NONPERTURBATIVE EXPANSIONS IN QUANTUM FIELD THEORY

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A brief review of variational perturbation theory and some its applications is presented.

1 Introduction

In theoretical consideration of many physical problems one often uses an approximation of a quantity under consideration by a finite number of terms of a certain series. In quantum field theory this is conventionally an expansion into a perturbative series. This approach combined with the renormalization procedure is now a basic method for computations. As is well-known, perturbative series for many interesting models including realistic models are not convergent. Nevertheless, at small values of the coupling constant these series may be considered as asymptotic series and could provide a useful information. However, even in the theories with a small coupling constant, for instance, in quantum electrodynamics there exist problems which cannot be resolved by perturbative methods. In quantum chromodynamics there are many problems whose solution requires nonperturbative approaches.

Many approaches have been devoted to the development of nonperturbative methods. Among them is the summation of a perturbative series (see reviews [1] and monograph [2]). The difficulty is that the procedure of summation of asymptotic series is not unique as it contains a functional arbitrariness. A correct formulation of the problem of summation is ensured by further information on the sum of a series [3]. At present information of that kind is known only for the simplest field-theoretical models [4]. Moreover, in many cases of physical interest, the series of perturbative theory is not Borel summable.

There have been approaches that are not directly based on the perturbative series or used some other expansions to get an approximation of a quantity under consideration [5, 6, 7, 8]. Many of nonperturbative approaches make use of a variational procedure for finding the leading contribution. However, in this case there is no always an algorithm of calculating corrections to the value found by a variational procedure, and this makes difficult to answer the question how adequate is the so-called main contribution to the object under investigation and what is the range of applicability of the obtained estimations.

To the study of the nonperturbative structure of quantum field theory there are methods that combine an expansion of a given quantity in a series that defines the algorithm of calculating the correction with an optimizing procedure. The nonperturbative Gaussian effective potential for a quantum system has been constructed by an approach of that sort in [9, 10, 11, 12]. There exist the various optimizing procedures. In [13, 14], for example, the principle of minimal sensitivity has been applied to the third-order calculation of the e^+e^- annihilation ratio. Different ways of constructing the variational procedures for scalar models of quantum field theories are discussed in [15, 16, 17]. However, even if the algorithm of calculating corrections, i.e. terms of a certain approximating series, exists,

it is not still sufficient. Here of fundamental importance are the properties of convergence of a series. Indeed, unlike the case when even a divergent perturbative series in the weak coupling constant approximates a given object as an asymptotic series, the approximating series in the absence of a small parameter should obey more strict requirements.

In this paper we consider the method of variational perturbation theory (VPT). In spite of the term "perturbation" the VPT approach does not use the coupling constant as a small expansion parameter and can be used to go beyond the weak-coupling regime. This method allows one to systematically determine the low energy structure in quantum chromodynamics. In this case, we shall construct the expansion which is based on a new small parameter and apply this method to the nonperturbative renormalization group analysis in quantum chromodynamics. Applications to the definition of the QCD running coupling in the timelike domain and to the semileptonic decay of the τ lepton will be considered. The main results concerning the method of variational theory and some its applications can be found in the papers [9, 14, 15, 16, 17] and [18, 19, 20, 21, 22, 23, 24] (see also references therein). We shall consider the method of the VPT series construction by using variational procedures of an harmonic and anharmonic types. In the case of the harmonic variational probe, it has been observed empirically [25] that the results seem to converge if the variational parameter is chosen, in each order, according to the principle of minimal sensitivity. This induced-convergence phenomenon is discussed in detail in [26]. In [27, 28] the proof of convergence of an optimized δ–expansion is given in the cases of zero and one dimensions. The proof of convergence of variational series in the case of anharmonic procedure is given in [16].

2 Anharmonic oscillator

Let us consider a quantum-mechanical anharmonic oscillator (AO) as an example of exploiting the VPT method. The AO from a point of view of the continual integral formalism is a one-dimensional φ^4-model. The Euclidean action looks as follows

$$S[\varphi] = S_0[\varphi] + \frac{m^2}{2} S_2[\varphi] + g S_4[\varphi], \tag{1}$$

where

$$S_0[\varphi] = \frac{1}{2}\int dx (\partial\varphi)^2, \quad S_2[\varphi] = \int dx\, \varphi^2, \quad S_4[\varphi] = \int dx\, \varphi^4. \tag{2}$$

Green functions. The Green functions are expressed in terms of the functional integral

$$G_{2\nu} = \int D\varphi \{\varphi^{2\nu}\} \exp\bigl(-S[\varphi]\bigr), \tag{3}$$

where $\{\varphi^{2\nu}\}$ denotes the product of fields $\varphi(x_1)\cdots\varphi(x_n)$.

It is convenient to pass to dimensionless variables: $\varphi \longrightarrow g^{-1/6}\varphi, x \longrightarrow g^{-1/3}x$, then the functional of action reads

$$S[\varphi] = S_0[\varphi] + \frac{\omega^2}{2} S_2[\varphi] + S_4[\varphi], \tag{4}$$

where $\omega^2 = m^2 g^{-2/3}$. The dimensionless Green functions $G_{2\nu}$ will be represented through (3) with the action (4).

Like the ordinary perturbation theory, the VPT method uses only Gaussian quadratures:

$$\int D\varphi \exp\Big\{-\Big[\frac{1}{2} < \varphi K \varphi > + < \varphi J >\Big]\Big\} =$$
$$= \Big(\det \frac{K}{-\partial^2 + m^2}\Big)^{-1/2} \exp\Big(\frac{1}{2} < J K^{-1} J >\Big). \tag{5}$$

Possible polynomial in the integrand of (5) can be obtained as usually by the corresponding number of differentiations of the exponential with respect to the source $J(x)$.

The variational probe of the system can be constructed by using the functional

$$A[\varphi] = \theta S_0[\varphi] + \frac{\chi}{2} S_2[\varphi], \tag{6}$$

where θ and χ are variational-type parameters.

We shall first consider the harmonic variational procedure. In this case the action functional splits as follows

$$S[\varphi] = S_0^h[\varphi] + S_{\rm int}^h[\varphi], \tag{7}$$

where

$$S_0^h[\varphi] = S_0[\varphi] + \frac{\omega^2}{2} S_2[\varphi] + A[\varphi], \quad S_{\rm int}^h[\varphi] = S_{\rm int}[\varphi] - A[\varphi]. \tag{8}$$

The expansion in the VPT series reads

$$G_{2\nu} = \sum G_{2\nu,n}(\theta, \chi), \tag{9}$$

$$G_{2\nu,n}[\theta, \chi] = \frac{(-1)^n}{n!} \int D\varphi \{\varphi^{2\nu}\} \Big(S_{\rm int}^h[\varphi]\Big)^n \exp\Big(-S_0^h[\varphi]\Big). \tag{10}$$

The functional integral in (10) is Gaussian. It is convenient to use the ordinary coefficients of a perturbative series when it is calculated. For this aim, let us rewrite (10) as

$$G_{2\nu,n} = \sum_{k=0}^{n} \frac{1}{k!(n-k)!} \Big(-\frac{\partial}{\partial \alpha}\Big)^{n-k} \int D\varphi \{\varphi^{2\nu}\} (-S_4)^k \exp\Big[-(S_0 + \frac{\omega^2}{2} S_2 + \alpha A)\Big], \tag{11}$$

where the parameter α is to be set to 1 after differentiation. Keeping in mind the intermediate dimensional regularization and making the change $\varphi \longrightarrow \varphi/\sqrt{1+\alpha\theta}$ we obtain

$$G_{2\nu,n} = \sum_{k=0}^{n} \frac{1}{(n-k)!} \Big(-\frac{\partial}{\partial \alpha}\Big)^{n-k} \frac{g_{2\nu,k}(z^2)}{(1+\alpha\theta)^{\nu+2k}}, \tag{12}$$

where

$$g_{2\nu,k}(z^2) = \frac{1}{k!} \int D\varphi \{\varphi^{2\nu}\} (-S_4)^k \exp\Big[-\Big(S_0 + \frac{z^2}{2} S_2\Big)\Big] \tag{13}$$

are the ordinary perturbative expansion coefficients for the Green functions (3). To calculate them, the standard Feynman diagrams, for example, can be used. The quantity z^2 in (13) looks as follows

$$z^2 = \frac{\omega^2 + \alpha\chi}{1 + \alpha\theta}. \tag{14}$$

Properties of the series (9) are determined by the asymptotics of the functional integral

$$\frac{(-1)^n}{n!}\int D\varphi\Big(S_4[\varphi]-A[\varphi]\Big)^n\exp\Big[-\Big(S_0[\varphi]+\frac{\omega^2}{2}S_2[\varphi]+A[\varphi]\Big)\Big] \tag{15}$$

at large n.

The investigation of the asymptotic behaviour of expression (15) in the leading order in n is, in fact, equivalent to finding the ordinary perturbative series coefficients. The series (9) turns out to be asymptotic like the ordinary one. Actually, its behaviour may be influenced by the θ and χ parameters, to attain the greater stability of results as compared with standard perturbation theory. However, one compelled to remain in the region of the weak coupling constant, mainly, as it turns out to be impossible for arbitrary values of the dimensionless coupling constant g/m^3 to gain, within the harmonic variational procedure, the stable results with respect to corrections. The latter is explained by the fact that at large n a sensible contribution to (15) comes from such field configurations at which the quantity $|\varphi(x)|$ is large. In this case the compensation by the harmonic addition $A[\varphi]$ of large $S_{\rm int}[\varphi]$ containing the fourth power of the field proves to be not sufficient.

Under the anharmonic variational procedure the action functional is represented as follows

$$S[\varphi]=S_0^a[\varphi]+S_{\rm int}^a[\varphi]\,, \tag{16}$$

$$S_0^a[\varphi]=S_0[\varphi]+\frac{\omega^2}{2}S_2[\varphi]+A^2[\varphi]\,,\quad S_{\rm int}^a[\varphi]=S_{int}[\varphi]-A^2[\varphi]\,. \tag{17}$$

Now, the field power in the variational probe is the same as in the interaction action $S_{\rm int}[\varphi]$. Keeping in mind that we have also the variational parameters at our disposal, we may anticipate that the convergence of the VPT series will be improved. As a concrete example, we shall consider the ground state energy for the anharmonic oscillator which is connected with the four-point Green function $G_4(0,0,0,0)$.

Ground state energy. We will proceed from the partition function represented by the path integral

$$\exp(-TE)=\int D\varphi\,\exp(-S[\varphi])\,, \tag{18}$$

where the integration in (18) runs over $\varphi(t)$ with condition: $\varphi(-T/2)=\varphi(T/2)$, the functional of action is given by (1) and integration runs over t from $-T/2$ to $T/2$.

The ground state energy E_0 follows from (18) in the limit $T\to\infty$. It is convenient to pass from the functional integrals typical of the statistical mechanics to the functional integrals of the Euclidean field theory. To this end consider the quantity dE_0/dg that is exprassed in terms of the four-point Euclidean Green function. So, passing to the dimensionless variables from (18) we obtain

$$\frac{dE_0}{dg}=g^{-2/3}G_4(0)\,, \tag{19}$$

where

$$G_4(0) = N^{-1}\int D\varphi\,\varphi^4(0)\,\exp\Big[-(S_0+\frac{\omega^2}{2}\,\tilde S_2+S_4)\Big]\,, \tag{20}$$

$$N = \int D\varphi\,\exp[-(S_0+\omega^2\,\tilde S+S_I)]\,. \tag{21}$$

The VPT expansion for the Green function (20) has the following form

$$G_4(0) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(1+m)A_{1+m}}{(n-m)!}\Big[\Gamma(1+m/2)\,\Gamma(1+3m/2)\Big]^{-1}\left(\frac{d}{d\alpha}\right)^{n-m} F_m(\theta,\chi,\alpha)\,, \quad (22)$$

where

$$F_m(\theta,\chi,\alpha) = \int_0^{\infty} dx\, x^{m/2}\exp(-x)\int_0^{\infty} dy\, y^{3m/2}\exp\Big[-\omega^2 y - (1-\alpha)\cdot(x\theta + y\chi)\Big]\,. \quad (23)$$

In the strong coupling limit we can set $\omega^2 = 0$ in (23). However, it is to be noticed that expanding $\exp(-\omega^2 y)$ in powers of ω^2 we can determine corrections to the main contribution.

From (22) and (23) we obtain for the ground state energy in the Nth VPT order in the strong coupling limit

$$E_0^N = 3g^{1/3}\sum_{n=0}^{N}\sum_{m=0}^{n} \frac{(1+m)A_{1+m}}{(n-m)!}\left(\frac{16}{9}\theta\right)^{1/3+m/2}[\Gamma(1+m/2)\Gamma(1+3m/2)]^{-1}\,R_{n,m}(\theta)\,, \quad (24)$$

where

$$R_{n,m}(\theta) = \int_0^{\infty} dx\, x^{m/2}\exp(-x)\int_0^{\infty} dy\, y^{3m/2}(\theta x + y)^{2(n-m)}\exp[-(\theta x + y)^2]\,. \quad (25)$$

Optimal value of the parameter θ in different versions of the optimization procedure turns out to be small, therefore, in the first VPT order we get from (24) and (25)

$$E_0^{(1)} = g^{1/3}(\varepsilon_0 + \varepsilon_1)\,, \quad (26)$$

where

$$\varepsilon_0 = \frac{3}{2}A_1\sqrt{\pi}\,x^2\,, \quad \varepsilon_1 = \frac{3}{4}A_1\sqrt{\pi}\,x^2 + \frac{4\,\Gamma(5/4)}{\sqrt{\pi}}A_2\,x^5\,, \quad x = \left(\frac{16}{9}\theta\right)^{1/6}\,. \quad (27)$$

Finding the variational parameter x by using the principle of minimal sensitivity (PMS) we get $E_0^{(1)}(x_2) = 0.660g^{1/3}$. The corresponding numerical result is $E_0^{\text{exact}} = 0.668g^{1/3}$ [29]. The stability of the VPT series with respect of the order N for values of θ close to the optimal one is indicated in Table 1.

To find corrections to the strong coupling calculations one has to perform the ω^2-expansion. As a result we get (below we do not use the condition $\theta_{\text{opt}} \ll 1$)

$$E_0^{(1)} = g^{1/3}(\,0.676 + 0.1407\,\omega^2 - 0.0085\,\omega^4 + \cdots)\,. \quad (28)$$

The corresponding numerical result is [29]

$$E_0^{\text{exact}} = g^{1/3}(\,0.668 + 0.1437\,\omega^2 - 0.0088\,\omega^4 + \cdots)\,. \quad (29)$$

Propagator. Let us calculate the parameter μ^2 which is connected with the two-point Green function, $\mu^{-2} = G_2(p=0)$, where

$$G_2(p=0) = \int dt \int D\varphi\,\varphi(t/2)\varphi(-t/2)\,\exp(-S[\varphi])\,. \quad (30)$$

Table 1: *The behaviour* $E_0^{(N)}/E_0^{\text{exact}}$ *vs.* N

$E_0^{(N)}/E_0^{\text{exact}}$				
N	$\theta = 0.020$	$\theta = 0.028$	$\theta = 0.032$	$\theta = 0.036$
0	0.956	1.063	1.107	1.144
1	0.981	1.006	1.011	1.013
2	0.995	1.005	1.006	1.006
3	1.000	1.004	1.004	1.004
4	1.001	1.003	1.002	1.002
5	1.001	1.002	1.001	1.001
6	1.000	1.000	1.000	1.000

The VPT series for the function $G_2(0)$ is as follows

$$G_2(0) = g^{-2/3}\frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{\Gamma(n+1/2-m/4)}{(n-m)!}\cdot\frac{B_m}{\Gamma(1+3m/2)}\,x^{2+3m}, \tag{31}$$

where B_m are dimensionless coefficients of the standard perturbation theory. For the considered first nontrivial VPT order we need the two values, $B_0 = 1$ and $B_1 = -6$. In the first VPT order we get from (31)

$$G_2^{(1)} = g^{-2/3}\,(G_{20} + G_{21})\,,$$

where

$$G_{20} = \frac{\sqrt{\pi}}{2}\,x^2\,, \qquad G_{21} = \frac{\sqrt{\pi}}{4}\,x^2 - 4\,\Gamma\left(\frac{5}{4}\right)\,x^5\,.$$

The PMS optimization ($\partial G_2^{(1)}/\partial x = 0$) leads to the value $\mu^2 = 3.078 g^{2/3}$. The corresponding numerical result is $\mu^2_{\text{exact}} = 3.009 g^{2/3}$.[1]

Effective potential. Consider the generating functional for the Green function (we employ the pseudo-Euclidean signature in the n-dimensional space, keeping in mind applications in field theory)

$$W[J] = \int D\varphi\,\exp\Big\{i\Big[\,S[\varphi] + < J\varphi >\Big]\Big\}\,, \tag{32}$$

where

$$< J\varphi >= \int dt\,J(t)\cdot\varphi(t) \quad \text{and} \quad S[\varphi] = S_0 - m^2\,\tilde{S} - g\,S_I\,. \tag{33}$$

The effective potential is usually constructed in the quasiclassical approximation based on the expansion in powers of the number of loops. In our case this method gives the one-loop potential of the form $V_{\text{eff}}^{1-\text{loop}} = \sqrt{m^2 + 12 g\varphi_0^2}\,/2$ that is completely unfit for the description of the nonperturbative region.

[1] Some other details connected with consideration of the anharmonic oscillator in the framework of the VPT method can be found in [30].

We will compute the effective potential by the VPT method.To this end, we introduce variational parameter a^2 rewriting the action to the form

$$S[\varphi] = \left[S_0 - m^2\,\tilde{S} - \frac{a^2}{T}\,\tilde{S}^2\right] - \left[g\,S_I - \frac{a^2}{T}\,\tilde{S}^2\right]. \tag{34}$$

The effective potential is obtained from the effective action when the field configurations are constant and, in this case, the variational parameter introduced in the form a^2/T will be independent of the "volume" T of x-space.

Expanding the exponential of (32) in powers of $g\,S_I - a^2\,\tilde{S}^2/T$ we get

$$\begin{aligned} W[J] &= \exp(-i\pi/4)T^{1/2}\int\limits_{-\infty}^{\infty}\frac{dv}{2\sqrt{\pi}}\exp(-iTv^2/4) \qquad (35)\\ &\times \sum_{n=0}^{\infty} i^n \sum_{k=0}^{n}\frac{1}{(n-k)!}\Big(i\frac{d}{d\varepsilon}\Big)^{n-k}\frac{1}{k!}\int D\varphi(-gS_I)^k \exp\Big\{i\Big[S_0 - M^2\tilde{S} + <J\varphi>\Big]\Big\}, \end{aligned}$$

where $M^2 = m^2 + \sqrt{\varepsilon}av$. Completing differentiations with respect to ε we have to set $\varepsilon = 1$. Denoting the perturbative expansion coefficients for the functional $W[J]$ by $\omega_k[J, M^2]$:

$$\omega_k[J, M^2] = \frac{(-i\,g)^k}{k!}\Big[\int dt\,\frac{\delta^4}{\delta J^4(t)}\Big]^k \cdot \exp\Big[\frac{i}{2} <J\,\Delta\,J>\Big], \tag{36}$$

where $\Delta(p) = (p^2 - M^2 + i\,0)^{-1}$. In the Nth VPT order we obtain

$$\begin{aligned} W^{(N)}[J] &= \exp(-i\,\pi/4)\,T^{1/2}\int\limits_{-\infty}^{\infty}\frac{dv}{2\sqrt{\pi}}\exp(-i\,T\,v^2/4)\\ &\times \sum_{n=0}^{N}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{(n-k)!}\Big(\frac{d}{d\varepsilon}\Big)^{n-k}\cdot\Big[\det\frac{\partial^2+M^2}{\partial^2+m^2}\Big]^{-1/2}\omega_k[J, M^2]. \qquad (37) \end{aligned}$$

The functional determinant in (37) is calculated by the relation $\det(\cdots) = \exp[\mathrm{Tr}\ln(\cdots)]$ and the result is

$$\Big[\det\frac{\partial^2+M^2}{\partial^2+m^2}\Big]^{-1/2} = \exp\Big\{-i\,\frac{T}{2}\,[(M^2)^{1/2} - (m^2)^{1/2}]\Big\}. \tag{38}$$

In the leading VPT order we get

$$W^{(1)}[J] = \exp(-i\pi/4)T^{1/2}\int\limits_{-\infty}^{\infty}\frac{dv}{2\sqrt{\pi}}\exp(-i\,T\,v^2/4)\Big[1 + \Big(\frac{\omega_1}{\omega_0} - \frac{d}{d\varepsilon}\ln\omega_0\Big)\Big], \tag{39}$$

where

$$\begin{aligned} S(v)) &= \frac{1}{2}\frac{J^2}{M^2} + \frac{v^2}{4} - \frac{1}{2}\,[(M^2)^{1/2} - (m^2)^{1/2}], \qquad (40)\\ \tilde{\omega}_0 &= \exp\Big\{i\,T\Big[\frac{J^2}{2\,M^2} - \frac{1}{2}((M^2)^{1/2} - (m^2)^{1/2})\Big]\Big\}, \qquad (41)\\ \frac{\omega_1}{\omega_0} &= -i\,g\,T\Big[\frac{3}{4}\,\frac{1}{M^2} + 3\,\frac{J^2}{(M^2)^{5/2}} + \frac{J^4}{(M^2)^4}\Big]. \qquad (42) \end{aligned}$$

In these expressions we take constant sources, $J = const$, which is required for constructing the effective potential.

To construct the effective potential introduce the generating functional of the connected Green functions $Z[J] = (iT)^{-1} \ln W[J]$. For the effective potential we obtain

$$V_{\text{eff}}[\varphi_0] = J\,\varphi_0 - Z[J]\,, \tag{43}$$

where J is derived from the equation $\varphi_0 = dZ[J]/dJ$.

The integrand in Eq. (39) contains a large parameter, T, in the exponential and thus that integral may be computed by the asymptotic method of a stationary phase. Then in the first VPT order in the strong coupling limit ($m^2 = 0$) we get

$$\begin{aligned} Z^{(1)}[J] &= Z_0[J] + Z_1[J]\,, \\ Z_0[J] &= \frac{3}{4}\frac{J^2}{M^2} - \frac{3}{8}(M^2)^{1/2}\,, \\ Z_1[J] &= \frac{1}{4}\frac{J^2}{M^2} + \frac{1}{8}(M^2)^{1/2} - g\Big[\frac{3}{4}\frac{1}{M^2} + 3\frac{J^2}{(M^2)^{5/2}} + \frac{J^4}{(M^2)^4}\Big]\,, \end{aligned} \tag{44}$$

where we introduced a new variational parameter, M^2, which is computed by an optimization procedure.

To compare with numerical results for E_0 and μ^2, we should know the expansion of $V_{eff}(\varphi_0)$ near the extremum. Solving the equation of optimization, $Z_1 = 0$ we get

$$M^2 = M_0^2\Big[1 + \frac{3}{4}\frac{J^2}{(M_0^2)^{3/2}} + O(J^4)\Big]\,,$$

where $M_0^2 = (6\,g)^{2/3}$.

The effective potential reads

$$V_{\text{eff}}^{(1)}(\varphi_0) = E_0^{(1)} + \frac{\mu_{(1)}^2}{2}\varphi_0^2 + O(\varphi_0^4)\,,$$

where

$$E_0^{(1)} = \frac{3}{8}(6\,g)^{1/3} = 0.681\,g^{1/3}\,, \qquad \mu_{(1)}^2 = M_0^2 = 3.302\,g^{2/3}\,.$$

3 $\varphi^4_{(4)}$-model

In this section, we apply the VPT approach to derive the nonperturbative β-function for the φ^4-model in four dimensions [18]. The corresponding Euclidean action has the form

$$S[\varphi] = S_0[\varphi] + S_I[\varphi]\,, \tag{45}$$

where

$$S_0[\varphi] = \frac{1}{2}\int dx\ \varphi(-\partial^2)\,\varphi\,, \qquad S_I[\varphi] = \frac{(4\pi)^2}{4!}\,g\int dx\ \varphi^4\,. \tag{46}$$

The series of perturbation theory for the generating functional of the Green functions

$$W[J] = \int D\varphi\,\exp\Big\{-S[\varphi] + \int dx\ J\cdot\varphi\Big\} \tag{47}$$

diverges. A formal argument consists in a meaningless functional integral for the negative coupling constant. The function $W[J]$ as a function of g is not analytic at $g = 0$. The concrete asymptotic behavior of higher-order terms of perturbation theory can be determined by the functional saddle-point method [31] In this case, the main contribution to the functional integral comes from the configurations of fields φ which are proportional to a positive power of the large saddle-point parameter n, and, therefore, the functional S_I in Eq. (46) cannot be considered as the perturbative term in the comparison with expression S_0.

The idea of the VPT method consists in the construction of a new effective functional interaction S_I'. We expect that this functional can be considered as a small value when compared with a new functional S_0'. In realization of this idea we should ensure the possibility of the calculations. Practically, we must use only the Gaussian functional integrals, i.e. the form of $\tilde{S}[\varphi]$ should be such that the functional integral in (47) can be reduced to Gaussian quadratures.

Let us take the VPT functional in the following form

$$\tilde{S}[\varphi] = \theta^2 S_0^2[\varphi] \tag{48}$$

which has the same degree of the field as the functional S_I. Consider a new splitting of the total action (45)

$$S[\varphi] = S_0'[\varphi] + \eta\, S_I'[\varphi]\,, \tag{49}$$

where

$$S_0'[\varphi] = S_0[\varphi] + \tilde{S}[\varphi]\,, \qquad S_I'[\varphi] = S_I[\varphi] - \tilde{S}[\varphi]\,. \tag{50}$$

In this case, the expansion of expression (47) is carried out in powers of η. After all calculations we should put $\eta = 1$. The parameter θ^2 in Eq. (48) is a parameter of variational type. The initial functional (47) certainly does not depend on this parameter. We may take θ^2 so as to provide the best approximation with a finite number of VPT series terms.

It is convenient to define the new parameter t by the relation

$$\theta^2 = 4\, C_S\, \frac{(4\pi)^2}{4!}\, g \cdot t\,. \tag{51}$$

Here $C_S = 4!/(16\pi)^2$ is a constant entering into the Sobolev inequality (see, for example, [32, 33] and [2])

$$\int dx\ \varphi^4 \le C_S \Big[\int dx \varphi(-\partial^2)\,\varphi\Big]^2.$$

The parameter t is fixed if we require the contribution of higher order terms of the VPT series to be minimal. This way of determining a variational parameter called the asymptotic optimization of VPT series gives the value $t = 1$.

After expansion in powers of η we obtain that the remainder contains the $\tilde{S}[\varphi]$ in the exponential and, consequently, we have a non-Gaussian form of the functional integral. However, the problem is easily solved by implementing the Fourier transformation. As a result, the Green function $G_{2\nu}$ in Nth order of VPT takes the following form

$$G_{2\nu}^{(N)} = \int_0^\infty d\alpha\ \alpha^{\nu-1} \exp(-\alpha - \theta^2\alpha^2) \sum_{n=0}^{N} \eta^n\, \alpha^{2n} \sum_{k=0}^{n} \frac{(\theta^2)^{n-k}}{(n-k)!}\, \frac{g_{2\nu}^k}{\Gamma(2k+\nu)}\,, \tag{52}$$

where the functions $g^k_{2\nu}$ are ordinary perturbative coefficients for the Green function $G_{2\nu}$. To calculate them, the standard Feynman diagrams can be used.

It should be stressed that the expansion of expression (52) in powers of the coupling constant g contains all powers of g. The first N terms of this expansion coincide with N terms of a perturbative series.

Let us consider the procedure of renormalization. Instead of the field φ and coupling constant g we introduce the bare field φ_0 and bare coupling constant g_0. The field φ_0 is connected with the renormalized field by the relation: $\varphi_0 = Z^{1/2}\varphi$. The divergent constants Z and g_0 are obtained from the VPT expansion. The constant Z can be calculated by using the propagator G_2. We will employ the constant Z in the first order of the VPT series. From Eq. (52) we find

$$Z^{(1)} = \Gamma(1)\, J_1(\theta_0^2) + \eta\, \theta_0^2\, \Gamma(3)\, J_3(\theta_0^2)\,, \tag{53}$$

where we defined

$$J_\nu(\theta^2) = \frac{1}{\Gamma(\nu)} \int_0^\infty d\alpha\; \alpha^{\nu-1} \exp(-\alpha - \alpha^2\theta^2)\,. \tag{54}$$

The function $J_\nu(\theta^2)$ is normalized by the condition $J_\nu(0) = 1$. The connected part of the four-point Green function in the second order of VPT has the form

$$-G_4^{(2)}(\mu^2) = \eta\, g_0\, J_4(\theta_0^2) + \eta^2 \left[g_0\, \frac{\theta_0^2}{1!}\, \frac{\Gamma(6)}{\Gamma(4)}\, J_6(\theta_0^2) - \frac{3}{2}\, g_0^2\, J_6(\theta_0^2)\, \ln \frac{\Lambda^2}{\mu^2} \right]. \tag{55}$$

In this expression we wrote out only the divergent part, we need in the following. We use the renormalization scheme with a symmetric normalization point μ^2 . For the bare coupling constant g_0 we write down the VPT expansion $g_0 = g(1 + \eta\,\alpha + \ldots)$. The VPT expansions for θ_0^2 and $J_\nu(\theta_0^2)$ are introduced in a similar manner. The divergent coefficient α is defined by expressions (53), (55) and the requirement for the function $-Z^2 G_4(\mu^2)$ being finite. If we change the normalization point $\mu \to \mu'$ and use the bare coupling constant being independent of μ we find the connection between g and g': $g' = g + \eta\beta(g)\ln(\mu'^2/\mu^2)$, where the β-function is expressed as

$$\beta(g) = \frac{3}{2}\, g^2 \frac{J_6(\theta^2)/J_4(\theta^2)}{1 - \theta^2 \Big\{ [\Gamma(6)\, J_6(\theta^2)/\Gamma(4)\, J_4(\theta^2)] - 2\,[\Gamma(3)\, J_3(\theta^2)/\Gamma(1)\, J_1(\theta^2)] \Big\}}. \tag{56}$$

Here the parameter θ^2 is connected with the renormalized coupling constant g by Eq. (51) with the optimal value $t = 1$.

The expansion of β-function (56) in the perturbation series contains all powers of the coupling constant g. It is interesting to compare the first coefficients of the VPT β-function (56) with the well-known coefficients of perturbation theory. From Eq. (56) we get

$$\beta(g) = 1.5\, g^2 - 2.25\, g^3 + 14.63\, g^4 - 134.44\, g^5 + \cdots\,. \tag{57}$$

In the considered massless case, we use counterterms containing only divergent parts. In the framework of the dimensional regularization this conforms only to the pole part for counterterms. Corresponding β-function in the four-loop approximation looks as follows [34]

$$\beta_{\rm pert}(g) = 1.5\, g^2 - 2.83\, g^3 + 16.27\, g^4 - 135.80\, g^5 + \cdots\,. \tag{58}$$

Note that in constructing the β-function (56) we used only the lowest order of VPT. For this approximation the expressions (57) and (58) are in agreement.

As follows from expression (56), the β-function is monotonously increasing and has no the ultraviolet stable point. For a large coupling constant, the β-function has the asymptotic behaviour

$$\beta(g) \simeq \frac{3}{10}\,\frac{\sqrt{\pi}}{3\pi/8-1}\,g^{3/2}. \tag{59}$$

The degree of g in Eq. (59) is larger than the linear increase of the β-function obtained in [35], and is smaller than the square increase found in [36].

4 Quantum chromodynamics

In the case of QCD we will apply the harmonic variational procedure which leads to a new small expansion parameter. To represent a simple explanation of the basic idea of the method, let us first make a start with very transparent example.

Toy model. Consider the following integral

$$W(g) = \int_{-\infty}^{\infty} dx\, \exp(-S[x])\,. \tag{60}$$

The expression (60) can be considered as the zero-dimensional analog of the ϕ^4-model. The function $S[x]$ plays the role of "the action functional"

$$S[x] = S_0[x] + S_I[x] = x^2 + g\,x^4. \tag{61}$$

In the quantum field theory we can calculate the Gaussian functional integrals. Let us imagine that in this simple case we have to operate with Gaussian integrals as well. Thus, we can try to evaluate the quantity (60) by using the Gaussian integrals of the sort

$$\int dx\, P(x)\, \exp(-a\,x^2) \tag{62}$$

with some polynomial $P(x)$ of x.

The standard method of calculations is the expansion of the expression $\exp(-S[x])$ in the power series of the "coupling constant" g. Indeed, in this case, one uses the Gaussian integrals (62) and obtains the standard asymptotic perturbative series

$$W(g) = \sum_{k=0}^{\infty} \omega_k \tag{63}$$

with the coefficients

$$\omega_k = \frac{1}{k!}\int_{-\infty}^{\infty} dx\,(-g\,x^4)^k\,\exp(-S_0[x])\,. \tag{64}$$

Whereas the expansion of the function (60) in the series (63) with coefficient (64) is unique, the inverse procedure of finding the sum of the series (63) without using additional information about the function (60) is nonunique. For example, the same series (63) has also the function $W(g) + \exp(-1/g)$ that has different from $W(g)$ asymptotic behavior at

large values of the coupling constant g. The reason for the incorrectness of the summation procedure is the asymptotic nature of the perturbative expansion (63). Therefore, the perturbation series by itself without any additional information about its sum cannot be used to evaluate the function (60) for sufficiently large values of the coupling constant. Of course, in this simple case, we know the needed additional information about $W(g)$ and can apply to the series (63) some method of summation, for example, the Borel method. But, in the real field theory models, we do not know this information about function that is represented by functional integral and the problem requires special attention.

The variational perturbation theory approach makes it possible to construct different expansion for the function (60) and for quantum field models using the Gaussian quadratures. In this section we will demonstrate how the VPT idea allows one to construct a nonperturbative expansion which is based on a new small expansion parameter. By using a new split of the action let us rewrite Eq. (61) in the form

$$S[x] = S'_0[x] + S'_I[x], \tag{65}$$

where we have introduced a new free action $S'_0[x] = \zeta^{-1}x^2$ and an action of interaction $S'_I[x] = gx^4 - (\zeta^{-1} - 1)x^2$. Here ζ is an auxiliary parameter of a variational type. Actually, the original quantity $W(g)$ does not depend on this parameter, therefore, when studying a finite number of the terms of the series it is possible to choose the variational parameter on the basis of some principle of optimization [16, 30].

The VPT series for (60) can be written down as follows

$$W(g) = \sum_{n=0}^{\infty} W_n, \tag{66}$$

where the terms of the VPT expansion have the form

$$\begin{aligned} W_n &= \frac{1}{n!} \int dx \, (-S'_I[x])^N \exp(-S'_0[x]) \\ &= \sum_{k=0}^{n} \frac{1}{(n-k)!\,k!} \int dx \, (-g\,x^4)^k \left[(\zeta^{-1} - 1)\, x^2 \right]^{n-k} \exp(-S'_0[x]). \end{aligned} \tag{67}$$

It is convenient to rewrite the free action as follows

$$S'_0[x] = \zeta^{-1} x^2 \Rightarrow \left[1 + \kappa(\zeta^{-1} - 1) \right] x^2 \tag{68}$$

and set $\kappa = 1$ after all calculations. In this case, any power of $[(\zeta^{-1} - 1)x^2]$ in Eq. (67) can be obtained by differentiations with respect to κ. The remaining polynomial $(-g\,x^4)^k$ has the standard perturbative form, therefore, we have a possibility to apply to calculations the standard diagram technique with modified propagator

$$\Delta = \frac{1}{1 + \kappa(\zeta^{-1} - 1)}. \tag{69}$$

For $\kappa = 1$, one finds $\Delta = \zeta$.

The terms of the VPT expansion can be written down in the form

$$W_n = \sum_{k=0}^{n} \frac{1}{(n-k)!} \left(-\frac{\partial}{\partial \kappa} \right)^{n-k} \omega_k, \tag{70}$$

Table 2: *The relative error* $D(g) = |W_{\text{theor}}(g)/W_{\text{exper}}(g) - 1|$

N	0	1	2	3	4	6	8
C	1.14	2.64	3.56	5.46	6.12	8.71	11.33
D(g=10) %	2.76	4.83	0.26	0.73	0.038	0.006	0.0012
D(g=1000) %	5.01	6.52	0.56	1.13	0.089	0.017	0.0033

where the coefficients

$$\omega_k = \frac{1}{k!} \int dx \, (-g\, x^4)^k \exp\left(-x\, \Delta^{-1}\, x\right) \tag{71}$$

are given by the standard diagrams of perturbation theory with the propagator (69).

Consider a structure of the VPT term (70). First of all, note that the differentiation with respect to parameter κ gives the additional factor $(1-\zeta)$

$$\frac{1}{m!}\left(-\frac{\partial}{\partial\kappa}\right)^m \Delta(\kappa=1) = (1-\zeta)^m\, \Delta(\kappa=1). \tag{72}$$

Secondly, it is easy to see that in this model the number of internal lines (L) in any diagram (here, all diagrams are vacuum diagrams) equal to the double number of vertices (V): $L = 2\,V$. The internal line corresponds to propagator and leads to the factor ζ, and the vertex gives the factor g. Thus, schematically, one can write down

$$W_n \sim (g\zeta^2)^n + (1-\zeta)(g\zeta^2)^{n-1} + \cdots + (1-\zeta)^{n-1}(g\zeta^2) + (1-\zeta)^n. \tag{73}$$

From (73), we can see that if the value of $(1-\zeta)$ will be proportional to $(g\zeta^2)$, the expression W_n will contain the common factor $(1-\zeta)^n$. So, let the parameter ζ obeys to equation

$$1 - \zeta = C g \zeta^2 \tag{74}$$

with some positive constant C. We see from (74) that for all values of the initial coupling constant g the new expansion parameter $a = 1-\zeta$ obeys the inequality: $0 \le a < 1$. The remained parameter C is independent of the value of the coupling constant g and can be found by different ways. For example, if we consider the first non-trivial order $W^{(1)}(g) = W_0(g) + W_1(g)$ and use so-called "fastest apparent convergence", from point of view of which an absolute value of the last calculated term in the expansion should be minimal or vanishes, and require that $W_1 = 0$, we find $C = 3/2$. In this case, we have the approximation $W(g)$ by the expression $W^{(1)}(g)$ (with $W_1 = 0$) with an accuracy better than six percent for all interval of g. In particular, at $g \to \infty$, the relative error of approximation is about 5.1%. Similar results can be obtained if one uses the principle of minimum sensitivity, or a normalization at some "experimental" value $W(g_0) = W_{\text{exper}}$. If one includes to our consideration the next orders of the VPT expansion we will obtain a best approximation of $W(g)$.

In Table 2 we can see a dependence of the parameter C on the order expansion and the relative error $D(g) = |W_{\text{theor.}}(g)/W_{\text{exper.}}(g) - 1|$. Here, to find the parameter C one makes use of the normalization condition $\min|W(g_0) - W_{\text{exper.}}| = \min|W(g_0) - W_{\text{exact}}(g_0)|$ at $g_0 = 1$.

Small expansion parameter in QCD. To explain the basic idea of the method in the case of QCD, let us first consider the pure Yang–Mills theory. The Lagrangian density has the form

$$\begin{aligned} L_{YM} &= -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2}\,g\,F_{\mu\nu}\,[A_\mu \times A_\nu] + \frac{1}{4}\,g^2\,[A_\mu \times A_\nu]^2 + L_{g.f.} + L_{F.P.} \\ &= L_0(A) + g\,L_3(A) + g^2\,L_4(A)\,, \end{aligned} \tag{75}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $L_{g.f.}$ and $L_{F.P.}$ are gauge fixing and Faddeev-Popov terms.

The term $L_3(A)$ generates the three-gluon and ghost-gluon-ghost vertices. This interaction is the Yukawa type interaction. The term $L_4(A)$ generates the four-gluon vertices. Let us introduce the $\chi_{\mu\nu}$ field and transform the term $L_4(A)$ to the Yukawa type diagrams

$$\exp\left\{i\frac{g^2}{4}\int dx\;[A_\mu \times A_\nu]^2\right\} = \int D\chi \exp\left\{-\frac{i}{2}\int dx\chi^2_{\mu\nu} + i\frac{g}{\sqrt{2}}\int dx\chi_{\mu\nu}[A_\mu \times A_\nu]\right\}. \tag{76}$$

The action functional can be written in the form

$$S = S_0(\chi) + S(A,\chi) + S^{Yuk.}_{YM}(A)\,, \tag{77}$$

where

$$S(A,\chi) = \frac{1}{2}\int dx\,dy\;A^a_\mu(x)\left[D^{-1}(x,y|\chi)\right]^{ab}_{\mu\nu} A^b_\nu(y) \tag{78}$$

and the gluon propagator $D(x,y|\chi)$ in the χ–field is defined as

$$\left[D^{-1}(x,y|\chi)\right]^{ab}_{\mu\nu} = \left[-\partial^2 g_{\mu\nu}\delta_{ab} + g\sqrt{2}f_{abc}\chi^c_{\mu\nu} + \text{gauge terms}\right]\delta(x-y). \tag{79}$$

The Green functions can be written as

$$G(\cdots) = \langle G_{Yuk.}(\cdots|\chi)\rangle\,, \tag{80}$$

where

$$G_{Yuk.}(\cdots|\chi) = \int DA[\cdots]\exp\left\{i\left[S(A,\chi) + S^{Yuk.}_{YM}(A)\right]\right\}, \tag{81}$$

and

$$\langle\cdots\rangle = \int D\chi\,[\cdots]\exp\left[i\,S_0(\chi)\right]. \tag{82}$$

The Green functions $G_{Yuk.}(\cdots|\chi)$ contain only the Yukawa type diagrams appearing inside the brackets $\langle\cdots\rangle$ with the gluon propagator $D(x,y|\chi)$. In Fig. 1 (a), the full gauge propagator is shown. The expansion $D(x,y|\chi)$ in perturbation theory generates the four-gluon graphs [Fig. 1 (b)] that are added to the Yukawa diagrams, and in this case we obtain the standard perturbation expansion [Fig. 1 (c)].

Let us rewrite the Lagrangian in the form

$$\begin{aligned} L(A,\chi) &= L_0(A,\chi) + L_I(A,\chi)\,, \\ L_0(A,\chi) &= \zeta^{-1}\,L(A,\chi) + \xi^{-1}\,L(\chi)\,, \\ L_I(A,\chi) &= \eta\left[g\,L^{Yuk.}_{YM}(A) - (\zeta^{-1}-1)\,L(A,\chi) - (\xi^{-1}-1)\,L(\chi)\right], \end{aligned} \tag{83}$$

where ζ and ξ are the parameters of variational type. The original quantity $L(A,\chi)$ does not depend on ζ and ξ. Therefore, the freedom in choosing ζ and ξ can be used to improve

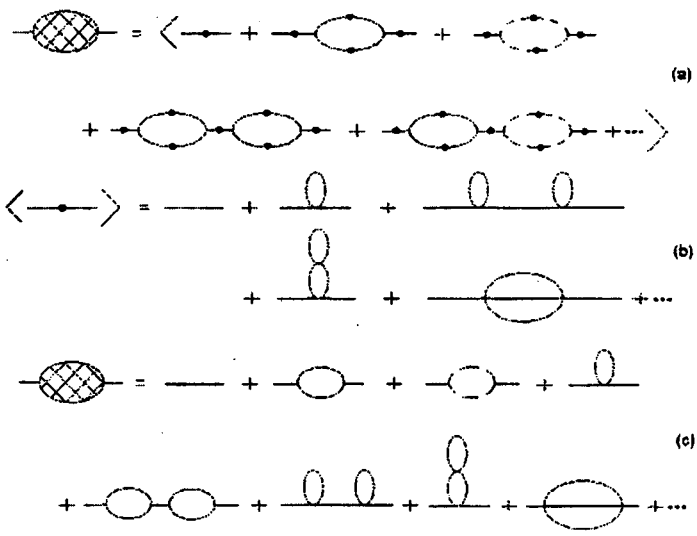

Figure 1: *The perturbation expansion of the full gluon propagator by using the* χ*-transformation. The gluon line with a point corresponds to the function* $D(\chi)$

the series properties . In the variational perturbation series a new action of interaction is used for constructing the expansion. It is clear that if the parameters $0 < \zeta < 1$ and $0 < \xi < 1$, we "strengthen" the new free Lagrangian and, at the same time, "weaken" the Lagrangian of interaction. After all calculations we put $\eta = 1$. This parameter will be also written in the propagator $D(x, y|\chi)$ in the combination with the coupling constant. The VPT series for the Green function is given by

$$\begin{aligned} G(\cdots) &= \sum_n G_n(\cdots), \\ G_n(\cdots) &= \frac{1}{n!}\eta^n \int D\chi DA[\cdots][i\,S_I(A,\chi)]^n \exp\left[i\,S_0(A,\chi)\right], \\ &= (i\eta)^n \sum_{k=0}^{n} \frac{1}{(n-k)!\,k!} \int D\chi DA[\cdots]\left[g\,S_{YM}^{Yuk.}(A)\right]^k \\ &\quad \left[(\zeta^{-1}-1)\,S(A,\chi) + (\xi^{-1}-1)\,S(\chi)\right]^{n-k} \exp\left[i\,S_0(A,\chi)\right]. \end{aligned} \tag{84}$$

We redefine the $L_0(A,\chi)$ for convenience of calculations as follows:

$$L_0(A,\chi) \Rightarrow L_0'(A,\chi) = [1+\kappa(\zeta^{-1}-1)]L(A,\chi) + [1+\kappa(\xi^{-1}-1)]L(\chi). \tag{85}$$

In this case, any power of $[(\zeta^{-1}-1)\,S(A,\chi) + (\xi^{-1}-1)\,S(\chi)]$ in (84) can be obtained by the corresponding number of differentiation of the expression $\exp[iS_0'(A,\chi,\kappa)]$ with respect to κ. After all calculations we set $\kappa = 1$.

From Eqs. (84) and (85) we have

$$G_n = \eta^n \sum_{k=0}^{n} \frac{1}{(n-k)!}\left(-\frac{\partial}{\partial\kappa}\right)^{n-k} \langle\, g_k(\kappa)\,\rangle\,, \tag{86}$$

where the functions

$$g_k(\kappa) = \frac{i^k}{k!} \int DA[\cdots] \left[g\, S_{YM}^{Yuk}(A) \right]^k \exp\left\{ i[1 + \kappa(\zeta^{-1} - 1)] \int dx\, L(A, \chi) \right\} \qquad (87)$$

correspond to the Yukawa diagrams of the Yang–Mills theory with gluon propagator

$$\frac{1}{1 + \kappa(\zeta^{-1} - 1)} D(x, y|\chi) \to \zeta D(x, y|\chi)$$

for $\kappa = 1$. The propagator of χ-field includes the factor $[1 + \kappa(\xi^{-1} - 1)]^{-1}$ transformed into ξ for $\kappa = 1$.

The operator of differentiation $(-\partial/\partial\kappa)^l / l!$ leads to the factor $(1 - \zeta)^l$ for the gluon propagator and $(1 - \xi)^l$ for the propagator of the χ-field.

It is easy to verify that the Nth order of the VPT series contains the Nth order of a perturbation series with the correction $O(g^{N+1})$, therefore, the VPT expansion does not contradict the perturbative results obtained for the small coupling constant.

The diagrams corresponding to the new expansion of the full gluon propagator are shown in Fig. 2. The gluon line with slash signifies the differentiation over κ and contains the factor $(1 - \zeta)$. If this line arises due to the χ-field propagator, the corresponding factor is $(1 - \xi)$.

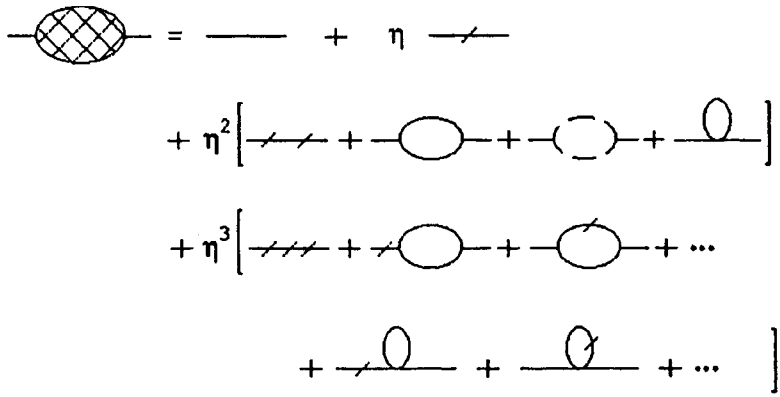

Figure 2: *The diagrams for the VPT expansion of the full gluon propagator*

The outline of the VPT expansion structure can be written as

$$\begin{aligned} 1 \;&+\eta(1 - \zeta) + \eta^2 \left[(1 - \zeta)^2 + g^2\zeta^3 + g^2\xi \right] \\ +\; &\eta^3 \left[(1 - \zeta)^3 + g^2\zeta^3(1 - \zeta) + g^2\xi(1 - \zeta) + g^2\xi(1 - \xi) \right] + \cdots \end{aligned} \qquad (88)$$

The construction of expansions for the Green functions corresponding to three-, four-gluon, ghost-gluon-ghost vertices are introduced in similar manner.

If we choose $\xi = \zeta^3$ and $(1 - \zeta)^2 = C\lambda\zeta^3$, where C is a positive constant, we obtain that the nth order term of our series contains the factor $(1 - \zeta)^n$ and the expansion parameter

$a = (1 - \zeta) < 1$ for all values of the initial coupling constant. Now, one can perform the renormalization procedure and define the renormalization constants of a power series of a [19, 20].

Consider the connection between the perturbative and nonperturbative regimes of the running coupling constant $\alpha_s(Q^2)$. To fix the parameter C we will use nonperturbative information from meson spectroscopy and derive $\alpha_s(Q^2)$ in the perturbative region at large Q^2. In other words, we will find the connection between the universal tension σ in the linear part of the quark-antiquark static potential $V_{\text{lin}}(r) = \sigma r$, which can be determined from meson spectroscopy, and the description of high energy physics. If, as usual, we assume that the quark potential in momentum space can be written as $V(q^2) = -16\pi\alpha_s(q^2)/3q^2$, where $\alpha_s(q^2)$ describes both large and small momentum, and that $\alpha_s(q^2)$ has the singular infrared asymptotics $\alpha_s(q^2) \sim q^{-2}$, we obtain, by taking the three-dimensional Fourier transform, the large-distance linear potential in coordinate space. The corresponding singular infrared behaviour of $\lambda = \alpha_s/(4\pi)$ conforms to the asymptotics of the β-function: $\beta(\lambda) \to -\lambda$ for a large coupling constant.

In the framework of this approach consider the functions $\beta^{(2)}$, $\beta^{(3)}$, $\beta^{(4)}$ and $\beta^{(5)}$ that are obtained if we take into consideration the terms $O(a^2)$, $O(a^3)$, $O(a^4)$ and $O(a^5)$ in the corresponding renormalization constant Z_λ. As has been shown [20], the values of $-\beta^{(k)}(\lambda)/\lambda$ as functions of the coupling constant for parameters $C_2 = 0.977$, $C_3 = 4.1$, $C_4 = 10.4$ and $C_5 = 21.5$ go to 1 at sufficiently large λ. The increase of C_k with the order of the expansion is explained by the necessity to compensate the high order contribution. A similar situation takes place also in zero- and one-dimensional models. The behaviour of the functions $-\beta^{(k)}(\lambda)/\lambda$ gives evidence for the convergence of the results, in accordance with the phenomenon of induced convergence. At large coupling, $-\beta^{(k)}(\lambda)/\lambda \simeq 1$, which corresponds to $\alpha_s(Q^2) \sim Q^{-2}$ at small Q^2.

The value of the coefficient σ in the linear part of the quark-antiquark static potential $V_{\text{lin}}(r) = \sigma r$ is $\sigma \simeq 0.15 \div 0.20$ GeV^2. At a small value of Q^2 the corresponding behaviour of $\alpha_s(Q^2)$ is $\alpha_s(Q^2) \simeq 3\sigma/2Q^2$. Here we will use this equation at a certain normalization point Q_0 and the value $\sigma = 0.1768$ GeV^2 which has been obtained in [37].

The renormalization group method gives an equation for the Q^2-evolution of the expansion parameter $a = a(Q^2)$ [19, 20]. In an appropriate region of the momentum, the value of $\sigma(Q^2) = 2/3Q^2\alpha_s(Q^2)$ is almost independent of the choice of Q_0 and lies in the interval $0.15 \div 0.20$ GeV^2. This result agrees with the phenomenology of meson spectroscopy. Thus, we have found all the parameters and can now consider the behaviour of the effective coupling constant at large Q^2. For example, we find $\alpha_{\text{eff}}(m_Z) = 0.126$. It should be stressed that we have obtained this result by evolution of the effective coupling starting from a very low energy scale. Taking into account this fact the value of $\alpha_{\text{eff}}(m_Z)$ obtained in such a way seems to be quite reasonable.

5 Conclusion

We have considered an approach to quantum field theory – the method of variational perturbation theory. The original action functional is rewritten using some variational addition and an expansion in the effective interaction is made. Therefore, in contrast to many nonperturbative approaches, in the VPT the quantity under consideration from the very beginning is written in the form of a series which makes it possible to calculate the

needed corrections. The VPT method thereby allows for the possibility of determining the degree to which the principal contribution found variationally using some variational principle adequately reflects the problem in question and determining the region of applicability of the results obtained.

The possibility of performing calculations using this approach is based on the fact that the VPT, like standard perturbation theory, uses only Gaussian functional quadratures. Here, of course, the VPT series possesses a different structure and, in addition, some of the Feynman rules are modified at the level of the propagators and vertices. The form of diagrams themselves does not change, which is very important technically. The diagrams contributing to the Nth order of the VPT expansion are of the same form as those contributing to the Nth order of ordinary perturbation theory.

The variational parameters arising in the VPT method allow the convergence properties of the VPT series to be controlled. In [12, 16] it has been shown that in the case of the anharmonic variational procedure for the scalar φ^4 model there is a finite region of parameter values in which the VPT series converges for all positive values of the coupling constant. For the harmonic variational procedure there are indications that VPT series can be also converged on the sense of the so-called induced convergence, by fine-tuning the variational parameters from order to order. Note also, that a possibility of constructing Leibnitz series in field models is interesting, because, in this case, the first few terms of the series can be used to obtain two-sided estimates of the sum of the series, and existence of variational parameters makes it possible to narrow these estimates the maximum amount in a given order of VPT (see [30]).

Here, we have considered the application of the method to quantum chromodynamics, where the VPT idea leads to an expansion with a new small expansion parameter. This parameter obeys an equation whose solution is always smaller than unity for any value of the coupling constant. Therefore, while remaining within the limits of applicability of this expansion it is possible to deal with considerably lower energies than in the case of perturbation theory. An important feature of this approach is the fact that for sufficiently small value of the running coupling constant $\bar{\alpha}_s$ it reproduces perturbative predictions. Therefore, all the high-energy physics is preserved in the VPT method. In going to lower energies, where standard perturbation theory ceases to be valid, $\bar{\alpha}_s \simeq 1$, the VPT running expansion parameter $\bar{a}$ remains small and we do not find ourselves outside the region of applicability of the method.

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