ELSEVIER

Nuclear Physics B 518 (1998) 455–472

NUCLEAR PHYSICS B

Chiral and parity anomalies at finite temperature and density

A.N. Sissakian [a], O.Yu. Shevchenko [b,1], S.B. Solganik [b]

[a] *Bogolubov Theoretical Laboratory, Joint Institute for Nuclear Research, Dubna, Moscow region 141980, Russia*

[b] *Laboratory of Nuclear Problems, Joint Institute for Nuclear Research, Dubna, Moscow region 141980, Russia*

Received 1 August 1997; accepted 5 February 1998

Abstract

Two closely related topological phenomena are studied at finite density and temperature. These are the chiral anomaly and the Chern–Simons term. Using different methods it is shown that $\mu^2 = m^2$ is the crucial point for Chern–Simons at zero temperature. So when $\mu^2 < m^2$ the μ influence is absent and we obtain the usual Chern–Simons term. On the other hand, when $\mu^2 > m^2$ the Chern–Simons term vanishes because of the non-zero density of the background fermions. The chiral anomaly does not depend on density and temperature. The connection between parity anomalous Chern–Simons and the chiral anomaly is generalized at finite density. These results hold in any dimension in abelian and in non-abelian cases. © 1998 Published by Elsevier Science B.V.

PACS: 11.15.-q; 05.30.Fk; 11.30.Er; 11.10.Wx

Keywords: Chern–Simons; Chiral anomaly; Finite density; Finite temperature

1. Introduction

Topological objects play a great role in modern physics. In particular, here we are interested in Chern–Pontriagin and Chern–Simons (CS) secondary characteristic classes. This corresponds to chiral anomaly in even dimensions and to CS (parity anomaly) in odd dimensions. Both phenomena are very important in quantum physics. Chiral anomalies in quantum field theory have direct applications to the decay of π_0 into two

[1] E-mail: shevch@nusun.jinr.ru.

PII S0550-3213(98)00148-5

photons ($\pi_0 \to \gamma\gamma$) in the understanding and solution of the U(1) problem, etc. On the other hand, there are many effects caused by the CS secondary characteristic class. For example, the appearance of gauge particle mass in quantum field theory, applications to condensed matter physics such as the fractional quantum Hall effect and high T_c superconductivity, the possibility of theory construction free of metric tensors, etc.

It should be emphasized that these two phenomena are closely related. As shown (at zero density) in Refs. [1,2], the trace identities connect the even dimensional anomaly with the odd dimensional CS. The main goal of this paper is to explore these anomalous objects at finite density and temperature.

It was shown [3–5] in a conventional zero density and temperature gauge theory that the CS term is generated in the Euler–Heisenberg effective action by quantum corrections. Since the chemical potential term $\mu\bar{\psi}\gamma^0\psi$ is odd under charge conjugation we can expect that it would contribute to the P and CP non-conserving quantity, the CS term. As we will see, this expectation is completely justified. The zero density approach is usually a good quantum field approximation when the chemical potential is small compared with the characteristic energy scale of physical processes. Nevertheless, for the investigation of topological effects this is not the case. As we will see below, even a small density can lead to effects.

In the excellent paper of Niemi [1] it was emphasized that the charge density at $\mu \neq 0$ becomes a non-topological object, i.e. contains a topological part as well as a non-topological part. The charge density at $\mu \neq 0$ (non-topological, neither parity odd nor parity even object) [2] in QED$_3$ at finite density was calculated and exploited in Ref. [6]. It should be emphasized that in Ref. [6] the charge density (calculated in the constant pure magnetic field) contains in addition to the parity odd part corresponding to the CS term, the parity even part, which cannot be covariantized and does not contribute to the mass of the gauge field. Here we are interested in the influence of finite density and temperature on the covariant parity odd form in the action leading to gauge field mass generation, the CS topological term. Deep insight into this phenomenon at small densities was reported in Refs. [1,2]. The result for the CS term coefficient in QED$_3$ is $[\tanh \frac{1}{2}\beta(m-\mu) + \tanh \frac{1}{2}\beta(m+\mu)]$ (see Ref. [2], formula (10.18)). However, to obtain this result it was heuristically assumed that, at small densities, the index theorem could still be used and only the odd energy part of the spectral density is responsible for the parity non-conserving effect. Because of this it was stressed in Ref. [2] that the result holds only for small μ. However, as we will see below this result holds for any value of the chemical potential. Thus, to obtain reliable results at any value of μ one has to use a transparent procedure free of any restrictions on μ, which would allow us to perform calculations with arbitrary non-abelian background gauge fields.

It was shown at zero chemical potential [1–3] that the CS term in odd dimensions is associated with the chiral anomaly in even dimensions by trace identities. As we will see below it is possible to generalize a trace identity to the non-zero density case.

[2] For brevity, for the parity invariance properties of local objects, we will keep in mind the symmetries of the corresponding action parts.

The trace identity connects the chiral anomaly with the CS term which has a μ and T dependent coefficient. Despite the fact that the chemical potential and temperature give rise to a coefficient before the CS term they do not influence the chiral anomaly. Indeed, the anomaly is a short distance phenomenon which should not be affected by medium μ and T effects or, more quantitatively, as the anomaly has ultraviolet nature, the temperature and chemical potential should not give any ultraviolet effect since the distribution functions decrease exponentially with energy in the ultraviolet limit.

This paper is organized as follows. In Section 2 the independence of the chiral anomaly of temperature and background fermion density is discussed. It is shown in the two-dimensional Schwinger model that the chiral anomaly is not influenced by the chemical potential μ nor the Lagrange multiplier κ in the conservation of chiral charge constraint. In addition, we consider the appearance of the CS term at finite density in even dimensional theories. In Section 3 we obtain the CS term in thee-dimensional theory at finite density and temperature using different methods. In Section 4 we evaluate the CS term coefficient in five-dimensional theory and generalize this result to arbitrary non-abelian odd-dimensional theory. In Section 5 we generalize the trace identity to finite density on the basis of the previous calculations. Section 6 is devoted to the concluding remarks.

2. Chiral anomaly and Chern–Simons term in even dimensions

As is well known, the chemical potential can be introduced into a theory as a Lagrange multiplier following the corresponding conservation laws. In non-relativistic physics this is the conservation of the full number of particles. In relativistic quantum field theory it is conserving charges. The ground state energy can be obtained by use of the variational principle

$$\langle\psi^* \hat{H}\psi\rangle = \min \tag{1}$$

under the charge conservation constraint for the relativistic equilibrium system

$$\langle\psi^* \hat{Q}\psi\rangle = \text{const}, \tag{2}$$

where $\hat{H}$ and $\hat{Q}$ are the hamiltonian and charge operator, respectively. We can use the method of undetermined Lagrange multipliers instead and seek the absolute minimum of the expression

$$\langle\psi^* (\hat{H} - \mu\hat{Q})\psi\rangle, \tag{3}$$

where μ is the Lagrange multiplier. Since $\hat{Q}$ commutes with the hamiltonian $\langle\hat{J}_0\rangle$ is conserved.

On the other hand, we can impose another constraint which implies chiral charge conservation

$$\langle\psi^* \hat{Q}_5\psi\rangle = \text{const}, \tag{4}$$

i.e. in the Lagrange approach we have

$$\langle \psi^* (\hat{H} - \kappa \hat{Q}_5) \psi \rangle = \min, \tag{5}$$

where κ arises as the Lagrange multiplier at the $\langle \hat{J}_0^5 \rangle$ = const constraint. Thus, μ corresponds to the non-vanishing fermion density (number of particles minus number of antiparticles) in the background. κ is responsible for conserving asymmetry in the number of left- and right-handed background fermions.

It should be emphasized that the formal addition of a chemical potential to the theory appears like a simple gauge transformation with gauge function μt. However, it not only shifts the time component of the vector potential but also gives the corresponding prescription for handling Green's function poles. The correct introduction of the chemical potential redefines the ground state (Fermi energy), which leads to a new spinor propagator with the correct ϵ prescription for poles. Therefore, for the free spinor propagator we have (see, for example, Refs. [7,8])

$$G(p;\mu) = \frac{\tilde{\not p} + m}{(\tilde{p}_0 + i\epsilon \operatorname{sgn} p_0)^2 - \boldsymbol{p}^2 - m^2}, \tag{6}$$

where $\tilde{p} = (p_0 + \mu, \boldsymbol{p})$. Thus, when $\mu = 0$ one obtains the usual ϵ prescription immediately because of the positivity of $p_0 \operatorname{sgn} p_0$. In the Euclidean metric one has

$$G(p;\mu) = \frac{\tilde{\not p} + m}{\tilde{p}_0^{\,2} + \boldsymbol{p}^2 + m^2}, \tag{7}$$

where $\tilde{p} = (p_0 + i\mu, \boldsymbol{p})$. In the presence of a background Yang–Mills field we consequently obtain for the Green function operator (in Minkowski space)

$$\hat{G} = (\gamma\tilde{\pi} - m)\frac{1}{(\gamma\tilde{\pi})^2 - m^2 + i\epsilon(p_0 + \mu)\operatorname{sgn}(p_0)}, \tag{8}$$

where $\tilde{\pi}_\nu = \pi_\nu + \mu\delta_{\nu 0}$, $\pi_\nu = p_\nu - gA_\nu(x)$.

We will now consider the chiral anomaly. It was shown in Ref. [9] that the chiral anomaly does not depend on μ and T. In Ref. [9] direct calculations in four-dimensional gauge theory were performed by imaginary and real time formalisms using the Fujikawa method and perturbation theory. These calculations are rather cumbersome. To clearly understand the μ-independent nature of the anomaly (the T-independence will be discussed later) we will consider here the simplest case, two-dimensional QED, and re-derive the result of Ref. [9] by use of the Schwinger non-perturbative method [10]. One can write

$$J^\mu = -ig \operatorname{tr}\left[\gamma^\mu G(x,x') \exp\left(-ig\int_{x'}^{x} d\xi^\mu A_\mu(\xi)\right)\right]_{x'\to x}, \tag{9}$$

where $G(x,x')$ is the propagator satisfying the equation

$$\gamma^\mu(\partial_\mu^x - igA_\mu(x))G(x,x') = \delta(x - x'). \tag{10}$$

Following Schwinger we use the ansatz

$$G(x,x') = G^0(x,x')\exp[ig(\phi(x) - \phi(x'))], \tag{11}$$

where $G^0(x,x')$ is the free propagator

$$\gamma^\mu \partial^x_\mu G^0(x,x') = \delta(x - x').$$

Thus, for ϕ we can write $\gamma^\mu\partial_\mu\phi = \gamma^\mu A_\mu$. From (6) we have

$$\begin{aligned}
G^0(x,x') &= \int \frac{d^2p}{(2\pi)^2} e^{ip(x-x')} \frac{\not p}{p^2 + i\varepsilon(p_0+\mu)\operatorname{sgn} p_0} \\
&= -i\not\partial \Bigg[\int \frac{d^2p}{(2\pi)^2} e^{ip(x-x')} \frac{1}{p^2 + i\varepsilon} - 2\int_{-\infty}^{+\infty} \frac{dp_1}{2\pi} \\
&\quad \times \int_{-\infty}^{+\infty} \frac{dp_0}{2\pi} \theta(-\tilde{p}_0 \operatorname{sgn} p_0) e^{ip(x-x')} \mathrm{m} \frac{1}{p^2+i\varepsilon} \Bigg].
\end{aligned} \tag{12}$$

So, beside the usual zero density part, a μ-dependent part appears. Further, we have to remove the regularization in the current by use of the symmetrical limit $x \to x'$. After some simple calculations it is clearly seen that all μ-dependent terms disappear after removing the limit. Thus, the contribution to the current arises from the μ-independent part only. Therefore

$$\begin{aligned}
J^\mu &= i\frac{g^2}{2\pi}\left(\delta^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right) A_\nu, \\
J_5^\mu &= i\frac{g^2}{2\pi}\left(\varepsilon^{\mu\nu} - \varepsilon^{\mu\alpha}\frac{\partial^\alpha\partial^\nu}{\partial^2}\right) A_\nu,
\end{aligned} \tag{13}$$

and we obtain the usual anomaly in the chiral current

$$\begin{aligned}
&\partial_\mu J^\mu = 0, \\
&\partial_\mu J_5^\mu = i\frac{g^2}{2\pi}\varepsilon^{\mu\nu}\partial_\mu A_\nu = i\frac{g^2}{4\pi}{}^*F.
\end{aligned} \tag{14}$$

Let us now consider the influence of κ on the chiral anomaly. Since, as we have seen above, κ is directly associated with the chiral charge it would be natural to expect some effect of κ on the chiral anomaly. However, a rather amazing situation occurs. The demand of chiral charge conservation (instead of the usual charge conservation) on the quantum level does not influence the chiral anomaly. Actually, in two dimensions the introduction of the Lagrange multiplier κ in the chiral charge conservation gives the term $\kappa\bar\psi\gamma^5\gamma^0\psi = \kappa\bar\psi\gamma^1\psi$ in the Lagrangian. Therefore, κ has the same effect as μ, i.e. κ does not influence the chiral anomaly (this can also be seen from direct calculations similar to those presented above for the case of μ). This can be explained due to the ultraviolet nature of the chiral anomaly, while κ (μ) does not introduce new divergences into the theory.

From the above calculations the principal difference between the chiral anomaly and CS is clearly seen. The ultraviolet regulator, P exponent, gives rise to the anomaly, but (as we will see below) does not influence CS. Thus, it is natural that the anomaly does not depend on μ, κ and T because it has ultraviolet regularization origin, while neither density nor temperature influence the ultraviolet behavior of the theory. General and clear proof of axial anomaly temperature independence will be presented in Section 5 on the basis of trace identities.

We now consider CS in even dimensional theory. From the definition one has

$$\frac{\partial I_{\rm eff}}{\partial \kappa} = \int d^D x \langle J_5^0 \rangle. \tag{15}$$

Since the axial anomaly does not depend on κ, the effective action contains the term proportional to the anomalous Q_5 charge with κ as a coefficient. The same is true for the chiral theory. There, the effective action contains the term proportional to the anomalous Q charge with μ as coefficient; see, for example, Refs. [11–13]. Therefore, we have

$$\Delta I_{\rm eff} = -\kappa \int dx_0\, W[A] \tag{16}$$

in conventional gauge theory and

$$\Delta I_{\rm eff}^{\rm chiral} = -\mu \int dx_0\, W[A] \tag{17}$$

in the chiral theory. Here $W[A]$ is the CS term. Thus we obtain the CS with the Lagrange multiplier as a coefficient.

It is well known that at non-zero temperature in the $\beta \to 0$ limit the dimensional reduction effect occurs. So the extra t-dependence of the CS term in (16) disappears and CS can be treated as a mass term in three-dimensional theory with coefficient $i\kappa/T$ (the same as for chiral theory with μ; see Ref. [11]). For the anomalous parts of the effective action we have

$$\begin{aligned} \Delta I_{\rm eff} &= -i\kappa\beta W[A], \\ \Delta I_{\rm eff}^{\rm chiral} &= -i\mu\beta W[A] \end{aligned} \tag{18}$$

in conventional and chiral gauge theories respectively. The only problem arising in treating CS as a mass term is that the coefficient is imaginary; see discussions on this theme in Refs. [11,13]. One can see that the results (16)–(18) hold in arbitrary even dimensions. Let us stress that we do not need any complicated calculations to obtain (16)–(18). The only thing we need is knowledge of the independence of the chiral anomaly on μ, κ and β.

3. CS in three-dimensional theory

3.1. Constant magnetic field

Let us first consider a (2+1)-dimensional abelian theory. Here we will use a constant magnetic background. We will evaluate the fermion density by performing a direct summation over Landau levels. As a starting point, we will use the formula for fermion number at finite density and temperature [1]

$$N = -\frac{1}{2}\sum_n \mathrm{sgn}\left(\frac{1}{2}\beta\lambda_n\right) + \sum_n \left[\frac{\theta(\lambda_n)}{\exp(-\beta(\mu-\lambda_n))+1} - \frac{\theta(-\lambda_n)}{\exp(-\beta(\lambda_n-\mu))+1}\right] = \frac{1}{2}\sum_n \tanh\frac{1}{2}\beta(\mu-\lambda_n) \overset{\beta\to\infty}{\longrightarrow} \frac{1}{2}\sum_n \mathrm{sgn}(\mu-\lambda_n). \tag{19}$$

Landau levels in a constant magnetic field have the form [14]

$$\lambda_0 = -m\,\mathrm{sgn}(eB), \qquad \lambda_n = \pm\sqrt{2n|eB|+m^2}, \tag{20}$$

where $n = 1, 2, \ldots$ It is also necessary to take into account in (19) the degeneracy of the Landau levels. Namely, the number of degenerate states for each Landau level is $|eB|/2\pi$ per unit area. Even now we can see that only zero-modes (because of $\mathrm{sgn}(eB)$) can contribute to the parity odd quantity. So, for zero temperature, by using the identity

$$\mathrm{sgn}(a-b) + \mathrm{sgn}(a+b) = 2\,\mathrm{sgn}(a)\theta(|a|-|b|)$$

one obtains for the zero-modes

$$\frac{|eB|}{4\pi}\mathrm{sgn}(\mu + m\,\mathrm{sgn}(eB)) = \frac{|eB|}{4\pi}\mathrm{sgn}(\mu)\theta(|\mu|-|m|) + \frac{|eB|}{4\pi}\mathrm{sgn}(eB)\mathrm{sgn}(m)\theta(|m|-|\mu|), \tag{21}$$

and for the non-zero-modes

$$\frac{1}{2}\frac{|eB|}{2\pi}\sum_{n=1}^{\infty}\mathrm{sgn}(\mu-\sqrt{2n|eB|+m^2}) + \mathrm{sgn}(\mu+\sqrt{2n|eB|+m^2}) = \frac{|eB|}{2\pi}\mathrm{sgn}(\mu)\sum_{n=1}^{\infty}\theta(|\mu|-\sqrt{2n|eB|+m^2}). \tag{22}$$

Combining contributions of all modes we obtain for the fermion density

$$\rho = \frac{|eB|}{2\pi}\mathrm{sgn}(\mu)\sum_{n=1}^{\infty}\theta(|\mu|-\sqrt{2n|eB|+m^2}) + \frac{1}{2}\frac{|eB|}{2\pi}\mathrm{sgn}(\mu)\theta(|\mu|-|m|)$$

$$+\frac{1}{2}\frac{eB}{2\pi}\text{sgn}(m)\theta(|m|-|\mu|)$$
$$=\frac{|eB|}{2\pi}\text{sgn}(\mu)\left(\text{Int}\left[\frac{\mu^2-m^2}{2|eB|}\right]+\frac{1}{2}\right)\theta(|\mu|-|m|)$$
$$+\frac{eB}{4\pi}\text{sgn}(m)\theta(|m|-|\mu|). \quad (23)$$

Here we see that the zero-modes contribute to the parity odd as well as to the parity even part, while non-zero-modes contribute to the parity even part only (note that under parity transformation $B\to -B$). Thus, the fermion density contains a parity odd part leading to the CS term in action after covariantization, as for the parity even part. It is straightforward to generalize the calculations to the finite temperature case. Substituting zero-modes into (19) one obtains

$$N_0=\frac{|eB|}{2\pi}\frac{1}{2}\tanh\left[\frac{1}{2}\beta\left(\mu+m\ \text{sgn}(eB)\right)\right]$$
$$=\frac{|eB|}{4\pi}\left[\frac{\sinh(\beta\mu)}{\cosh(\beta\mu)+\cosh(\beta m)}+\text{sgn}(eB)\frac{\sinh(\beta m)}{\cosh(\beta\mu)+\cosh(\beta m)}\right], \quad (24)$$

so, extracting the parity odd part, one obtains for CS at finite temperature and density

$$N_{\text{CS}}=\frac{eB}{4\pi}\frac{\sinh(\beta m)}{\cosh(\beta\mu)+\cosh(\beta m)}=\frac{eB}{4\pi}\tanh(\beta m)\frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}. \quad (25)$$

Therefore, the result coincides with the result of Niemi [2] for the CS term coefficient obtained for small μ, $[\tanh\frac{1}{2}\beta(m-\mu)+\tanh\frac{1}{2}\beta(m+\mu)]$. The limit to zero temperature is obvious. The disadvantage of this method is that it works only for abelian and constant field cases.

The result at zero temperature can be obtained by use of the Schwinger proper-time method. Consider $(2+1)$-dimensional theory in the abelian case and choose the background field in the form

$$A^\mu=\tfrac{1}{2}x_\nu F^{\nu\mu}, \qquad F^{\nu\mu}=\text{const}.$$

To obtain the CS term in this case, it is necessary to consider the background current

$$\langle J^\mu\rangle=\frac{\delta S_{\text{eff}}}{\delta A_\mu}$$

rather than the effective action itself. This is because the CS term formally vanishes for such a choice of A^μ but its variation with respect to A^μ produces a non-vanishing current. So, consider

$$\langle J^\mu\rangle=-ig\,\text{tr}[\gamma^\mu G(x,x')]_{x\to x'}, \quad (26)$$

where

$$G(x,x')=\exp\left(-ig\int_{x'}^{x}d\zeta_\mu\,A^\mu(\zeta)\right)\langle x|\hat{G}|x'\rangle. \quad (27)$$

Let us rewrite the Green function (8) in a more appropriate form

$$\hat{G} = (\gamma\tilde{\pi} - m)\left[\frac{\theta((p_0+\mu)\mathrm{sgn}(p_0))}{(\gamma\tilde{\pi})^2 - m^2 + i\epsilon} + \frac{\theta(-(p_0+\mu)\mathrm{sgn}(p_0))}{(\gamma\tilde{\pi})^2 - m^2 - i\epsilon}\right]. \tag{28}$$

We now use the well-known integral representation of denominators

$$\frac{1}{\alpha \pm i0} = \mp i \int_0^\infty ds\, \mathrm{e}^{\pm i\alpha s},$$

which corresponds to introducing the "proper-time" s into the calculation of the Euler–Heisenberg Lagrangian by the Schwinger method [15]. We obtain

$$\hat{G} = (\gamma\tilde{\pi} - m)\Bigg[-i\int_0^\infty ds \exp(is[(\gamma\tilde{\pi})^2 - m^2 + i\epsilon])\theta((p_0+\mu)\mathrm{sgn}(p_0))$$
$$+i\int_0^\infty ds \exp(-is[(\gamma\tilde{\pi})^2 - m^2 - i\epsilon])\theta(-(p_0+\mu)\mathrm{sgn}(p_0))\Bigg]. \tag{29}$$

For simplicity, we restrict ourselves to the magnetic field case, where $A_0 = 0$, $[\tilde{\pi}_0, \tilde{\pi}_\mu] = 0$. We can then easily factorize the time dependent part of the Green function. By using the obvious relation

$$(\gamma\tilde{\pi})^2 = (p_0+\mu)^2 - \boldsymbol{\pi}^2 + \tfrac{1}{2}g\sigma_{\mu\nu}F^{\mu\nu} \tag{30}$$

one obtains

$$G(x,x')|_{x\to x'} = -i\int \frac{dp_0}{2\pi}\frac{d^2p}{(2\pi)^2}(\gamma\tilde{\pi} - m)$$
$$\times \int_0^\infty ds[\mathrm{e}^{is(\tilde{p}_0^2-m^2)}\mathrm{e}^{-is\boldsymbol{\pi}^2}\mathrm{e}^{isg\sigma F/2} - \theta(-(p_0+\mu)\mathrm{sgn}(p_0))$$
$$\times(\mathrm{e}^{is(\tilde{p}_0^2-m^2)}\mathrm{e}^{-is\boldsymbol{\pi}^2}\mathrm{e}^{isg\sigma F/2} + \mathrm{e}^{-is(\tilde{p}_0^2-m^2)}\mathrm{e}^{is\boldsymbol{\pi}^2}\mathrm{e}^{-isg\sigma F/2})]. \tag{31}$$

Here the first term corresponds to the usual μ-independent case and there are two additional μ-dependent terms. In the calculation of the current the following trace arises:

$$\mathrm{tr}[\gamma^\mu(\gamma\tilde{\pi} - m)\mathrm{e}^{isg\sigma F/2}] = 2\pi^\nu g^{\nu\mu}\cos(g|{}^*F|s) + 2\frac{\pi^\nu F^{\nu\mu}}{|{}^*F|}$$
$$\times\sin(g|{}^*F|s) - 2im\frac{{}^*F^\mu}{|{}^*F|}\sin(g|{}^*F|s),$$

where ${}^*F^\mu = \varepsilon^{\mu\alpha\beta}F_{\alpha\beta}/2$ and $|{}^*F| = \sqrt{B^2 - E^2}$. Since we are interested in the calculation of the parity odd part (CS term) it is sufficient to consider only terms proportional to the dual strength tensor ${}^*F^\mu$. On the other hand, the term $2\pi^\nu g^{\nu\mu}\cos(g|{}^*F|s)$ at $\nu = 0$

(see expression for the trace; we consider here that there is only a magnetic field) also gives a non-zero contribution to the current J^0 [6]

$$J^0_{\text{even}} = g\frac{|gB|}{2\pi}\left(\text{Int}\left[\frac{\mu^2-m^2}{2|gB|}\right]+\frac{1}{2}\right)\theta(|\mu|-|m|). \tag{32}$$

This part of the current is parity invariant because under parity $B \to -B$. It is clear that this parity even object does not contribute to the parity anomaly nor to the mass of the gauge field. Moreover, this term has the magnetic field in the denominator of the argument of the cumbersome function – the integer part. So, the parity even term appears to be "non-covariantizable", i.e. it cannot be converted to the covariant form in the effective action. Since we are exploring the parity anomalous topological CS term, we will not consider this parity even term. Therefore, only the term proportional to the dual strength tensor ${}^*F^\mu$ gives rise to CS. The relevant part of the current after spatial momentum integration reads

$$J^\mu_{\text{CS}} = \frac{g^2}{4\pi^2}m^*F^\mu \int_{-\infty}^{+\infty} dp_0 \int_0^\infty ds[\mathrm{e}^{is(\tilde{p}_0^2-m^2)} - \theta(-\tilde{p}_0\,\text{sgn}(p_0))$$
$$\times(\mathrm{e}^{is(\tilde{p}_0^2-m^2)} + \mathrm{e}^{-is(\tilde{p}_0^2-m^2)m}t))]. \tag{33}$$

Thus, we obtain in addition to the usual CS part [4], the μ-dependent part also. It is easy to calculate using the formula

$$\int_0^\infty ds\,\mathrm{e}^{is(x^2-m^2)} = \pi\left(\delta(x^2-m^2)+\frac{i}{\pi}\mathcal{P}\frac{1}{x^2-m^2}\right),$$

and we eventually obtain

$$J^\mu_{\text{CS}} = \frac{m}{|m|}\frac{g^2}{4\pi}{}^*F^\mu[1-\theta(-(m+\mu)\text{sgn}(m))-\theta(-(m-\mu)\text{sgn}(m))]$$
$$= \frac{m}{|m|}\theta(m^2-\mu^2)\frac{g^2}{4\pi}{}^*F^\mu. \tag{34}$$

Let us now discuss the non-abelian case. Then $A^\mu = T_a A^\mu_a$ and

$$\langle J^\mu_a\rangle = -ig\,\text{tr}[\gamma^\mu T_a G(x,x')]_{x\to x'}.$$

It is well known [4,16] that there exist only two types of constant background fields. The first is the "abelian" type (it is easy to see that the self-interaction $f^{abc}A^\mu_b A^\mu_c$ disappears under this choice of background field)

$$A^\mu_a = \eta_a \tfrac{1}{2}x_\nu F^{\nu\mu}, \tag{35}$$

where η_a is an arbitrary constant vector in color space, $F^{\nu\mu}$ = const. The second is the pure "non-abelian" type

$$A^\mu = \text{const}. \tag{36}$$

Here the derivative terms (abelian part) vanish from the strength tensor and it contains only the self-interaction part $F_a^{\mu\nu} = gf^{abc}A_b^{\mu}A_c^{\mu}$. It is clear that to obtain the abelian part of the CS term we should consider the background field (35), whereas for the non-abelian (not containing a derivative, cubic in A) part we have to use (36).

Calculations in the "abelian" case reduce to the previous analysis, except for the trivial adding of the color indices to the formula (34)

$$J_a^{\mu} = \frac{m}{|m|}\theta(m^2 - \mu^2)\frac{g^2}{4\pi}{}^*F_a^{\mu}. \tag{37}$$

In the case of (36) all calculations are similar. The only difference is that the origin of the term $\sigma_{\mu\nu}F^{\mu\nu}$ in (30) is not the linearity A in x (as in the abelian case) but the pure non-abelian A^{μ} = const. Here the term $\sigma_{\mu\nu}F^{\mu\nu}$ in (30) becomes quadratic in A and we have

$$J_a^{\mu} = \frac{m}{|m|}\theta(m^2 - \mu^2)\frac{g^3}{4\pi}\varepsilon^{\mu\alpha\beta}\mathrm{tr}[T_a A^{\alpha} A^{\beta}]. \tag{38}$$

Combining formulas (37) and (38) and integrating over field A_a^{μ} we eventually obtain

$$S_{\rm eff}^{\rm CS} = \frac{m}{|m|}\theta(m^2 - \mu^2)\pi W[A], \tag{39}$$

where $W[A]$ is the CS term

$$W[A] = \frac{g^2}{8\pi^2}\int d^3x\,\varepsilon^{\mu\nu\alpha}\,\mathrm{tr}\left(F_{\mu\nu}A_{\alpha} - \frac{2}{3}gA_{\mu}A_{\nu}A_{\alpha}\right).$$

In conclusion note that it may seem that the covariant notation used throughout this section is rather artificial. However, the covariant notation is useful here because it helps us to extract the Levi–Civita tensor corresponding to the parity anomalous CS term.

3.2. Arbitrary gauge field background

One can see that the procedures we have used above to calculate CS are non-covariant. Indeed, both of them use a constant magnetic background. Here we will use a procedure completely covariant free of any restriction on the gauge field, which allows us to perform calculations immediately in the non-abelian case. We will employ the perturbative expansion. The zero-temperature case within this procedure has been explored in Ref. [17].

Let us first consider non-abelian three-dimensional gauge theory. The only graphs whose P-odd parts contribute to the parity anomalous CS term are shown in Fig. 1.

Therefore, the part of the effective action containing the CS term is

$$I_{\rm eff}^{\rm CS} = \frac{1}{2}\int\limits_x A_{\mu}(x)\int\limits_p \mathrm{e}^{-ixp}A_{\nu}(p)\Pi^{\mu\nu}(p)$$
$$+\frac{1}{3}\int\limits_x A_{\mu}(x)\int\limits_{p,r} \mathrm{e}^{-ix(p+r)}A_{\nu}(p)A_{\alpha}(r)\Pi^{\mu\nu\alpha}(p,r), \tag{40}$$

Fig. 1. Graphs whose P-odd parts contribute to the CS term in non-abelian 3D gauge theory.

where the polarization operator and vertices have the standard form

$$\Pi^{\mu\nu}(p) = g^2 \int_k \mathrm{tr}[\gamma^\mu S(p+k;\mu)\gamma^\nu S(k;\mu)],$$

$$\Pi^{\mu\nu\alpha}(p,r) = g^3 \int_k \mathrm{tr}[\gamma^\mu S(p+r+k;\mu)\gamma^\nu S(r+k;\mu)\gamma^\alpha S(k;\mu)]. \tag{41}$$

$S(k;\mu)$ is the Euclidean fermion propagator at finite density and temperature (7) and the following notation is used: $\int_x = i\int_0^\beta dx_0 \int d\boldsymbol{x}$ and $\int_k = (i/\beta)\sum_{n=-\infty}^{\infty}\int d\boldsymbol{k}/(2\pi)^2$. First consider the second-order term (Fig. 1a). It is well known that the only object giving the possibility of constructing the P and T odd form in the action is the Levi–Civita tensor.[3] Thus, we will drop all terms not containing the Levi–Civita tensor. The signal for mass generation (CS term) is $\Pi^{\mu\nu}(p^2=0) \neq 0$. So we obtain

$$\Pi^{\mu\nu} = g^2 \int_k (-i2m e^{\mu\nu\alpha} p_\alpha)\frac{1}{(\tilde{k}^2+m^2)^2}. \tag{42}$$

After some simple algebra one obtains

$$\begin{aligned}\Pi^{\mu\nu} &= -i2mg^2 e^{\mu\nu\alpha} p_\alpha \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^2k}{(2\pi)^2}\frac{1}{(\tilde{k}^2+m^2)^2}\\ &= -i2mg^2 e^{\mu\nu\alpha} p_\alpha \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\frac{i}{4\pi}\frac{1}{\omega_n^2+m^2},\end{aligned} \tag{43}$$

where $\omega_n = (2n+1)\pi/\beta + i\mu$. Performing a summation we obtain

$$\Pi^{\mu\nu} = i\frac{g^2}{4\pi} e^{\mu\nu\alpha} p_\alpha \tanh(\beta m)\frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}. \tag{44}$$

It is easily seen that in the $\beta \to \infty$ limit we will obtain a zero-temperature result [17]

$$\Pi^{\mu\nu} = i\frac{m}{|m|}\frac{g^2}{4\pi} e^{\mu\nu\alpha} p_\alpha \theta(m^2-\mu^2). \tag{45}$$

In the same manner, handling the third-order contribution (Fig. 1b) one obtains

[3] In three dimensions it arises as a trace of three γ-matrices (Pauli matrices).

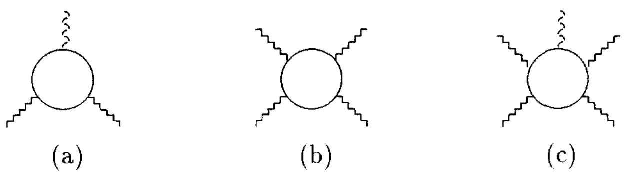

Fig. 2. Graphs whose P-odd parts contribute to the CS term in non-abelian 5D theory.

$$\Pi^{\mu\nu\alpha} = -2g^3 i e^{\mu\nu\alpha} \frac{i}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} \frac{m(\tilde{k}^2+m^2)}{(\tilde{k}^2+m^2)^3}$$

$$= -i2mg^3 e^{\mu\nu\alpha} \frac{i}{\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^2k}{(2\pi)^2} \frac{1}{(\tilde{k}^2+m^2)^2}, \tag{46}$$

and, further, all calculations are identical to second order

$$\Pi^{\mu\nu\alpha} = i\frac{g^3}{4\pi} e^{\mu\nu\alpha} \tanh(\beta m) \frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}. \tag{47}$$

Substituting (44) and (47) into the effective action (40) we eventually obtain

$$I_{\rm eff}^{\rm CS} = \tanh(\beta m) \frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)} \frac{g^2}{8\pi}$$

$$\times \int d^3x\, e^{\mu\nu\alpha} \,\mathrm{tr}\left(A_\mu \partial_\nu A_\alpha - \tfrac{2}{3} g A_\mu A_\nu A_\alpha\right). \tag{48}$$

Thus, we have the CS term with a temperature- and density-dependent coefficient.

4. Chern–Simons in an arbitrary odd dimension

Let us now consider five-dimensional gauge theory. Here the Levi–Civita tensor is five-dimensional $e^{\mu\nu\alpha\beta\gamma}$ and the relevant graphs are shown in Fig. 2.

The part of the effective action containing the CS term reads

$$I_{\rm eff}^{\rm CS} = \tfrac{1}{3} \int\limits_x A_\mu(x) \int\limits_{p,r} \mathrm{e}^{-ix(p+r)} A_\nu(p) A_\alpha(r) \Pi^{\mu\nu\alpha}(p,r)$$

$$+\tfrac{1}{4} \int\limits_x A_\mu(x) \int\limits_{p,r,s} \mathrm{e}^{-ix(p+r+s)} A_\nu(p) A_\alpha(r) A_\beta(s) \Pi^{\mu\nu\alpha\beta}(p,r,s)$$

$$+\tfrac{1}{5} \int\limits_x A_\mu(x) \int\limits_{p,r,s,q} \mathrm{e}^{-ix(p+r+s+q)} A_\nu(p) A_\alpha(r)$$

$$\times A_\beta(s) A_\gamma(s) \Pi^{\mu\nu\alpha\beta\gamma}(p,r,s,q). \tag{49}$$

All calculations are similar to the three-dimensional case. First consider the third-order contribution (Fig. 2a)

$$\Pi^{\mu\nu\alpha}(p,r) = g^3 \int_k \mathrm{tr}[\gamma^\mu S(p+r+k;\mu)\gamma^\nu S(r+k;\mu)\gamma^\alpha S(k;\mu)]. \tag{50}$$

Taking into account that the trace of five γ-matrices in five dimensions is

$$\mathrm{tr}[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\rho] = 4ie^{\mu\nu\alpha\beta\rho},$$

we extract the parity odd part of the vertices

$$\Pi^{\mu\nu\alpha} = g^3 \frac{i}{\beta}\sum_{n=-\infty}^{\infty}\int \frac{d^4k}{(2\pi)^4}(i4me^{\mu\nu\alpha\beta\sigma}p_\beta r_\sigma)\frac{1}{(\tilde{k}^2+m^2)^3}, \tag{51}$$

or, in a more transparent way,

$$\Pi^{\mu\nu\alpha} = i4mg^3 e^{\mu\nu\alpha\beta\sigma}p_\alpha r_\sigma \frac{i}{\beta}\sum_{n=-\infty}^{+\infty}\int \frac{d^4k}{(2\pi)^4}\frac{1}{(\omega_n^2+\boldsymbol{k}^2+m^2)^3}$$

$$= i4mg^3 e^{\mu\nu\alpha\beta\sigma}p_\alpha r_\sigma \frac{i}{\beta}\sum_{n=-\infty}^{+\infty}\frac{-i}{64\pi^2}\frac{1}{\omega_n^2+m^2}. \tag{52}$$

Performing summation one arrives at

$$\Pi^{\mu\nu\alpha} = i\tanh(\beta m)\frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}\frac{g^3}{16\pi^2}e^{\mu\nu\alpha\beta\sigma}p_\alpha r_\sigma. \tag{53}$$

In the same way from graphs Fig. 2b,c one obtains

$$\Pi^{\mu\nu\alpha\beta} = i\tanh(\beta m)\frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}\frac{g^4}{8\pi^2}e^{\mu\nu\alpha\beta\sigma}s_\sigma \tag{54}$$

and

$$\Pi^{\mu\nu\alpha\beta\gamma} = i\tanh(\beta m)\frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}\frac{g^5}{16\pi^2}e^{\mu\nu\alpha\beta\sigma}. \tag{55}$$

Substituting (53)–(55) in the effective action (49) we obtain the final result for CS in five-dimensional theory

$$I_{\mathrm{eff}}^{\mathrm{CS}} = \tanh(\beta m)\frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}\frac{g^3}{48\pi^2}\int_x e^{\mu\nu\alpha\beta\gamma}$$

$$\times\mathrm{tr}\left(A_\mu\partial_\nu A_\alpha\partial_\beta A_\gamma + \tfrac{3}{2}gA_\mu A_\nu A_\alpha\partial_\beta A_\gamma + \tfrac{3}{5}g^2 A_\mu A_\nu A_\alpha A_\beta A_\gamma\right). \tag{56}$$

It is remarkable that all parity odd contributions are finite in three-dimensional and five-dimensional cases. Thus, all values in the effective action are renormalized in a standard way, i.e. the renormalizations are determined by conventional (parity even) parts of the vertices.

From the above direct calculations it is clearly seen that the chemical potential and temperature-dependent coefficient is the same for all parity odd parts of the diagrams and does not depend on the space dimension. Therefore, the influence of finite density and temperature on CS term generation is the same in any odd dimension

$$\begin{aligned} I_{\text{eff}}^{\text{CS}} &= \tanh(\beta m)\frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}\pi W[A] \\ &\overset{\beta\to\infty}{\longrightarrow} \frac{m}{|m|}\theta(m^2-\mu^2)\pi W[A], \end{aligned} \tag{57}$$

where $W[A]$ is the CS secondary characteristic class in any odd dimension. Since only the lowest orders of the perturbative series contribute to the CS term at finite density and temperature (the same situation is well known at zero density), the result obtained using the formally perturbative technique appears to be non-perturbative. Thus, the μ- and T-dependent CS term coefficient reveals the amazing property of universality. Namely, it does not depend on either the dimension of the theory nor the abelian or non-abelian gauge theory studied.

The arbitrariness of μ gives us the possibility of observing the CS coefficient behavior at any mass. It is very interesting that $\mu^2 = m^2$ is the crucial point for CS at zero temperature. Indeed, it is clearly seen from (57) that when $\mu^2 < m^2$ the influence of μ disappears and we obtain the usual CS term $I_{\text{eff}}^{\text{CS}} = \pi W[A]$. On the other hand, when $\mu^2 > m^2$ the situation is completely different. One can see that here the CS term disappears because of the non-zero density of the background fermions. We would like to emphasize the important massless case $m = 0$ considered in many reports; see, for example, Refs. [2,4,18]. Here, even negligible density or temperature, which always occur in any physical process, leads to the disappearance of the parity anomaly. Let us stress again that we have used no restrictions on μ. Thus we not only confirm the result of Ref. [2] for CS in QED_3 at small density, but also expand it to the arbitrary μ, non-abelian case and arbitrary odd dimension.

5. Trace identity

Here, we will consider the trace identity at finite temperature and density. First, using the well-known trace identity at finite temperature [1,2], we will present simple reasons why the chiral anomaly does not depend on temperature in any even dimension. Indeed, at finite temperature and zero density the trace identity still holds and one has [1,2]

$$\begin{aligned} \langle N\rangle_\beta = -\frac{1}{2\beta}\sum_{-\infty}^{+\infty}\frac{m}{m^2+\omega_n^2}\Bigg(\int dx\,(\text{anomaly}) \\ + \int dx\,\partial_i\,\mathrm{tr}\Big\langle x\Big|i\Gamma_i\Gamma^c\frac{1}{H_0+i\sqrt{m^2+\omega_n^2}}\Big\rangle\Bigg). \end{aligned} \tag{58}$$

The second term on the left-hand side is a surface term, which does not contribute to the topological part of the trace identity [1,2]. Thus, for the topological part which we are interested in, the trace identity takes the form

$$\langle N\rangle_\beta^{\text{topological}} = -\frac{1}{2\beta}\sum_{-\infty}^{+\infty}\frac{m}{m^2+\omega_n^2}\left(\int dx\,(\text{anomaly})\right). \tag{59}$$

We know the result for the left-hand side of Eq. (59) in an arbitrary odd dimension. Substituting (57) in

$$\langle N\rangle_\beta^{\text{CS}} = \langle N\rangle_\beta^{\text{topological}} = \frac{\delta I_{\text{eff}}^{\text{CS}}}{g\delta A_0}, \tag{60}$$

and taking into account that

$$\frac{1}{2\beta}\sum_{n=-\infty}^{+\infty}\frac{m}{\omega_n^2+m^2} = \frac{1}{4}\frac{\sinh(\beta m)}{1+\cosh(\beta m)}, \tag{61}$$

one can see that the only possibility of reconciling the left and right sides of Eq. (59) is have the anomaly temperature independent. Thus, we have that the axial anomaly does not depend on temperature in any even-dimensional theory.

Further, we can generalize the trace identity for the topological part to an arbitrary finite density. From (57) and (60) we obtain

$$\langle N\rangle_{\beta,\mu}^{\text{CS}} = -\frac{1}{4}\tanh(\beta m)\frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}\int dx\,(\text{anomaly}), \tag{62}$$

where $\langle N\rangle_{\beta,\mu}^{\text{CS}}$ is the odd part of the fermion number in D-dimensional theory at finite density and temperature, and (anomaly) is the axial anomaly in $(D-1)$-dimensional theory. On the other hand, as we have seen above, the anomaly does not depend on μ in two and four dimensions (and does not depend on T in any even-dimensional theory). Our comprehension of the problem allows us to generalize this to an arbitrary even dimension. Indeed, the anomaly is the result of ultraviolet regularization, while μ (and T) have no effect on the ultraviolet behavior of the theory. Taking into account (62) and that at finite density

$$\frac{1}{2\beta}\sum_{n=-\infty}^{+\infty}\frac{m}{\omega_n^2+m^2} = \frac{1}{4}\tanh(\beta m)\frac{1}{1+\cosh(\beta\mu)/\cosh(\beta m)}, \tag{63}$$

we can identify $\langle N\rangle_{\beta,\mu}^{\text{topological}}$ and $\langle N\rangle_{\beta,\mu}^{\text{CS}}$. Therefore, we obtain the trace identity generalized to finite density for the topological part of the fermion number

$$\langle N\rangle_{\beta,\mu}^{\text{CS}} = \langle N\rangle_{\beta,\mu}^{\text{topological}} = -\frac{1}{2\beta}\sum_{-\infty}^{+\infty}\frac{m}{m^2+\omega_n^2}\left(\int dx\,(\text{anomaly})\right). \tag{64}$$

The physical grounds of formula (64) can be more clearly understood if we recall the calculations performed in Section 3.1 by use of a summation over Landau levels. We have seen that only zero-modes contribute to the P-odd part in contrast to the P-even part which is contributed by all modes. Therefore, the index theorem and trace identities hold only for the parity odd (topological) part of the fermion number at finite density.

Thus, Eq. (64) connects the CS term and the chiral anomaly in arbitrary dimensional theory at finite density and temperature.

6. Conclusions

The influence of finite temperature and density on CS term generation is obtained in any odd-dimensional theory such as abelian and non-abelian cases. It is of interest that $\mu^2 = m^2$ is the crucial point for CS at zero temperature. Indeed, it is clearly seen from (57) that when $\mu^2 < m^2$ the influence of μ disappears and we obtain the usual CS term $I_{\rm eff}^{\rm CS} = \pi W[A]$. On the other hand, when $\mu^2 > m^2$ the CS term disappears because of the non-zero density of the background fermions.

The μ- and T-dependent CS term coefficient exhibits the property of universality. Namely, it does not depend on either the dimension of the theory or whether abelian or non-abelian gauge theory is considered. It must be stressed that, at $m = 0$, even negligible density or temperature, which always occurs in any physical process, leads to the disappearance of the parity anomaly.

Medium effects such as the influence of finite density and temperature on the chiral anomaly have been studied. The simple and general argument that the chiral anomaly is independent of temperature has been presented. It is shown that even if we introduce conservation of chiral charge as a constraint, the chiral anomaly is not affected. By using the fact that the chiral anomaly does not depend on temperature and density we explore the appearance of the CS number in even-dimensional theories under two type of constraint. These are charge conservation with Lagrange multiplier μ (conventional chemical potential) and chiral charge conservation with Lagrange multiplier κ, which corresponds to the conservation of the left(right)-handed fermion asymmetry in the background.

On the other hand, the independence of the chiral anomaly on density and temperature, together with our direct calculations of the CS coefficient, permit the simple generalization of the trace identity to the finite density case. Thus, the connection between the CS term and the chiral anomaly at finite density and temperature is obtained in arbitrary dimensional theory.

References

[1] A.J. Niemi, Nucl. Phys. B 251 (1985) 155.
[2] A.J. Niemi and G.W. Semenoff, Phys. Rep. 135 (1986) 99.
[3] A.J. Niemi and G.W. Semenoff, Phys. Rev. Lett. 51 (1983) 2077.
[4] A.N. Redlich, Phys. Rev. D 29 (1984) 2366.
[5] L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B 234 (1984) 269.
[6] J.D. Lykken, J. Sonnenschein and N. Weiss, Phys. Rev. D 42 (1990) 2161; Int. J. Mod. Phys. A 6 (1991) 1335, 5155;
A.M.J. Schakel, Phys. Rev. D 43 (1991) 1428;
V.Y. Zeitlin, Mod. Phys. Lett. A 8 (1993) 1821.
[7] E.V. Shuryak, Phys. Rep. 61 (1980) 73.
[8] A. Chodos, K. Everding and D.A. Owen, Phys. Rev. D 42 (1990) 2881.
[9] A. Gomez Nicola and R.F. Alvarez-Estrada, Int. J. Mod. Phys. A 9 (1994) 1423.
[10] J. Schwinger, Phys. Rev. 128 (1962) 2425.
[11] A.N. Redlich and L.C.R. Wijewardhana, Phys. Rev. Lett. 54 (1985) 970;
K. Tsokos, Phys. Lett. B 157 (1985) 413.

[12] V.A. Rubakov and A.N. Tavkhelidze, Theor. Mat. Fiz. 65 (1985) 250;
V.A. Rubakov, Prog. Theor. Phys. 75 (1986) 366.
[13] A.R. Rutherford, Phys. Lett. B 182 (1986) 187.
[14] M.H. Johnson and B.A. Lippmann, Phys. Rev. 76 (1949) 828.
[15] J. Schwinger, Phys. Rev. 82 (1951) 664.
[16] L.S. Brown and W.I. Weisberger, Nucl. Phys. B 157 (1979) 285.
[17] A.N. Sissakian, O.Yu. Shevchenko and S.B. Solganik, Phys. Lett. B 403 (1997) 75.
[18] R. Jackiw, *in* Relativity, Groups and Topology II, Proc. Les Houches Summer School XL, 1983, ed. B.S. DeWitt and R. Stora (North-Holland, New York, 1984).