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NONPERTURBATIVE EXPANSION METHOD IN QCD AND ITS APPLICATIONS

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1. An approximation of a quantity under consideration by a finite number of terms of a certain series is often used in analysis of many physical problems. In quantum field theory this is conventionally an expansion into a perturbative series. This approach combined with the renormalization procedure is now a basic method for computations. As is well known, perturbative series for many interesting models including realistic models are not convergent. Nevertheless, at small values of the coupling constant these series may be considered as asymptotic series and could provide a useful information. However, there exist problems which cannot be resolved by perturbative methods. In quantum chromodynamics there are many problems whose solution requires nonperturbative approaches.

There are methods that combine an expansion of a given quantity in a series that defines the algorithm of calculating the correction with an optimizing procedure (see [1, 2] and references therein). Different ways of constructing the variational procedures for scalar models of quantum field theories are discussed in [3–5]. In this paper we apply the idea of variational perturbation theory (VPT) to QCD. In spite of the term «perturbation» the VPT approach does not use the coupling constant as a small expansion parameter and can be used to go beyond the weak-coupling regime. This method allows one to systematically determine the low energy structure in quantum chromodynamics. In this case, we shall construct the expansion which is based on a new small parameter. The results concerning the method of variational perturbation theory and some its applications can be found in the papers [6–9] and [10–12].

2. In the case of QCD we will apply the harmonic variational procedure which leads to a new small expansion parameter. To explain the basic idea of the method we consider the pure Yang–Mills theory. The Lagrangian density has the form

$$\begin{aligned} L_{YM} &= -\frac{1}{4}\,(F_{\mu\nu})^2 - \frac{1}{2}\,g\,F_{\mu\nu}\,[A_\mu{\times}A_\nu] + \frac{1}{4}\,g^2\,[A_\mu{\times}A_\nu]^2 + L_{g.f.} + L_{F.P.} = \\ &= L_0(A) + g\,L_3(A) + g^2\,L_4(A)\,, \end{aligned} \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $L_{g.f.}$ and $L_{F.P.}$ are gauge fixing and Faddeev–Popov terms.

The term $L_3(A)$ generates the three-gluon and ghost-gluon-ghost vertices. This interaction is the Yukawa type interaction. The term $L_4(A)$ generates the four-gluon vertices. Let us introduce the $\chi_{\mu\nu}$ field and transform the term $L_4(A)$ to the Yukawa type interaction

$$\exp\left\{ i\frac{g^2}{4}\int dx\ [A_\mu \times A_\nu]^2 \right\} =$$

$$= \int D\chi \exp\left\{ -\frac{i}{2}\int dx \chi^2_{\mu\nu} + i\frac{g}{\sqrt{2}}\int dx \chi_{\mu\nu}[A_\mu \times A_\nu] \right\} . \qquad (2)$$

The action functional can be written in the form

$$S = S_0(\chi) + S(A,\chi) + S^{Yuk}_{YM}(A)\,, \qquad (3)$$

where

$$S(A,\chi) = \frac{1}{2}\int dx\, dy\ A^a_\mu(x)\ \left[D^{-1}(x,y|\chi)\right]^{ab}_{\mu\nu}\ A^b_\nu(y) \qquad (4)$$

and $D(x,y|\chi)$ is the gluon propagator in the χ-field.

The Green functions can be represented as $G(\cdots) = \langle G_{Yuk}(\cdots|\chi)\rangle$, where

$$G_{Yuk}(\cdots|\chi) = \int DA\,[\cdots]\,\exp\left\{\, i\ \left[\, S(A,\chi) + S^{Yuk}_{YM}(A)\right]\,\right\} \qquad (5)$$

and

$$\langle\cdots\rangle = \int D\chi\,[\cdots]\,\exp\left[\, i\, S_0(\chi)\,\right] . \qquad (6)$$

The Green functions $G_{Yuk}(\cdots|\chi)$ contain only the Yukawa type diagrams appearing inside the brackets $\langle\cdots\rangle$ with the gluon propagator $D(x,y|\chi)$.

Rewrite the Lagrangian in the form

$$\begin{aligned} L(A,\chi) &= L_0(A,\chi) + L_I(A,\chi)\,, \\ L_0(A,\chi) &= \zeta^{-1}\, L(A,\chi) + \xi^{-1}\, L(\chi)\,, \\ L_I(A,\chi) &= \eta\left[\, g\, L^{Yuk}_{YM}(A) - (\zeta^{-1} - 1)\, L(A,\chi) - (\xi^{-1} - 1)\, L(\chi)\right] , \end{aligned} \qquad (7)$$

where ζ and ξ are the parameters of variational type. The original quantity $L(A,\chi)$ does not depend on ζ and ξ. The freedom in choosing ζ and ξ can be

used to improve the series properties. The VPT series for the Green function is given by

$$\begin{aligned} G(\cdots) &= \sum_n G_n(\cdots)\,, \\ G_n(\cdots) &= \frac{1}{n\,!}\,\eta^n \int D\chi DA\,[\cdots]\,[\,i\,S_I(A,\chi)]^n \exp\,[\,i\,S_0(A,\chi)] = \\ &= (i\,\eta)^n \sum_{k=0}^{n} \frac{1}{(n\,-\,k)\,!\,k\,!} \int D\chi DA\,[\cdots]\,\big[\,g\,S_{YM}^{Yuk}(A)\,\big]^k \times \\ &\times \quad \big[\,(\zeta^{-1}\,-\,1)\,S(A,\chi)\,+\,(\xi^{-1}\,-\,1)\,S(\chi)\,\big]^{n-k} \exp\,[\,i\,S_0(A,\chi)]\,. \end{aligned} \tag{8}$$

Redefine the $L_0(A,\chi)$ for convenience of calculations as follows

$$L_0(A,\chi) \Rightarrow L_0'(A,\chi) = \big[1+\kappa(\zeta^{-1}-1)\big]\,L(A,\chi) + \big[1+\kappa(\xi^{-1}-1)\big]\,L(\chi)\,. \tag{9}$$

Any power of $[(\zeta^{-1}-1)S(A,\chi)+(\xi^{-1}-1)S(\chi)]$ in (8) can be obtained by the corresponding number of differentiation with respect to κ (after all calculations we set $\kappa=1$).

From Eqs. (8) and (9) we have

$$G_n \,=\, \eta^n \sum_{k=0}^{n} \frac{1}{(n\,-\,k)\,!} \Big(-\frac{\partial}{\partial\,\kappa}\Big)^{n\,-\,k} \,\langle\, g_k(\kappa)\,\rangle\,, \tag{10}$$

where the functions

$$g_k(\kappa) = \frac{i^k}{k!} \int DA\,[\cdots]\,\big[\,g\;S_{YM}^{Yuk}(A)\,\big]^k \exp\Big\{i\,[1+\kappa(\zeta^{-1}-1)] \int dx\,L_0(A,\chi)\Big\} \tag{11}$$

correspond to the Yukawa diagrams of the Yang–Mills theory with gluon propagator

$$\frac{1}{1+\kappa(\zeta^{-1}-1)}\,D(x,y|\chi) \to \zeta\,D(x,y|\chi)$$

for $\kappa\,=\,1$. The propagator of χ-field includes the factor $[1+\kappa(\xi^{-1}-1)]^{-1}$ which is transformed into ξ for $\kappa=1$.

The operator of differentiation $(-\partial/\partial\kappa)^l/l!$ leads to the factor $(1-\zeta)^l$ for the gluon propagator and $(1-\xi)^l$ for the propagator of the χ-field. Thus, the outline of the VPT expansion structure can be written as

$$\begin{aligned} 1 \quad &+\eta\,(1\,-\,\zeta)\,+\,\eta^2\,\Big[\,(1\,-\,\zeta)^2\,+\,g^2\,\zeta^3\,+\,g^2\,\xi\,\Big]\,+ \\ +\quad &\eta^3\,\Big[\,(1\,-\,\zeta)^3\,+\,g^2\,\zeta^3\,(1\,-\,\zeta)\,+\,g^2\,\xi\,(1\,-\,\zeta)\,+\,g^2\,\xi\,(1\,-\,\xi)\,\Big]\,+\,\cdots \end{aligned} \tag{12}$$

If we choose $\xi = \zeta^3$ and $(1-\zeta)^2 = C\lambda\zeta^3$, where C is a positive constant, we obtain that the nth order term of our series contains the factor $(1-\zeta)^n$ and the expansion parameter $a = (1-\zeta) < 1$ for all values of the initial coupling constant. To fix the parameter C, it is possible to use a nonperturbative information coming from the meson spectroscopy (see details in [7, 8]).

The renormalization group method gives an equation for the Q^2-evolution of the expansion parameter $a = a(Q^2)$. By solving the renormalization group equation one finds the momentum dependence of the running expansion parameter as a solution of the following transcendental equation

$$\ln \frac{Q^2}{Q_0^2} = \frac{C}{2\,\beta_0}\,[\,f(a) - f(a_0)\,]\,. \tag{13}$$

For the function $f(a)$, by finding the renormalization constants in the massless renormalization scheme with an accuracy $O(a^3)$, we get

$$f(a) = \frac{2}{a^2} - \frac{6}{a} - 48\,\ln a - \frac{18}{11}\,\frac{1}{1-a} +$$

$$+\frac{624}{121}\,\ln(1-a) + \frac{5184}{121}\,\ln\left(1+\frac{9}{2}\,a\right)\,. \tag{14}$$

For any values of Q^2, this equation has a unique solution $a(Q^2)$ in the interval between 0 and 1.

It is easy to verify that the Nth order of the VPT series contains the Nth order of a perturbation series with the correction $O(\,g^{N+1}\,)$, therefore, the VPT expansion does not contradict the perturbative results obtained for the small coupling constant.

3. As an example of applications of the a-expansion approach consider a description of the inclusive decay of the τ lepton taking into account renormalon contributions [13, 14]. The R_τ ratio can be parameterized by the Adler D-function $D(Q^2) = -Q^2 d\Pi/dQ^2$. The two-loop perturbative approximation is given by $D(t,\lambda) = 1 + 4\lambda(\mu^2)$, where $t = Q^2/\mu^2$. Standard renormalization group improvement leads to the substitution $\lambda(\mu^2) \to \bar{\lambda}(t,\lambda)$, which implies a summation of the leading logarithmic contributions. However, due to the ghost pole of the running coupling at $Q^2 = \Lambda^2$ this substitution breaks the analytic properties of the D-function in the complex $q^2 = -Q^2$ plane, namely that the D-function should only have a cut on the positive real q^2 axis. We may correct this feature by noting that the above solution of the renormalization group equation is not unique. The general solution is a function of the running coupling with the asymptotic behaviour $1 + 4\lambda$, for small λ. To maintain the ana-

lytic properties* of the D-function we can write it as the dispersion integral of $R(s) = (1/\pi)\mathrm{Im}\Pi(s+i\epsilon)$, and use RG improvement on the integrand rather than D itself. This method leads to $D(t,\lambda) = 1+4\lambda_{\rm eff}(t,\lambda)$. The Borel representation of $\lambda_{\rm eff}(t,\lambda)$ has the form

$$\lambda_{\rm eff}(t,\lambda) \,=\, \int_0^\infty db\, \mathrm{e}^{-b/\bar\lambda(t,\lambda)}\, B(b)\,, \tag{15}$$

with $B(b) = \Gamma(1+b\beta_0)\Gamma(1-b\beta_0)$. Here $\beta_0 = 11-2/3N_f$ is the first coefficient of the β-function, and N_f is the number of active flavors. Thus, in the Borel plane there are singularities at $b\beta_0 = -1, -2, ...$ and $b\beta_0 = 1, 2, ...$ corresponding to ultraviolet and infrared (IR) renormalons, respectively.

The first IR singularity at $b\beta_0 = 1$ is probably absent since there is no corresponding operator in the operator product expansion. Although this issue is not currently settled, it seems reasonable to assume that the first IR renormalon occurs at $b = 2/\beta_0$, and we would like to use this property of the operator product expansion as an additional constraint on the choice of solution to the renormalization group equation. This can be simply achieved, and as a result we obtain the following expression for $\lambda_{\rm eff}$:

$$\lambda_{\rm eff}(t,\lambda) = \int_0^\infty d\,\tau\,\omega(\tau)\frac{\bar\lambda(kt,\lambda)}{1+\bar\lambda(kt,\lambda)\beta_0\,\ln\tau}\,, \tag{16}$$

in which the factor k reflects the renormalization scheme ambiguity and the function $\omega(\tau) = 2\tau/(1+\tau)^3$ describes the distribution of virtuality usually associated with renormalon chains. The function $B(b)$ in the Borel transform of (16) has the form

$$B(b) \,=\, \Gamma(1\,+\,b\beta_0)\,\Gamma(2\,-\,b\beta_0)\,. \tag{17}$$

Thus in this representation for $\lambda_{\rm eff}$ the positions of all ultraviolet singularities remain unchanged, but the first IR renormalon singularity at $b = 1/\beta_0$ is absent.

In order to render Eq. (16) integrable we must combine this method with the nonperturbative a-expansion in which from the beginning the running coupling has no ghost pole. Separating the QCD contribution to R_τ-ratio as Δ_τ and writing $R_\tau = R_\tau^0\,(1+\Delta_\tau)$, where R_τ^0 is the well-known electroweak factor, we obtain the expression

$$\Delta_\tau = 48\int_0^{M_\tau^2}\frac{ds}{M_\tau^2}\left(\frac{s}{M_\tau^2}\right)^2\left(1-\frac{s}{M_\tau^2}\right)\tilde\lambda(ks)\,, \tag{18}$$

*It has been shown [15] (see also [16]) that requiring the correct analytic properties for the running coupling is associated with the inclusion of nonperturbative power corrections of the form $\exp\{-1/[\bar\lambda(Q^2)\beta_0]\}$. The role of these contributions in the context of the τ-decay has been analyzed in [17, 18].

in which the factor k again parametrizes the renormalization scheme and $\tilde{\lambda} = a^2(1+3a)/C$. In what follows we shall use the $\overline{MS}$ scheme, in which $k = \exp(-5/3)$. Note that the renormalon representation obtained for the coupling modifies the polynomial in the integral so that the maximum now occurs near $s = (2/3)M_\tau^2$.

Taking as input the experimental value of $R_\tau^{\exp} = 3.56 \pm 0.03$, three active quark flavors and the variational parameter $C = 4.1$ as in [8], we find $\alpha_s(M_\tau^2) = 0.339 \pm 0.015$ which differs significantly from that obtained ($\alpha_s(M_\tau^2) = 0.40$ in leading order [11]) without the renormalon-inspired representation* for the coupling. The method, applying the matching procedure in the physical s-channel [10] and using standard heavy quark masses, leads to $R_Z = 20.90 \pm 0.03$, which agrees well with experimental data.

4. We have considered an approach to quantum chromodynamics – the method of variational perturbation theory. The original action functional is rewritten using some variational addition and an expansion in the effective interaction is made. Therefore, in contrast to many nonperturbative approaches, in the VPT the quantity under consideration from the very beginning is written in the form of a series which makes it possible to calculate the needed corrections. The VPT method thereby allows for the possibility of determining the degree to which the principal contribution found variationally using some variational principle adequately reflects the problem in question and determining the region of applicability of the results obtained. The variational parameters arising in the VPT method allow the convergence properties of the VPT series to be controlled. For the harmonic variational procedure used here there are indications that VPT series can be converged on the sense of the so-called induced convergence, by fine-tuning the variational parameters from order to order.

Here, we have considered the application of the method to quantum chromodynamics, where the VPT idea leads to an expansion with a new small expansion parameter. This parameter obeys an equation whose solution is always smaller than unity for any value of the coupling constant. Therefore, while remaining within the limits of applicability of this expansion it is possible to deal with considerably lower energies than in the case of perturbation theory. An important feature of this approach is the fact that for sufficiently small value of the running coupling constant $\bar{\alpha}_s$ it reproduces perturbative predictions. Therefore, all the high-energy physics is preserved in the VPT method. In going to lower energies, where standard perturbation theory ceases to be valid, $\bar{\alpha}_s \simeq 1$, the VPT running expansion parameter $\bar{a}$ remains small and we do not find ourselves outside the region of applicability of the method.

*The analysis of the heavy quark bound state spectrum using the a-expansion approach and the renormalon representation has been performed in [19].

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