation lines [11]. This means that ρ_0 can not be achieved by analytical continuation of ρ_c (for instance, taking $E = 0$ in ρ_c for the semiclassical approximation).

It is evident, one should leave in the sum (2.34) the term with higher volume V_{W^i}, $i = 0, c$. This is just the domain of $u_c \in W^c$ trajectory, and one can put out ρ_0 since the sub-domain of $u_c^0 \in W^0$ is the point [4, 11]. Indeed, it will be shown that $\rho_c \sim V_{tr} = \infty$, where V_{tr} is the volume of time translations mode (zero frequency mode). At the same time, $\rho_0 \sim O(1)$. Therefore, iff the time translation invariance is unbroken, one can say that ρ_0 is realised on the measure $\sim O(1)/V_{tr} = 0$.

The ability to classify contributions by the trajectory topology classes becomes possible since there is not in (2.34) the u_c^0 and u_c interference term. This is evident consequence of the orthogonality of corresponding Hilbert spaces. Therefore, the choice of solution means choice of corresponding vacuum.

3 WKB perturbation theory

It can be shown that (2.23) restores ordinary WKB expansion. The first step of this calculations is to find the solution of inhomogeneous equation (2.28), Sec.3.1. Then we may find that this perturbation theory counts positive powers of g, Sec.3.2. At the end the zero frequency modes problem will be discussed.

3.1 Tree decomposition

Let us consider the tree decomposition (2.29) more carefully for the potential

$$v(u;g) = \frac{1}{2}w_0^2u^2 + \frac{1}{4}gu^4. \tag{3.1}$$

It is evident:

$$v(u;g) = \frac{1}{g}v(g^{1/2}u) \tag{3.2}$$

The decomposition (2.29) can be written in the form:

$$u_j(t) = u_c(t) + \sum_{n=1}^{\infty}\frac{1}{n!}\int\prod_{k=1}^{n}\{dt_i j(t_i)\}\,G_n(t,t_1,....,t_n;u_c) \tag{3.3}$$

It easy to show that the n-point Green function [1]

$$G_n = O(g^{(n-1)/2}). \tag{3.4}$$

Indeed, inserting (3.3) into the equation:

$$\ddot{u} + \omega_0^2u + gu^3 = j \tag{3.5}$$

we find:

$$(\partial_t^2 + \omega_0^2 + 3gu_c^2)G_1(t,t_1;u_c) = \delta(t-t_1) \tag{3.6}$$

The operator $(\partial_t^2 + \omega^2 + 3gu_c^2)$ is g independent since $u_c = O(1/g^{1/2})$ and, therefore, $G_1 = O(g^0)$.

Note also, the operator $(\partial_t^2 + \omega^2 + 3gu_c^2)$ is translationally noninvariant. By this reason considered perturbation theory is sufficiently complicated so that only first corrections has been computed till now.

The equation for G_2 have the form:

$$(\partial_t^2 + \omega^2 + 3gu_c^2)G_2(t,t_1,t_2;u_c) + 6gu_cG_1(t,t_1;u_c)G_1(t,t_2;u_c) = 0. \tag{3.7}$$

Therefore, in accordance with (3.4), $G_2 = O(g^{1/2})$. In result, the analysis of higher orders over j would justify (3.4).

The interactions generating functional V_T computed for the case (3.1) has the form:

$$\tilde{V}_T(u,x) = 2g\mathrm{Re}\int_{C_+}dtx^3(t)u(t) + O(\varepsilon), \tag{3.8}$$

where the $O(\varepsilon)$ term is proportional to the imaginary part of S_{cl}.

The operator $\mathbf{K}$ is linear over $\hat{x} = \delta/\delta x$. Therefore, action of $\exp\{-i\mathbf{K}\}$ will give:

$$\rho_c \sim : e^{-2i\tilde{\mathbf{V}}_T(u,\hat{j}/2i)} : e^{iS_{P(T)}(u)} e^{-2i\tilde{H}_T(u;\tau)} \delta(E + \varepsilon - H_T(u_j)), \tag{3.9}$$

where the colons prescribe normal product, when the operator should stay to the left of all functions on which it may act, and the unimportant for present analyses integrations were not mentioned.. The expansion of the operator exponent gives the perturbation series:

$$\rho_c \sim \sum_n \frac{(-2i)^n}{n!} : \tilde{\mathbf{V}}_T^n(u_j, \hat{j}/2i) : e^{iS_{P(T)}(u_j)} e^{-2i\tilde{H}_T(u_j;\tau)} \delta(E + \varepsilon - H_T(u_j)). \tag{3.10}$$

Let us consider the self-interaction part for the beginning. This means that the shifting energy levels [8, 4] renormalisation of S_P and $\tilde{H}_T$ is not considered. In other words, we omit the action of operators $\prod \hat{j}(t_i)$ on $\exp\{iS_P(u_j) - 2i\tilde{H}_T(u_j;\tau)\}$:

$$\rho_c \sim \delta(E + \varepsilon - H_T(u_j)) e^{iS_{P(T)}(u_c)} e^{-2i\tilde{H}_T(u_c;\tau)} \sum_n \frac{(-21)^n}{n!} : \tilde{\mathbf{V}}_T^n(u_j, \hat{j}/2i) : .$$

Then the lowest order contribution is $\sim \tilde{\mathbf{V}}$. So, in the first order we find:

$$\sim \hat{j}^3 u_j \sim G_3 = O(g^2), \tag{3.11}$$

where the estimation (3.4) was used and the prescription that the auxiliary variable j should be taken equal to zero was taken into account. In the second order $\sim \tilde{\mathbf{V}}_T^2 = O(\hat{j}^6)$ contribution have following order over g:

$$\sim g^2 \hat{j}^6 u_j^2 = O(g^4), \tag{3.12}$$

and so on.

In result, one can find that the n-th order in expansion of $\exp\{-i\mathbf{K}\}$ gives $O(g^{2n})$ expansion if the self-interactions only are included in $\rho(E)$.

As follows from decomposition (3.3) and estimation (3.4) the action of operator $\hat{j}$ on u_j gives coefficient $\sim g^{1/2}$. The renormalisation of S_P and $\tilde{H}_T$ start from $\sim 1/g$ terms, but higher orders would contain the positive powers of g since they are produced by the actions of $\prod \hat{j}(t_i)$.

3.2 Connection with WKB expansion

One can show another argument that considered above perturbation theory is nothing new but is the ordinary expansion around u_c developed in early publications [1, 7]. Let us use for this purpose the substitution:

$$u(t) \to u_c(t) + u(t) \tag{3.13}$$

Then

$$\rho_c(E) = 2\pi \int_0^\infty dT\, e^{-i\mathbf{K}(\varepsilon\tau, \mathbf{jx})} \int DM(u_c, u) \delta(E + \varepsilon - H_T(u_c + u)) \times$$
$$\times e^{iS_{P(T)}(u_c+u)} e^{-2i\tilde{H}_T(u_c+u;\tau) - 2i\tilde{V}_T(u_c+u,x)}, \tag{3.14}$$

where

$$DM(u_c,u)=\prod_t du\delta\left(\frac{\delta S(u_c+u)}{\delta u}+j\right) \tag{3.15}$$

We should take into account that u_c depends on the integration constants ξ and η. Therefore, if (ξ,η) form the manifold W^c, as was mentioned in Sec.2.3, one should sum over all solutions $u_c\in W^c$, see Sec.3.3.

We want to show now that (3.14) may be reduced to the product of two path integrals. Indeed, using (2.26) and (2.16), (2.18) we find from (3.14) that

$$\rho_c(E)=2\int_0^\infty dT\int_{-\infty}^{+\infty}d\tau' e^{-i\mathbf{K}}\int DuDx' e^{2i(E+\varepsilon-H_T(u_c+u))\tau'}e^{2iH_T(u_c+u)\tau}\times$$
$$\times e^{S_{cl}(u_c+u\pm x';T\pm\tau')}e^{2i\mathrm{Re}\int dtx(\partial S(u_c+u)/\partial u)}e^{-2i\mathrm{Re}\int dtx\{(\partial S(u_c+u)/\partial u)+j\}}. \tag{3.16}$$

The action of operator $\exp\{-i\mathbf{K}\}$ leads to substitutions: $x\to x'$, $\tau\to\tau'$ and $\varepsilon\to 0$, $j\to 0$. Taking this into account we find:

$$\rho_c(E)=\left|\int_0^\infty dT e^{iET}\int Due^{iS_{C_+}(u_c+u)}\right|^2, \tag{3.17}$$

where the functional integral should be calculated perturbatively over u. Note, calculation of amplitudes is useful since eliminates the doubling of degrees of freedom.

3.3 Zero modes

The defined by eq.(3.17) $\rho_c(E)$ stay undefined till the procedure of summation over all $u_c\in W$ is not formulated. Following to the equality:

$$\sum_{\{u_c\}}=\int_W d\xi d\eta\sigma(u;\xi,\eta)$$

we should define the density $\sigma(u;\xi,\eta)$ of states in the domain $(\xi,\xi+d\xi;\eta,\eta+d\eta)$. The Faddeev-Popov *ansatz* is used for this purpose [7].

By definition, (ξ,η) are the constants of integration and they may be chosen arbitrarily. For example, wee may take (ξ,η) as the initial coordinate and momentum of particle on the trajectory u_c.

But the 'field-theoretical' definitions would be much more useful for us, see Sec.4. One may note that the dependence on (ξ,η) indicates the symmetry breaking. Then η may be taken as the generator J of broken symmetry and ξ as the canonically conjugate to it coordinate Θ. It will be important for us that (ξ,η) define the solution u_c unambiguously. In other words, we will use the ordinary mechanical statement [11] that (ξ,η) form a manifold W^c and u_c belongs to it *completely*. So, we would assume that the equations:

$$\xi=\Theta(u_c,\dot{u_c}),\ \eta=J(u_c,\dot{u_c}) \tag{3.18}$$

define the integration constants of u_c unambiguously.

Then, to define the density σ, we may insert into the initial representation (2.23) the unite (Faddeev-Popov *ansatz*):

$$1=\int_W\prod_t d\xi d\eta\delta(\xi-\Theta(u,\dot{u}))\delta(\eta-J(u,\dot{u}) \tag{3.19}$$

Note, by definition η should coincide with conserved generator. But nevertheless we consider $\eta = \eta(t)$ and the same for ξ. This assumption is necessary since the quantum case is considered.

We can change order of integration and integrate firstly over u using the δ-function of the measure DM. Lagrange equation (2.30) should be solved taking into account the constraints (3.18):

$$\rho_c(E) = 2\pi \int_W d\xi(0) \int_0^\infty dT e^{-i\mathbf{K}(\varepsilon\tau,\mathbf{jx})} \int DM_c(u)\delta(E+\varepsilon-H_T(u)) e^{iS_{P(T)}(u)} e^{-2i\tilde{H}_T(u;\tau)-2i\tilde{V}_T(u,x)}, \quad (3.20)$$

where $\xi(0)$ is the initial phase and the constraint measure

$$DM_c(u) = \prod_t du\delta\left(\frac{\delta S(u)}{\delta u} + j\right)\delta\left(\xi_0 - \Theta(u,\dot{u})\right) \quad (3.21)$$

was introduced. In our problem the value of J is restricted by $\delta(E+\varepsilon-H_T(u))$. It was used in (3.20) that DM_c is ξ independent since Lagrange equation is invariant against ξ variations and $j = j(t)$ is the auxiliary variable. This means that $\rho_c(E)$ defined in (3.20) is proportional to the volume

$$V_{tr} = \int d\xi(0)$$

of the time translation mode.

It is important here to trace on the following question. One can note that (3.20) gives $\rho_c \sim V_{tr}^1$. On other hand, as follows from (3.17), one may expect $\rho_c \sim V_{tr}^2$. It is evident that this discrepancy is the consequence of loaded into formalism condition of the orthogonality of Hilbert spaces, see Sec.2.3.

Remembering definition of ρ as the squire of amplitudes, we may insert the Faddeev-Popov's unite defined on the whole time contour $C = C_+ + C_-$, see (2.27), to take into account the input condition that the trajectories $u_+(t \in C_+)$ and $u_-(t \in C_-)$ are absolutely independent. This means that, generally speaking, the boundary conditions for this trajectories should not coincide and, therefore, if we introduce $\xi(t \in C_\pm)|_{t=0} \equiv \xi_\pm$, one should have in mind that, generally speaking, $\xi_+ \neq \xi_-$. Then integration over ξ_+ and ξ_- should be performed independently.

But we have considered the closed-path contributions, see (2.13). This gives restriction for the $u_\pm$ boundary properties. Taking into account (2.15), we can find, considering the periodic orbits, that

$$\xi_+ = \xi_- \pm kP_1(E),\ k = 0, 1, 2, ..., \quad (3.22)$$

where $P_1(E)$ is the elementary period. Just this solution leads to $S_P \neq 0$ and the necessary summation over k gives the energy levels quantisation condition (2.4), see also [8, 4].

4 Mapping on the cotangent manifold

The necessity to search a new form of the perturbation theory is caused by extremal complexity of the WKB perturbation theory described above.

The quantum nature of collective variables (ξ, η) was mentioned previously by many authors [1, 12]. We would like continue this idea considering them as a new quantum variables. For this purpose we will use the δ-like definition of measure, the definition of the interactions generating functional $\tilde{V}_T$ and the perturbations generating functional $\exp\{\mathbf{K}\}$, to count the possible excitations of the field $u_c \in W^c$, see Sec.4.1. In Sec.4.2 we will show the structure of new perturbation theory.

4.1 Procedure of mapping

Let us return to the Faddeev-Popov unite

$$1 = \int D\xi D\eta \prod_t \delta(\xi - \Phi(u,\dot{u}))\delta(\eta - J(u,\dot{u})) \tag{4.1}$$

It is assumed, as was offered in Sec.3.3,

$$\prod_t \equiv \prod_{t\in C=C_+ + C_-} .$$

The first order formalism will be useful for us . Corresponding measure

$$DM(u,p) = \prod_t dudp\delta\left(\dot{u} - \frac{\partial H_j(u,p)}{\partial p}\right)\delta\left(\dot{p} + \frac{\partial H_j(u,p)}{\partial u}\right), \tag{4.2}$$

where the total Hamiltonian

$$H_j(u,p) = \frac{1}{2}p^2 + v(u) - ju \tag{4.3}$$

includes the energy of quantum excitations ju. It is evident that the integration over p gives incident measure (2.25).

Inserting (4.1) into the functional integral with measure (4.2) we find that we have four equations for u and p:

$$\dot{u} = \frac{\partial H_j(u,p)}{\partial p},\ \dot{p} = -\frac{\partial H_j(u,p)}{\partial u} \tag{4.4}$$

and

$$\xi(t) = \Phi(u,\dot{u}),\ \eta(t) = J(u,\dot{u}). \tag{4.5}$$

In previous section the first pare of equations (4.4) was used to calculate the functional integral.

But now we would like use second one (4.5). It is possible iff u_c belongs to the space W^c completely and W^c is a manifold. This condition means that the eqs.(4.5) have unique solution (u_c, p_c) and this solution transform (4.4) into identity at least at $j = 0$. Let $u_c(\xi,\eta)$ and $p_c(\xi,\eta)$ are the solutions of (4.5). One can recognise in our description the ordinary canonical transformation (1.4), i.e. it defines the cotangent foliation $W^c = T^*\Omega$. But eqs.(4.4) and (4.5) should be solved simultaneously. So, inserting u_c, p_c into (4.4) we should use the 'excited' by j solutions $\xi_j(t)$ and $\eta_j(t)$.

So, we wish to adopt the statement that the random (Gaussian) walk, induced by the same operator $\exp\{-i\mathbf{K}\}$, covers both $W^c = (\xi,\eta)$ and $\Omega = (u,p)$ spaces densely. By this reason one may choose one of them arbitrarily.

The corresponding Jacobian of transformation Δ is δ-like:

$$\Delta = \prod_t \delta(\dot{u}_c - \frac{\partial H_j(u_c,p_c)}{\partial p_c})\delta(\dot{p}_c - \frac{\partial H_j(u_c,p_c)}{\partial u_c}), \tag{4.6}$$

and

$$det^{-1}(u_c,p_c) = \int \prod_t dudp\delta(\xi - \Phi(u,\dot{u}))\delta(\eta - J(u,\dot{u})) = 1 \tag{4.7}$$

since the transformation is canonical. This allows to diagonalise Δ and mapping into the W^c space leads to following path integral representation:

$$\rho_c(E) = 2\pi \int dT e^{-i\mathbf{K}} \int DM(\xi,\eta)\delta(E+\varepsilon - h(\xi,\eta;T)) e^{iS_{P(T)}(u_c)} e^{-2i\tilde{h}(u_c;\tau,T)-2i\tilde{V}_T(u_c,x)}, \tag{4.8}$$

where the measure

$$DM(\xi,\eta) = \prod_t d\xi d\eta \delta\left(\dot{\xi} - \frac{\partial h_j(\xi,\eta)}{\partial \eta}\right)\delta\left(\dot{\eta} + \frac{\partial h_j(\xi,\eta)}{\partial \xi}\right) \tag{4.9}$$

and h_j is the transformed Hamiltonian:

$$h_j(\xi,\eta) = h(\eta) - j u_c(\xi,\eta). \tag{4.10}$$

In result of mapping the problem of calculation of functional integral was reduced to solution of equations:

$$\dot{\xi} = \frac{\partial h_j(\xi,\eta)}{\partial \eta} = \omega(\eta) - j\frac{u_c(\xi,\eta)}{\partial \eta},\ \dot{\eta} = -\frac{\partial h_j(\xi,\eta)}{\partial \xi} = j\frac{u_c(\xi,\eta)}{\partial \xi}, \tag{4.11}$$

where one can choose, for example,

$$\omega(\eta) = \frac{\partial h(\eta)}{\partial \eta} = 1. \tag{4.12}$$

This means that in this case

$$\eta = H(u,p),\ \xi = \int^u \frac{dy}{\sqrt{2(H-v(y))}}. \tag{4.13}$$

It is evident that the solution of this equations gives $u_c(\xi,\eta)$ and $p_c(\xi,\eta)$ unambiguously.

4.2 Structure of transformed perturbation theory

We want to show now that $\rho(E)$, defined in (4.8), has the strong coupling expansion. Let us start for this purpose from the 'tree decomposition' of the equations (4.11):

$$\dot{\xi} = \frac{\partial h_j(\xi,\eta)}{\partial \eta} = 1 - j\frac{u_c(\xi,\eta)}{\partial \eta},\ \dot{\eta} = -\frac{\partial h_j(\xi,\eta)}{\partial \xi} = j\frac{u_c(\xi,\eta)}{\partial \xi}. \tag{4.14}$$

We will consider following decomposition of the solutions ξ_j and η_j:

$$\begin{aligned}\xi_j(t) &= \xi_0(t) + \sum_n \frac{1}{n!}\int \prod\{dt_i j(t_i)\}\xi_n(t;t_1,...,t_n),\\ \eta_j(t) &= \eta_0(t) + \sum_n \frac{1}{n!}\int \prod\{dt_i j(t_i)\}\eta_n(t;t_1,...,t_n).\end{aligned} \tag{4.15}$$

Inserting (4.15) into the (4.14) we find equation for the n-point Green functions $\xi_n(t;t_1,...,t_n)$ and $\eta_n(t;t_1,...,t_n)$. It can be shown:

$$\xi_n = O(g^{-n/2}),\ \eta_n = O(g^{-n/2}). \tag{4.16}$$

Indeed, in zero order over j we have:

$$\xi_0 = \xi(0) + t,\ \eta_0 = \eta(0) \tag{4.17}$$

since W^c is the homogeneous and isotropic manifold. Then, in the first order over j:

$$\dot{\xi}_1(t;t_1) = \delta(t-t_1)\frac{\partial u_c(\xi_0(t),\eta_0)}{\partial \eta_0} = O(g^{-1/2}),\ \dot{\eta}_1(t;t_1) = \delta(t-t_1)\frac{\partial u_c(\xi_0(t),\eta_0)}{\partial \xi_0} = O(g^{-1/2}) \tag{4.18}$$

since the derivatives of u_c are unable to change the g dependence. In second order we have the equations:

$$\dot{\xi}_2(t;t_1,t_2) = \delta(t-t_1)\left\{\xi_1(t;t_2)\frac{\partial u_c(\xi_0(t),\eta_0)}{\partial \eta_0 \partial \xi_0} + \eta_1(t;t_2)\frac{\partial u_c(\xi_0(t),\eta_0)}{\partial \eta_0 \partial \eta_0}\right\} = O(g^{-1}),$$
$$\dot{\xi}_2(t;t_1,t_2) = \delta(t-t_1)\left\{\xi_1(t;t_2)\frac{\partial u_c(\xi_0(t),\eta_0)}{\partial \xi_0 \partial \xi_0} + \eta_1(t;t_2)\frac{\partial u_c(\xi_0(t),\eta_0)}{\partial \xi_0 \partial \eta_0}\right\} = O(g^{-1}) \tag{4.19}$$

And so on. So, each power of $\hat{j}$ adds $g^{-1/2}$. This proves the estimation (4.16). It is important to note that eqs.(4.19) are trivially integrable. Therefore, we can calculate (ξ_n, η_n) for arbitrary n.

The operator $\mathbf{K}$ is linear over $\hat{x}$. So, the result of its action gives the normal ordered structure:

$$: e^{-2i\tilde{\mathbf{V}}_T(u_c,\hat{j}/2i)} : e^{iS_{P(T)}(u_c)} e^{-2i\tilde{H}_T(u_c;\tau)}\delta(E+\varepsilon - h(\xi_j,\eta_j;T)), \tag{4.20}$$

and at the very end of calculations one should take the auxiliary variables x equal to zero.

Let us consider once more the gu^4 theory. Then,

$$\tilde{V}_T(u_c,\hat{j}/2i) = O(\hat{j}^3),$$

Therefore, leaving the self-interaction parts only, in the lowest order we would have the contribution

$$\sim \tilde{V}_T(u_c,\hat{j}/2i) \sim g\hat{j}^3 u_c = O(1/g) \tag{4.21}$$

where (4.16) was used. So, the lowest order of new perturbation theory is $\sim 1/g$. In result, the n-th order is $\sim \tilde{V}_T(u_c,\hat{j}/2i)^n \sim 1/g^n$.

The action of $\tilde{V}_T^n(u_c,\hat{j}/2i)$ on $e^{iS_{P(T)}(u_c)}e^{-2i\tilde{H}_T(u_c;\tau)}\delta(E+\varepsilon-h(\xi_j,\eta_j;T))$ did not alter this conclusion since the derivative of u_c can not change the g dependence.

5 Conclusion

We conclude this paper by notation that it is impossible the transformed theory reduce to amplitude representation. Indeed, let us return to (4.8) and use the Fourier definition of the δ-functions:

$$\rho_c(E) = \int_0^\infty dT \int_{-\infty}^{+\infty} d\tau' e^{-i\mathbf{K}} \int D\xi D\eta Dx_\xi Dx_\eta e^{2i(E+\varepsilon-h(\eta;T))\tau'} \times$$
$$\times e^{-2i\mathrm{Re}\int_{C_+} dt x_\xi\{\dot{\xi}-\partial h_j(\xi,\eta)/\partial\eta\}} e^{-2i\mathrm{Re}\int_{C_+} dt x_\eta\{\dot{\eta}+\partial h_j(\xi,\eta)/\partial\xi\}} e^{iS_{P(T)}(u_c)} e^{-2i\tilde{h}(u_c;\tau,T)-2i\tilde{V}_T(u_c,x)}, \tag{5.1}$$

where, see (2.16),

$$-2\tilde{h}(u_c;\tau,T)=S_{cl}(u_c\pm x;T\pm\tau)-S_{cl}(u_c\pm x;T)+2\tau h(\eta). \tag{5.2}$$

Using this definition, and remembering that the action of operator $\exp\{-i\hat{e}\hat{\tau}/2\}$ gives $\tau=\tau'$ and $\varepsilon=0$, we find:

$$\rho_c(E)=\int_0^\infty dT\int_{-\infty}^{+\infty}d\tau e^{2iE\tau}e^{-i\mathrm{Re}\int_{C_+}dt\hat{j}\hat{x}/2}\int D\xi D\eta Dx_\xi Dx_\eta e^{iS_{cl}(u_c\pm x;T\pm\tau)-iS_{cl}(u_c\pm x;T)}\times$$
$$\times e^{-2i\mathrm{Re}\int_{C_+}dtx_\xi\delta S(u_c)/\delta\eta}e^{2i\mathrm{Re}\int_{C_+}dtx_\eta\delta S(u_c)/\delta\xi}e^{iS_{P(T)}(u_c)}e^{-2i\tilde{V}_T(u_c,x)}, \tag{5.3}$$

if the transformed action

$$S(u_c)=\int dt\{\eta\dot{\xi}-h(\eta)\}.$$

Action of the perturbation generating operator gives:

$$\rho_c(E)=\int_0^\infty dT\int_{-\infty}^{+\infty}d\tau e^{2iE\tau}\int D\xi D\eta Dx_\xi Dx_\eta e^{iS_{cl}(u_c\pm x_c;T\pm\tau)}\times$$
$$\times e^{-2i\mathrm{Re}\int_{C_+}dt\{x_\xi(\delta S(u_c)/\delta\eta)-x_\eta(\delta S(u_c)/\delta\xi)\}}e^{-2i\mathrm{Re}\int_{C_+}dtx_c(\delta S(u_c)/\delta u_c)}, \tag{5.4}$$

if (2.18) is used and, using the local coordinates of the W space,

$$x_c=x_\xi\frac{\partial u_c}{\partial\eta}-x_\eta\frac{\partial u_c}{\partial\xi}=\delta u_c\wedge\delta p_c \tag{5.5}$$

Now, if

$$\frac{\delta S(u_c)}{\delta\xi}=\frac{\partial u_c}{\partial\xi}\frac{\delta S(u_c)}{\delta u_c},\ \frac{\delta S(u_c)}{\delta\eta}=\frac{\partial u_c}{\partial\eta}\frac{\delta S(u_c)}{\delta u_c}, \tag{5.6}$$

then we can write:

$$\rho_c(E)=\int_0^\infty dT\int_{-\infty}^{+\infty}d\tau e^{2iE\tau}\int D\xi D\eta Dx_\xi Dx_\eta e^{iS_{cl}(u_c\pm x_c;T\pm\tau)}. \tag{5.7}$$

The quantities (x_ξ,x_η) and (ξ,η) have different meaning. First ones are the virtual variation of the 'field' u along the corresponding axis of W^c space, and the integrals over them should be calculated perturbatively, but last ones are the axis of the $W^c=T^*\Omega$ phase space. The closed path action

$$S_{cl}(u_c\pm x_c;T\pm\tau)=S_{C_+(T+\tau)}(u_c(\xi,\eta)+x_c(\xi,\eta;x_\xi,x_\eta))-S_{C_-(T-\tau)}(u_c(\xi,\eta)-x_c(\xi,\eta;x_\xi,x_\eta)). \tag{5.8}$$

It is evident from (5.7) the transformed representation can not be written in the factorized form of product of two amplitudes.

We interpret this conclusion as impossibility of the canonical transformations in the path integrals (2.8) since on the cotangent manifolds the quantum excitations induce the phase space flows in which all degrees of freedom are mixed.

In traditional terms this means the problem of time ordering of nonlinear operators. Our success is based on the observation that the unitarity condition unambiguously defines the perturbation theory in the ('linear') representation, where we may disentangle all time orderings. Fixing this procedure

in the structure of DM, βK and $\tilde{V}_T$ one can do arbitrary transformations. But, the payment for this success is necessity to work in terms of less habitual absorption part of amplitude and, by this reason, one should be careful interpreting our perturbation theory as a general, see [3].

But, in conclusion, quantising the nonlinear waves our strong coupling perturbation theory seems much more attractive since we can perform the calculation in this theory up to the end, choosing W^c as the homogeneous and isotropic space.

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