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FINITE DIMENSIONAL INTEGRABLE SYSTEMS

ON A GENERALIZED *D*-DIMENSIONAL OSCILLATOR: INTERBASIS EXPANSIONS

© 1998 Ye. M. Hakobyan, G. S. Pogosyan, A. N. Sissakian

Joint Institute for Nuclear Research, Dubna, Russia

This article deals with nonrelativistic study of a D-dimensional superintegrable system, which generalizes the ordinary isotropic oscillator system. The coefficients for the expansion between the hyperspherical and Cartesian bases (transition matrix), and vice-versa, are found in terms of the $SU(2)$ Clebsch–Gordan coefficients analytically continued to real values of their arguments. The diagram method, which allows one to construct a transition matrix for arbitrary dimension, is developed.

1. INTRODUCTION

The present article represents a generalization of the work done by one of the authors (G.P.) fifteen years ago together with Ter-Antonyan and Smorodinsky [1]. It could also be considered as a continuation of the work [2] published in these Proceedings. Here we study the superintegrable D-dimensional oscillator system [3] corresponding to the singular potential

$$V = \frac{1}{2}\sum_{i=1}^{D}\left(\Omega^2 x_i^2 + \frac{k_i^2 - \frac{1}{4}}{x_i^2}\right). \qquad (1)$$

The constants $\Omega > 0$ and k_i ($i = 1, 2, \ldots, D$) are chosen to be strictly positive. In the case of $k_i = 1/2$, equation (1) yields the well-known oscillator potential. The Schrödinger equation for the D-dimensional potential (1)

$$\left[-\frac{1}{2}\Delta + \frac{1}{2}\sum_{i=1}^{D}\left(\Omega^2 x_i^2 + \frac{k_i^2 - \frac{1}{4}}{x_i^2}\right)\right]\Psi = E\Psi \qquad (2)$$

is separable in the general ellipsoidal coordinates [4] and, particularly, in the Cartesian and polyspherical coordinates. The potential (1) for $D = 2$ and $D = 3$ belongs to the potentials, which were systematically investigated by Smorodinsky with collaborators in [5, 6] and later considered from a different point of view (quantization in different systems of coordinates, path integral treatment, invariant and noninvariant algebra, quadratic algebra, interbasis expansion) in [2, 7–13].

The plan of this article is as follows. Section 2 is devoted to the Schrödinger equation for the superintegrable D-dimensional oscillator in the Cartesian coordinates. In Section 3, by using the "tree" formalism [14] we construct the hyperspherical basis, which is the solution of the Schrödinger equation in the polyspherical coordinates. Section 4 is the calculation of the interbasis expansion coefficients between the hyperpspherical and Cartesian bases and determines the graphical method of constructing the transition matrix.

2. SOLUTION OF THE SCHRÖDINGER EQUATION

2.1. Cartesian Basis

The Cartesian wave functions, vanishing as $x_i \longrightarrow 0$ and $x_i \longrightarrow \infty$ ($i = 1, 2, \ldots, D$), have the following form [2, 10–12]:

$$\Psi_{\mathbf{n}}(\mathbf{x}) = \prod_{i=1}^{D}\psi_{n_i}(x_i, \pm k_i) = \prod_{i=1}^{D}\sqrt{\frac{\Omega^{1/2} n_i!}{\Gamma(n_i \pm k_i + 1)}}\times$$
$$\times\exp\left(-\frac{\Omega}{2}x_i^2\right)\left(\sqrt{\Omega x_i^2}\right)^{1/2 \pm k_i} L_{n_i}^{\pm k_i}(\Omega x_i^2), \qquad (3)$$

where $\mathbf{x} = (x_1, \ldots, x_D)$, $n = (n_1, \ldots, n_D)$, $n_i \in \mathbb{N}$ and $L_n^{\nu}(x)$ are associated Lagerre polynomials [15]. The wave function (3) is normalized in the domain $[0, \infty)$

$$\int_0^{\infty}\Psi_{\mathbf{n'}}(\mathbf{x})^*\Psi_{\mathbf{n}}(\mathbf{x})d\mathbf{x} = \frac{1}{2^D}\delta_{\mathbf{n'n}}, \qquad (4)$$

and the energy spectrum is

$$E = \Omega\sum_{i=1}^{D}(2n_i \pm k_i + 1) = \Omega\left(2N + D \pm \sum_{i=1}^{D}k_i\right), \qquad (5)$$

where $N = n_1 + n_2 + \ldots + n_D$ is the principal quantum number. Note that the positive sign at k_i has to be taken when $k_i > 1/2$, and both signs, positive and negative, must be taken into account if $0 < k_i \le 1/2$.

2.2. Hyperspherical Basis

Consider the system of coordinates

$$x_i = r\tilde{x}_i, \quad r = \sqrt{x_1^2 + x_2^2 + \ldots + x_D^2}, \qquad (6)$$

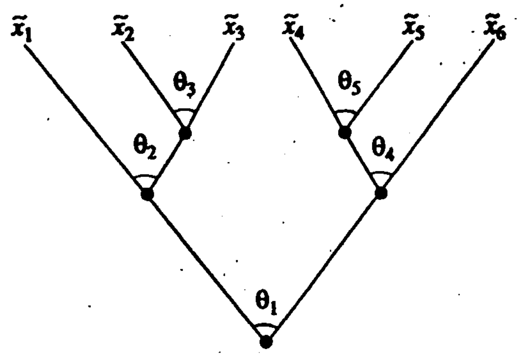


Figure.

where the Cartesian coordinates $\tilde{x}_i$ $(i = 1, 2, \ldots, D)$ denote the point on the $(D-1)$-dimensional unit sphere S_{D-1}: $\sum_{i=1}^{D} \tilde{x}_i^2 = 1$. Looking for the wave function $\Psi(\mathbf{x})$ in the form

$$\Psi(\mathbf{x}) = R(r)Y(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_D) \qquad (7)$$

after partial separation of variables in the Schrödinger equation (2) we come to the differential equation for the radial function $R(r)$

$$\frac{d^2R}{dr^2} + \frac{D-1}{r}\frac{dR}{dr} + \left\{2E - \frac{l(l+D-2)}{r^2} - \Omega^2 r^2\right\}R = 0, \qquad (8)$$

and to the equation describing the guantum motion on the S_{D-1} sphere with the Rosochatius potential [16]

$$\left[-\Delta + \sum_{i=1}^{D}\frac{k_i^2 - \frac{1}{4}}{\tilde{x}_i^2}\right]Y = l(l+D-2)Y, \qquad (9)$$

where l is the hyperspherical separation constant. Here Δ is the Laplacian on the sphere and has the form

$$\Delta = \sum_{1 \le i \le k}^{D} L_{ik}^2, \quad K_{ik} = \tilde{x}_i\partial_{\tilde{x}_k} - \tilde{x}_k\partial_{\tilde{x}_i}.$$

To solve equation (9) via separation of variables in the polyspherical system of coordinates, we follow the graphical method, which was developed by Vilenkin, Kuznetsov and Smorodinsky in paper [14]. According to this method it is useful to associate a polyspherical system of coordinates with definite graph, called "tree". In the D-dimensional Euclidean space with the coordinates $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_D$ any tree has D free points and $D-1$ nodes. To each node we ascribe a spherical angle θ_i $(i = 1, 2, \ldots, D-1)$ and for each line, opened (free) or closed, which goes to the right or left side, we write a function $\sin\theta_i$ or $\cos\theta_i$, respectively. In this case, the coordinate $\tilde{x}_i$ may be represented as a product of all the lines coming toward itself. For example, to the tree on Figure there correspond the following polyspherical coordinates:

$$\tilde{x}_1 = \cos\theta_1\cos\theta_2, \quad \tilde{x}_2 = \cos\theta_1\sin\theta_2\cos\theta_3,$$

$$\tilde{x}_3 = \cos\theta_1\sin\theta_2\sin\theta_3, \quad \tilde{x}_4 = \sin\theta_1\cos\theta_4\cos\theta_5,$$

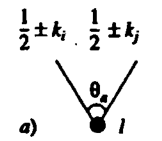
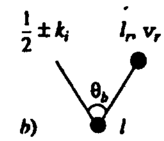
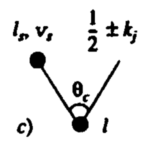
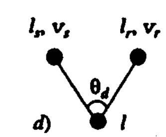
$$\tilde{x}_5 = \sin\theta_1\cos\theta_4\sin\theta_5, \quad \tilde{x}_6 = \sin\theta_1\sin\theta_4.$$

To construct the separated solution

$$Y(\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_D) = \prod_{k=1}^{D-1} f_k(\theta_k) \qquad (10)$$

for equation (9), we follow the paper [14] and introduce four types of vertices or elementary "cells" on a tree, as illustrated in the first line of Table 1.

Table 1

$\frac{1}{2}$				
θ	$0 \le \theta \le 2\pi$	$0 \le \theta \le \pi$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$0 \le \theta \le \frac{\pi}{2}$
$d\Omega(\theta)$	$d\theta$	$(\sin\theta)^{v_r}d\theta$	$(\cos\theta)^{v_s}d\theta$	$(\cos\theta)^{v_s}(\sin\theta)^{v_r}d\theta$
$Y_l(\theta)$	$f^l_{\frac{1}{2}\pm k_i, \frac{1}{2}\pm k_j}(\theta)$	$f^l_{\frac{1}{2}\pm k_i; l_r, v_r}(\theta)$	$f^l_{l_s, v_s; \frac{1}{2}\pm k_j}(\theta)$	$f^l_{l_s, v_s; l_r, v_r}(\theta)$
l	$2q \pm k_i \pm k_j + 1$	$2q \pm l_r \pm k_i + \frac{1}{2}$	$2q \pm l_s \pm k_j + \frac{1}{2}$	$2q + l_s + l_r$
$\prod_{i=1}^{D}(\tilde{x}_i)^{\tilde{n}_i}$	$(\cos\theta)^{\tilde{n}_i}(\sin\theta)^{\tilde{n}_j}$	$(\cos\theta)^{\tilde{n}_i}(\sin\theta)^{N_r}$	$(\cos\theta)^{N_s}(\sin\theta)^{\tilde{n}_j}$	$(\cos\theta)^{N_s}(\sin\theta)^{N_r}$

Consider the general cell *d*) with two closed endpoints on the first line of Table 1. Let l_s, l_r and l be separation constants corresponding to the nodes on the cell *d*) and the parameters v_s and v_r represent numbers of nodes above the origin of the cell to the left and right sides, respectively. Then, the separated equation corresponding to the angle $\theta = \theta_d$ is

$$\left[\frac{1}{\cos^{v_s}\theta\sin^{v_r}\theta}\frac{d}{d\theta}\cos^{v_s}\theta\sin^{v_r}\theta\frac{d}{d\theta} + l(l+v_r+v_s) - \frac{l_s(l_s+v_s-1)}{\cos^2\theta} - \frac{l_r(l_r+v_r-1)}{\sin^2\theta}\right]f(\theta) = 0. \quad (11)$$

Equation (11) is of a Pöschl–Teller type, and the corresponding solution [17], orthonormalized in the region $\theta \in [0, \pi/2]$, has the following form:

$$f(\theta) \equiv f^l_{l_s v_s; l_r v_r}(\theta) = N_q^{(\alpha_r,\alpha_s)}(\cos\theta)^{l_s}(\sin\theta)^{l_r}P_q^{(\alpha_r,\alpha_s)}(\cos 2\theta), \quad (12)$$

where $q = 0, 1, 2, \ldots$ is a spherical quantum number and

$$\alpha_r = l_r + \frac{v_r - 1}{2}, \quad \alpha_s = l_s + \frac{v_s - 1}{2}, \quad 2q = l - l_s - l_r. \quad (13)$$

The normalization constant is

$$N_q^{(\alpha,\beta)} = \sqrt{\frac{2(2q+\alpha+\beta+1)\Gamma(q+1)\Gamma(q+\alpha+\beta+1)}{\Gamma(q+\alpha+1)\Gamma(q+\beta+1)}}. \quad (14)$$

The solutions to the separated equations for other cells *a*), *b*) and *c*) on Table 1 may be found from equation (12) by adjusting to every free endpoint $\tilde{x}_i$, $v_i = 0$ and the "momentum" $l_i = 1/2 \pm k_i$. These functions are written in the fourth line of Table 1 and have the following form:

$$f^l_{\pm k_i;\pm k_j}(\theta) = \frac{1}{2}f^l_{\frac{1}{2}\pm k_i, 0; \frac{1}{2}\pm k_j, 0}(\theta) = \frac{N_q^{(\pm k_i,\pm k_j)}}{2}(\cos\theta)^{\frac{1}{2}\pm k_i}(\sin\theta)^{\frac{1}{2}\pm k_j}P_q^{(\pm k_j,\pm k_i)}(\cos 2\theta), \quad (15)$$

$$f^l_{\pm k_i; l_r}(\theta) = \frac{1}{\sqrt{2}}f^l_{\frac{1}{2}\pm k_i, 0; l_r, v_r}(\theta) = \frac{N_q^{(\pm k_i,\alpha_r)}}{\sqrt{2}}(\cos\theta)^{\frac{1}{2}\pm k_i}(\sin\theta)^{l_r}P_q^{(\alpha_r,\pm k_i)}(\cos 2\theta), \quad (16)$$

$$f^l_{l_s;\pm k_j}(\theta) = \frac{1}{\sqrt{2}}f^l_{l_s v_s; \frac{1}{2}\pm k_j, 0}(\theta) = \frac{N_q^{(\alpha_s,\pm k_j)}}{\sqrt{2}}(\cos\theta)^{l_s}(\sin\theta)^{\frac{1}{2}\pm k_j}P_q^{(\pm k_j,\alpha_s)}(\cos 2\theta). \quad (17)$$

Let us go to the radial equation (8), which has the orthonormalized solution

$$R(r) \equiv R_{n_r l}(r) = \sqrt{\frac{2\Omega^{l+D/2}n_r!}{\Gamma(n_r+l+D/2)}} \times \exp\left(-\frac{\Omega}{2}r^2\right)r^l L_{n_r}^{l+\frac{D-2}{2}}(\Omega r^2), \quad (18)$$

where $n_r \in \mathbb{N}$ is a radial quantum number and hypermomentum $l = \sum_{i=1}^{D-1}(q_i \pm k_i + 1/2) + (1/2 \pm k_D)$. The energy spectrum is given by equation (5), where the principal quantum number now is $N = n_r + q_1 + q_2 + \ldots + q_{D-1}$. The total hyperspherical wave function (7) is given by formulae (18) and (10)

$$\Psi_{n_r \mathbf{l}}(r, \boldsymbol{\theta}) = R_{n_r l}(r)Y_{\mathbf{l}}(\boldsymbol{\theta}), \quad (19)$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_{D-1})$, $\mathbf{l} = (l_1, l_2, \ldots, l_{D-1})$, and the connection between the spherical quantum number q and the separation constants l_i are represented by the fourth line in Table 1.

3. CONNECTING CARTESIAN AND HYPERSPHERICAL BASES

For the fixed value of energy we can write the expansion of the Cartesian basis $\Psi_{\mathbf{n}}(\mathbf{x})$ in terms of the hyperspherical basis $\Psi_{n_r\mathbf{l}}(r, \boldsymbol{\theta})$ in the form

$$\Psi_{\mathbf{n}}(\mathbf{x}) = \sum_{\mathbf{q}} W_{\mathbf{n}}^{N,\mathbf{q}}(\pm k_1, \ldots, \pm k_N)\Psi_{n_r\mathbf{l}}(r, \boldsymbol{\theta}). \quad (20)$$

Here, the sum is taken over $(D-1)$ quantum numbers $\mathbf{q} = (q_1, \ldots, q_{D-1})$ and determined by the condition $N = n_1 + \ldots + n_D = n_r + q_1 + \ldots + q_{D-1}$. By multiplying both sides of the expansion (20) by the factor r^{-D} and using the asymptotic formula for the associated Laguerre polynomials $L_n^{\alpha}(x)$ for large x

$$L_n(x) \sim \frac{(-x)^n}{n!}, \quad (21)$$

equation (20) yields an equation dependent only on variables $\boldsymbol{\theta}$. Then, by using the orthogonality property of $Y_{\mathbf{l}}(\boldsymbol{\theta})$ in the region $\theta_i \in [0, \pi/2]$, we obtain the integral representation for the transition matrix (20)

$$W_{\mathbf{n}}^{N,\mathbf{q}}(\pm k_1, \ldots, \pm k_N) = M\int d\Omega(\boldsymbol{\theta})Y_{\mathbf{l}}(\boldsymbol{\theta})\prod_{i=1}^{N}(\tilde{x}_i)^{\tilde{n}_i}, \quad (22)$$

where

$$M = \frac{(-1)^{N-n_r}}{\sqrt{2}}\sqrt{\frac{2^{2D}n_r!\Gamma(n_r+l+D/2)}{\prod_{i=1}^{D}[n_i!\Gamma(n_i \pm k_i + 1)]}} \quad (23)$$

and $\tilde{n}_i = (2n_i \pm k_i + 1/2)$.

To calculate the transition matrix (22), we must know exactly the form of the tree, corresponding to the polyspherical coordinates, and the contribution from each of the cells (see Table 1) to the functions $d\Omega(\boldsymbol{\theta})$, $Y_l(\boldsymbol{\theta})$ and $\prod_{i=1}^{D}(\tilde{x}_i)^{\tilde{n}_i}$. Therefore, the matrix (22) includes only four types of integrals

$$F^l_{\tilde{n}_i, \tilde{n}_j}(\pm k_i, \pm k_j) = \int_0^{\pi/2} (\cos\theta)^{\tilde{n}_i}(\sin\theta)^{\tilde{n}_j} f^l_{\pm k_i; \pm k_j}(\theta)d\theta, \quad (24)$$

$$F^l_{\tilde{n}_i, N_r}(\pm k_i; l_r, v_r) = \int_0^{\pi/2} (\cos\theta)^{\tilde{n}_i}(\sin\theta)^{N_r} f^l_{\pm k_i; l_r}(\theta)d\theta, \quad (25)$$

$$F^l_{N_r, \tilde{n}_j}(L_s, v_s; \pm k_j) = \int_0^{\pi/2} (\cos\theta)^{N_s}(\sin\theta)^{\tilde{n}_i} f^l_{l_s; \pm k_j}(\theta)d\theta, \quad (26)$$

$$F^l_{N_s, N_r}(l_s, v_s; l_r, v_r) = \int_0^{\pi/2} (\cos\theta)^{N_s}(\sin\theta)^{N_r} f^l_{l_s; l_r}(\theta)d\theta, \quad (27)$$

where N_s and N_r are the sums over all $\tilde{n}_i$ above the cell on the left and right sides, respectively.

Let us now calculate the general integral (27). Using the Rodrigues formula for the Jacobi polynomials [15], we obtain

$$F^l_{N_s; N_r}(l_s, v_s; l_r, v_r) = \frac{(-1)^{\frac{l-l_s-l_r}{2}}}{2^{\frac{N_s+N_r+l_s+l_r+v_s+v_r+2}{2}}} \times \sqrt{\frac{\left(l+\frac{v_s+v_r}{2}\right)}{\Gamma\left(\frac{l-l_s-l_r}{2}+1\right)\Gamma\left(\frac{l+l_s-l_r}{2}+\frac{v_r+1}{2}\right)}} \times$$
$$\times \sqrt{\frac{\Gamma\left(\frac{l+l_s+l_r}{2}+\frac{v_s+v_r}{2}\right)}{\Gamma\left(\frac{l-l_s+l_r}{2}+\frac{v_s+1}{2}\right)}} \int_{-1}^{1} (1+x)^{N_s-l_s}(1-x)^{N_r-l_r} \times \quad (28)$$
$$\times \frac{d^{\frac{l-l_s-l_r}{2}}}{dx^{\frac{l-l_s-l_r}{2}}}\left[(1+x)^{\frac{l+l_s-l_r}{2}+\frac{v_s-1}{2}}(1-x)^{\frac{l-l_s+l_r}{2}+\frac{v_r-1}{2}}\right]dx.$$

Comparing (28) with the integral representation of the Clebsh–Gordan coefficients $C^{c,\gamma}_{a,\alpha;b,\beta}$ for the group $SU(2)$ [18], we obtain

$$F^l_{N_s; N_r}(l_s, v_s; l_r, v_r) = \frac{(-1)^{q+\frac{l-l_s-N_r}{2}}}{\sqrt{2}} K^{l_s, v_s; l_r, v_r}_{N_s, N_r} C^{c,\gamma}_{a,\alpha;b,\beta} \quad (29)$$

with

$$K^{l_s, v_s; l_r, v_r}_{N_s, N_r} = \sqrt{\frac{\Gamma\left(\frac{N_s-l_s}{2}+1\right)\Gamma\left(\frac{N_s+l_s}{2}+\frac{v_s+1}{2}\right)}{\Gamma\left(\frac{N_r+N_s+l}{2}+\frac{v_r+v_s}{2}+1\right)}} \times \sqrt{\frac{\Gamma\left(\frac{N_r-l_r}{2}+1\right)\Gamma\left(\frac{N_r+l_r}{2}+\frac{v_r+1}{2}\right)}{\Gamma\left(\frac{N_s+N_r-l}{2}+1\right)}} \quad (30)$$

and

$$4a = l_s - l_r + N_s + N_r + v_s - 1,$$
$$4b = l_r - l_s + N_s + N_r + v_r - 1,$$
$$4\alpha = l_r + l_s + N_s - N_r + v_s - 1,$$
$$4\beta = l_r + l_s + N_r - N_s + v_r - 1,$$
$$2c = l + \frac{v_s-1}{2} + \frac{v_r-1}{2},$$
$$2\gamma = l_s + l_r + \frac{v_s-1}{2} + \frac{v_r-1}{2}.$$

By realizing that the integrals (24)–(26) may be expressed through the integral (27), we obtain

$$F^l_{\tilde{n}_i, \tilde{n}_j}(k_i, k_j) = \frac{1}{2}F^l_{\tilde{n}_i, \tilde{n}_j}\left(\frac{1}{2}\pm k_i, 0; \frac{1}{2}\pm k_j, 0\right), \quad (31)$$

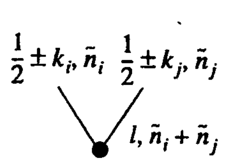
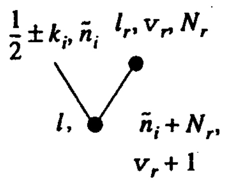
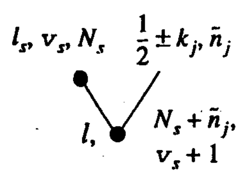
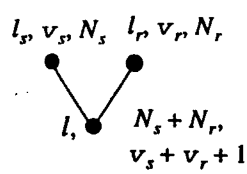
$$F^l_{N_r, \tilde{n}_j}(l_s, v_s, k_j) = \frac{1}{\sqrt{2}}F^l_{N_r, \tilde{n}_j}\left(l_s, v_s; \frac{1}{2}\pm k_j, 0\right), \quad (32)$$

$$F^l_{\tilde{n}_i, N_r}(k_i, l_r, v_r) = \frac{1}{\sqrt{2}}F^l_{\tilde{n}_i, N_r}\left(\frac{1}{2}\pm k_i, 0; l_r, v_r\right). \quad (33)$$

Thus, the contributions from the four types of cells in the matrix (22) are calculated and given by formulae (29) and (31)–(33).

Let us now construct the transition matrix (22). At first, we need to prove that the matrix (22) can be expressed in terms of the product of the Clebsch–Gordan coefficients only. Indeed, let "s", "r" and "d" be the quantum numbers corresponding to the cells "s" and

Table 2

			
$(-1)^{\frac{l-\frac{1}{2}\mp k_i-\tilde{n}_j}{2}} C^{c\gamma}_{a\alpha,b\beta}$	$(-1)^{\frac{l-\frac{1}{2}\mp k_i-N_r}{2}} C^{c\gamma}_{a\alpha,b\beta}$	$(-1)^{\frac{l-l_s-\tilde{n}_j}{2}} C^{c\gamma}_{a\alpha,b\beta}$	$(-1)^{\frac{l-l_s-N_r}{2}} C^{c\gamma}_{a\alpha,b\beta}$
$v_s=v_r=0;\ l_s=\frac{1}{2}\pm k_i;\ l_r=\frac{1}{2}\pm k_j$	$v_s=0;\ l_s=\frac{1}{2}\pm k_i$	$v_r=0;\ l_r=\frac{1}{2}\pm k_i$	

"r" corresponding to the left and right top in this cell and "d" being the origin of the cell. Therefore, using the relation $v_d = v_s + v_r + 1$, we can write

$$K^{l_r v_s; l_r v_r}_{N_r, N_r} \equiv K(s, r; d) = \frac{f(s)f(r)}{f(d)}, \quad (34)$$

where

$$f(i) = \sqrt{\Gamma\left(\frac{N_i-l_i}{2}+1\right)\Gamma\left(\frac{N_i+l_i}{2}+\frac{v_i+1}{2}\right)}. \quad (35)$$

The full contribution of constants (30) to the transition matrix (22) is equal to the product of the constants (34) upon all the cells

$$\prod_{i=1}^{D-1} K_i(s, r; d) = \frac{1}{f(d)}\prod_{i=1}^{D} f(i) = \sqrt{\frac{\prod_{j=1}^{D}\Gamma(n_j+1)\Gamma(n_j\pm k_j+1)}{\Gamma(n_r+l+D/2)\Gamma(n_r+1)}}, \quad (36)$$

where $n_s + N_r = 2n_r + 1$. Then, the contribution of all the constants (30) with the coefficient (23) is eliminated and for any tree the transition matrix (22) has the following form:

$$W^{N,q}_{\mathbf{n}}(\pm k_1, \ldots, \pm k_D) = \prod_{i=1}^{D-1}(-1)^{c_i-a_i-\beta_i} C^{c_i,\gamma_i}_{a_i,\alpha_i;b_i,\beta_i} \quad (37)$$

with $a_i, b_i, \alpha_i, \beta_i, c_i, \gamma_i$ given by formula (29), and the multiplication is taken upon all the cells. The quantum numbers in (37) for $k_i \neq 1/2$ are not integers or half or odd integers and, therefore, the coefficients in the matrix (37) may be considered as an analytic continuation of the $SU(2)$ Clebsch–Cordan coefficients for the real values of their arguments. For $D = 2$ and $D = 3$ we obtain the result from paper [2].

Determine now the graphical methods of constructing the matrix (37), which we call a transition "tree" and which is identical to the corresponding hyperspherical "tree". Let the "momentum" $1/2 \pm k_i$ and Cartesian quantum number $\tilde{n}_i = 2n_i \pm k_i + 1$ correspond to any free endpoint of the tree and the separation constants l_i, the numbers v_i and $N_i = \sum \tilde{n}_i$ correspond to the nodes. Then, after drawing the transition "tree" and multiplying the contributions from all the cells in the tree according to Table 2, we come to the final result in the form (37).

Because of the orthogonality properties for the $SU(2)$ Clebsch–Gordan coefficients, the inverse expansion could be written as

$$\Psi_{n_r,l}(r, \theta) = \sum_{n_1+n_2+\ldots+n_D=N} W^{*N,q}_{\mathbf{n}}(\pm k_1, \ldots, k_N)\Psi_{\mathbf{n}}(\mathbf{x}).$$

4. CONCLUSION

One of the main results of this paper is the construction of the hyperspherical wave function which is the solution of the Schrödinger equation for the motion on the $(D-1)$-dimensional sphere for the Rosochatius potential [16] and which generalizes the classical hyperspherical function for $k_i \neq 1/2$ [14].

We have also calculated the transition matrix between the hyperspherical and Cartesian bases and shown that the Clebsch–Gordan coefficients entering into this matrix are the analytic continuation of the $SU(2)$ Clebsch–Gordan coefficients for real values of their arguments. In addition, we propose the diagram method, the "transition tree", which allows one to construct a transition matrix for an arbitrary tree.

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