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HIDDEN SYMMETRY OF THE YANG–COULOMB MONOPOLE

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Bound system composed of the Yang monopole coupled to a particle of the isospin by the SU(2) and Coulomb interaction is considered. The generalized Runge–Lenz vector and the SO(6) group of hidden symmetry are established. It is also shown that the group of hidden symmetry makes it possible to calculate the spectrum of system by a pure algebraic method.

During the last few years attempts were made to find the electromagnetic duality phenomenon[1] in the framework of quantum mechanics.[2–10] One of the systems arising in these searches is the Yang–Coulomb monopole (YCM), a unique example of an integrable non-Abelian system.

The YCM is defined as a five-dimensional bound system composed of the Yang monopole[11] and a particle of the isospin. Both the monopole and particle are also assumed to have electric charges of the opposite signs. Thus, the monopole–particle coupling is realized not only by the SU(2) gauge field but also by the Coulomb interaction. At large distances the Coulomb structure becomes immaterial and YCM seems to be pure Yang monopole.

In this letter, we analyze the symmetry properties of the YCM. We establish the SO(6) group of hidden symmetry of YCM and use this symmetry for calculation of the energy spectrum of YCM by an algebraic method.

Let us introduce the space $\Omega = \mathbb{R}^5(x_j) \otimes S^3$ where $\mathbb{R}^5(x_j)$ and S^3 have the meaning of the configuration and the gauge space of YCM. We keep the following notation: $j = 0, 1, 2, 3, 4$; $a = 1, 2, 3$; $\hat{T}_a$ are the SU(2) gauge group generators; $\mathbf{A}^a$ is the triplet of Yang monopole's gauge potentials; σ^a are the Pauli matrices. The YCM is governed by the Hamiltonian

$$\hat{H} = \frac{1}{2m}\hat{\pi}^2 + \frac{\hbar^2}{2mr^2}\hat{T}^2 - \frac{e^2}{r},$$

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where $r = (x_j x_j)^{1/2}$, $\hat{T}^2 = \hat{T}_a \hat{T}_a$, $\hat{\pi}^2 = \hat{\pi}_j \hat{\pi}_j$,

$$\hat{\pi}_j = -i\hbar \frac{\partial}{\partial x_j} - \hbar A_j^a \hat{T}_a ,$$

$$A_i^a = \frac{2i}{r(r + x_0)} \tau_{ij}^a x_j$$

and τ^a are the 5×5 matrices having the following explicit form

$$\tau^1 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i\sigma^1 \\ 0 & i\sigma^1 & 0 \end{pmatrix}, \qquad \tau^2 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\sigma^3 \\ 0 & -i\sigma^3 & 0 \end{pmatrix},$$

$$\tau^3 = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma^2 & 0 \\ 0 & 0 & \sigma^2 \end{pmatrix}$$

and satisfying $[\tau^a, \tau^b] = i\varepsilon_{abc}\tau^c$.

Each term of the $\mathbf{A}^a$-triplet coincides with the gauge potential of the five-dimensional Dirac monopole with a unit topological charge and the line of singularity extended along the axis $x_0 \leq 0$.

It is obvious that $\tau_{ij}^a = -\tau_{ji}^a$ and, therefore, A_j^a are orthogonal to x_j. Moreover, it follows from

$$4\tau_{ij}^a \tau_{jk}^b = \delta_{ab}(\delta_{ik} - \delta_{i0}\delta_{k0}) + 2i\varepsilon_{abc}\tau_{ik}^c$$

that the vectors A_j^a are also orthogonal to each other

$$A_j^a A_j^b = \frac{r - x_0}{r^2(r + x_0)} \delta_{ab} .$$

The following fundamental commutation relations are valid:

$$[\hat{\pi}_i, x_k] = -i\hbar\delta_{ik} , \qquad [\hat{\pi}_i, \hat{\pi}_k] = i\hbar^2 F_{ik}^a \hat{T}_a ,$$

where

$$F_{ik}^a = \partial_i A_k^a - \partial_k A_i^a + \varepsilon_{abc} A_i^b A_k^c$$

is the triplet of gauge fields. In a more explicit form

$$F_{ik}^a = \frac{1}{r^2}[(x_k + r\delta_{k0})A_i^a - (x_i + r\delta_{i0})A_k^a - 2i\tau_{ik}^a] .$$

The last expression follows from the formula:

$$\varepsilon_{abc}\tau_{ij}^b \tau_{km}^c = \frac{i}{2}\big[(\delta_{i0}\delta_{k0} - \delta_{ik})\tau_{jm}^a - (\delta_{i0}\delta_{m0} - \delta_{im})\tau_{jk}^a + (\delta_{j0}\delta_{m0} - \delta_{jm})\tau_{ik}^a - (\delta_{j0}\delta_{k0} - \delta_{jk})\tau_{im}^a\big] .$$

The straightforward computation gives

$$F_{ij}^a F_{jk}^b = \frac{1}{r^6}(x_i x_k - r^2\delta_{ik})\delta_{ab} + \frac{1}{r^2}\varepsilon_{abc} F_{ik}^c .$$

Since YCM is a non-Abelian extension of the five-dimensional Coulomb problem, it is natural to try to construct an analog of the Runge–Lenz vector for YCM by passing from $\mathbb{R}^3$ to $\mathbb{R}^5(x_j)$ and taking account of the gauge field. The first step has been made many years ago[12]

$$\hat{M}_k = \frac{1}{2\sqrt{m}}\left(\hat{p}_i\hat{l}_{ik} + \hat{l}_{ik}\hat{p}_i + \frac{2me^2}{\hbar}\frac{x_k}{r}\right),$$

where $\hat{p}_i = -i\hbar\frac{\partial}{\partial x_i}$, $\hbar\hat{l}_{ik} = x_i\hat{p}_k - x_k\hat{p}_i$. The second step can be realized by the substitution[11]: $\hat{p}_i \to \hat{\pi}_i$, $\hat{l}_{ik} \to \hat{L}_{ik}$ where

$$\hat{L}_{ik} = \frac{1}{\hbar}(x_i\hat{\pi}_k - x_k\hat{\pi}_i) - r^2F^a_{ik}\hat{T}_a.$$

It is possible to verify that

$$[\hat{L}_{ik}, x_j] = i\delta_{ij}x_k - i\delta_{kj}x_i, \qquad [\hat{L}_{ik}, \hat{\pi}_j] = i\delta_{ij}\hat{\pi}_k - i\delta_{kj}\hat{\pi}_i,$$

$$[\hat{L}_{ij}, \hat{L}_{mn}] = i\delta_{im}\hat{L}_{jn} - i\delta_{jm}\hat{L}_{in} - i\delta_{in}\hat{L}_{jm} + i\delta_{jn}\hat{L}_{im},$$

i.e. $\hat{L}_{ik}$ are indeed the generators of the group SO(5) and $[\hat{H}, \hat{L}_{ik}] = 0$.

After some tedious calculations, we have $[\hat{H}, \hat{M}_k] = 0$ which means that $\hat{M}_k$ is in fact the analog of the Runge–Lenz vector for YCM. It can also be shown that

$$[\hat{L}_{ij}, \hat{M}_k] = i\delta_{ik}\hat{M}_j - i\delta_{jk}\hat{M}_i, \qquad [\hat{M}_i, \hat{M}_k] = -2i\hat{H}\hat{L}_{ik}.$$

These commutation rules generalize relations known from the theory of the Coulomb problem.[13]

Finally, let us introduce the 6×6 matrix

$$\hat{D} = \begin{pmatrix} \hat{L}_{ij} & -\hat{M}'_j \\ \hat{M}'_j & 0 \end{pmatrix},$$

where $\hat{M}'_i = (-2\hat{H})^{-1/2}\hat{M}_i$. The components $\hat{D}_{\mu\nu}$ (here $\mu, \nu = 0, \ldots, 5$) satisfy the commutation relations

$$[\hat{D}_{\mu\nu}, \hat{D}_{\lambda\rho}] = i\delta_{\mu\lambda}\hat{D}_{\nu\rho} - i\delta_{\nu\lambda}\hat{D}_{\mu\rho} - i\delta_{\mu\rho}\hat{D}_{\nu\lambda} + i\delta_{\nu\rho}\hat{D}_{\mu\lambda},$$

i.e. $\hat{D}_{\mu\nu}$ are the generators of the group SO(6). Since $[\hat{H}, \hat{D}_{\mu\nu}] = 0$, one concludes that YCM is provided by the SO(6) group of hidden symmetry.

The Casimir operators for SO(6) are[14]

$$\hat{C}_2 = \frac{1}{2}\hat{D}_{\mu\nu}\hat{D}_{\mu\nu},$$

$$\hat{C}_3 = \varepsilon_{\mu\nu\rho\sigma\tau\lambda}\hat{D}_{\mu\nu}\hat{D}_{\rho\sigma}\hat{D}_{\tau\lambda},$$

$$\hat{C}_4 = \frac{1}{2}\hat{D}_{\mu\nu}\hat{D}_{\nu\rho}\hat{D}_{\rho\tau}\hat{D}_{\tau\mu}.$$

According to Ref. 15, the eigenvalues of these operators can be taken as

$$C_2 = \mu_1(\mu_1+4) + \mu_2(\mu_2+2) + \mu_3^2\,,$$

$$C_3 = 48(\mu_1+2)(\mu_2+1)\mu_3\,,$$

$$C_4 = \mu_1^2(\mu_1+4)^2 + 6\mu_1(\mu_1+4) + \mu_2^2(\mu_2+2)^2 + \mu_3^4 - 2\mu_3^2\,,$$

where μ_1, μ_2 and μ_3 are positive integers or half-integers and $\mu_1 \geq \mu_2 \geq \mu_3$.

Direct calculations lead to the representation

$$\begin{aligned} \hat{C}_2 &= -\frac{e^4 m}{2\hbar^2 \hat{H}} + 2\hat{T}^2 - 4\,, \\ \hat{C}_3 &= 48\left(-\frac{me^4}{2\hbar^2\hat{H}}\right)^{1/2} \hat{T}^2\,, \\ \hat{C}_4 &= \hat{C}_2^2 + 6\hat{C}_2 - 4\hat{C}_2\hat{T}^2 - 12\hat{T}^2 + 6\hat{T}^4\,. \end{aligned} \tag{1}$$

From the last equation we can obtain another expression for the eigenvalue C_4

$$C_4 = [C_2 - 2T(T+1)]^2 + 6[C_2 - 2T(T+1)] + 2T^2(T+1)^2$$

and conclude that

$$C_2 - 2T(T+1) = \mu_1(\mu_1+4)\,, \tag{2}$$

$$\mu_2^2(\mu_2+2)^2 + \mu_3^4 - 2\mu_3^2 = 2T^2(T+1)^2\,. \tag{3}$$

The energy levels of YCM can be derived from (1) and (2)

$$\varepsilon_{\mu_1}^T = -\frac{me^4}{2\hbar^2(\mu_1+2)^2}\,. \tag{4}$$

The substitution of the eigenvalues of $\hat{H}$ and $\hat{T}^2$ in the equation for $\hat{C}_3$ gives one more formula for C_3

$$C_3 = 48(\mu_1+2)T(T+1)\,.$$

Now, we have two expressions for C_3 and the comparison leads to the relation

$$T(T+1) = (\mu_2+1)\mu_3\,. \tag{5}$$

Comparing this with (2), we have the equation

$$(\mu_2^2 - \mu_3^2)[(\mu_2+2)^2 - \mu_3^2] = 0\,.$$

Since $\mu_3 \leq \mu_2$, one concludes that $\mu_3 = \mu_2$. Then, from (5) it follows that $\mu_2 = T$ and, therefore, μ_1 in (4) takes only values $\mu_1 = T, T+1, T+2, \ldots,$ — the result known from our previous paper.[10]

Let us complete the present consideration of YCM by three remarks:

(a) It is known[16] that the five-dimensional Coulomb problem is provided with the hidden SO(6) symmetry. From this position, our letter states that the inclusion of the SU(2) gauge field does not break the hidden symmetry of the initial system. The same phenomenon takes place for the Abelian particle–charge system.[17]

(b) The elegant approach presented above to solve the YCM eigenvalue problem has recently been used for the same purpose in connection with the so-called SU(2) Kepler problem.[18]

(c) The eigenfunctions of YCM and the degeneracy of the energy levels has been considered in our recent publication.[10]

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