Variational Perturbation Theory in QCD and its Applications

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1 Introduction

In these lectures the nonperturbative approach to quantum field theory, variational perturbation theory (VPT), is briefly reviewed.

Solution of many physical problems is based on approximation of a quantity under consideration by a finite number of terms of a certain series. In quantum field theory this is conventionally an expansion into a perturbative series. This approach combined with the renormalization procedure is now a basic method for computations. As is well-known, perturbative series for many interesting models including realistic models are not convergent. Nevertheless, at small values of the coupling constant these series may be considered as asymptotic series and could provide a useful information. However, even in the theories with a small coupling constant, for instance, in quantum electrodynamics there exist problems which cannot be solved by perturbative methods. Also, a lot of problems of quantum chromodynamics require nonperturbative approaches.

Many approaches have been devoted to the development of nonperturbative methods. Among them is the summation of a perturbative series (see reviews [1] and monograph [2]). The difficulty is that the procedure of summation of asymptotic series is not unique as it contains a functional arbitrariness. A correct formulation of the problem of summation is ensured by further information on the sum of a series [3]. At present information of that kind is known only for the simplest field-theoretical models [4]. Moreover, in many cases of physical interest, the series of perturbative theory is not Borel summable.

There have been approaches that are not directly based on the perturbative series. Many of nonperturbative approaches make use of a variational procedure for finding the leading contribution. However, in this case there is no always an algorithm of calculating corrections to the value

found by a variational procedure, and this makes difficult to answer the question how adequate is the so-called main contribution to the object under investigation and what is the range of applicability of the obtained estimations.

Therefore, useful approaches to the study of the nonperturbative structure of quantum field theory are the methods that combine an expansion of a given quantity in a series that defines the algorithm of calculating the correction with an optimizing procedure. The nonperturbation Gaussian effective potential for a quantum system has been constructed by an approach of that sort in refs. [5, 6, 7, 8]. There exist the various optimizing procedures. In [9, 10], for example, the principle of minimal sensitivity has been applied to the third–order calculation of $R_{e^+e^-}$. Different ways of constructing the variational procedures for scalar models of quantum field theories are discussed in refs. [11, 12, 13]. However, even if the algorithm of calculating corrections, i.e. terms of a certain approximating series, exists, it is not still sufficient. Here of fundamental importance are the properties of convergence of a series. Indeed, unlike the case when even a divergent perturbative series in the weak coupling constant approximates a given object as an asymptotic series, the approximating series in the absence of a small parameter should obey more strict requirements. Reliable information in this case may be obtained only on the basis of convergent series.

We shall consider the method of a series construction with the aid of a variational procedure of the harmonic type. It has been observed empirically in [14] that the results seem to converge if the variational parameter is chosen, in each order, according to the principle of minimal sensitivity. This induced–convergence phenomenon is discussed in detail in ref. [15]. In ref. [16]] the proof of convergence of an optimized δ–expansion is given in the cases of zero and one dimensions. The proof of convergence of variational series in the case of anharmonic procedure is given in ref. [12] . Here, we discuss a method which allows one to systematically determine the low energy structure in quantum chromodynamics. We shall construct the expansion which is based on a new small parameter and apply this method to the nonperturbative renormalization group analysis in quantum chromodynamics. Applications to the definition of the QCD running coupling in the timelike domain and to the semileptonic decay of the τ lepton will be considered. The main results concerning the method of variational theory and some its applications can be found in the papers [5, 10, 11, 12, 13]

and [17, 18, 19, 20, 21, 22] (see also references therein).

2 Toy model

To represent a simple explanation of the basic idea of the method, let us first make a start with very transparent example, simple integral of the form

$$W(g) = \int_{-\infty}^{\infty} dx \, \exp(-S[x]) . \tag{2.1}$$

The expression (2.1) can be considered as the zero-dimensional analog of the ϕ^4-model. The function $S[x]$ plays the role of "the action functional"

$$S[x] = S_0[x] + S_I[x] = x^2 + g\,x^4 . \tag{2.2}$$

In the quantum field theory we can calculate the Gaussian functional integrals. Let us imagine that in this simple case we have to operate with Gaussian integrals as well. Thus, we can try to evaluate the quantity (2.1) by using the Gaussian integrals of the sort

$$\int dx \, P(x) \, exp(-a\,x^2) \tag{2.3}$$

with some polynomial $P(x)$ of x.

The standard method of calculations is the expansion of the expression $\exp(-S[x])$ in the power series of the "coupling constant" g. Indeed, in this case, one uses the Gaussian integrals (2.3) and obtains the standard asymptotic perturbative series

$$W(g) = \sum_{k=0}^{\infty} \omega_k \tag{2.4}$$

with the coefficients

$$\omega_k = \frac{1}{k!} \int_{-\infty}^{\infty} dx \, (-g\,x^4)^k \, \exp(-S_0[x]) . \tag{2.5}$$

Whereas the expansion of the function (2.1) in the series (2.4) with coefficient (2.5) is unique, the inverse procedure of finding the sum of the series (2.4) without using additional information about the function (2.1) is nonunique. For example, the same series (2.4) has also the function $W(g) + \exp(-1/g)$ that has different from $W(g)$ asymptotic behavior at

large values of the coupling constant g. The reason for the incorrectness of the summation procedure is the asymptotic nature of the perturbative expansion (2.4). Therefore, the perturbation series by itself without any additional information about its sum cannot be used to evaluate the function (2.1) for sufficiently large values of the coupling constant. Of course, in this simple case, we know the needed additional information about $W(g)$ and can apply to the series (2.4) some method of summation, for example, the Borel method. But, in the real field theory models, we do not know this information about function that is represented by functional integral and the problem requires special attention.

The VPT approach makes it possible to construct different expansion for the function (2.1) and for quantum field models using the Gaussian quadratures. In this section we will demonstrate how the VPT idea allows one to construct a nonperturbative expansion which is based on a new small expansion parameter.[1]

By using a new split of the action let us rewrite Eq. (2.2) in the form

$$S[x] = S'_0[x] + S'_I[x], \tag{2.6}$$

where we have introduced a new free action $S'_0[x] = \zeta^{-1}x^2$ and an action of interaction $S'_I[x] = gx^4 - (\zeta^{-1} - 1)x^2$. Here ζ is an auxiliary parameter of a variational type. Actually, the original quantity $W(g)$ does not depend on this parameter, therefore, when studying a finite number of the terms of the series it is possible to choose the variational parameter on the basis of some principle of optimization [12, 23].

The VPT series for (2.1) can written down as follows

$$W(g) = \sum_{n=0}^{\infty} W_n, \tag{2.7}$$

where the terms of the VPT expansion have the form

$$\begin{aligned} W_n &= \frac{1}{n!}\int dx\,(-S'_I[x])^N \exp(-S'_0[x]) \qquad (2.8)\\ &= \sum_{k=0}^{n} \frac{1}{(n-k)!\,k!}\int dx\,(-g\,x^4)^k \left[(\zeta^{-1}-1)\,x^2\right]^{n-k} \exp(-S'_0[x])\,. \end{aligned}$$

[1]Here, we use the so-called harmonic variational procedure. Other choices of the trial VPT functionals have been considered in [11, 12, 13, 22].

It is convenient to rewrite the free action as follows

$$S_0'[x] = \zeta^{-1} x^2 \Rightarrow \left[1 + \kappa(\zeta^{-1} - 1) \right] x^2 \tag{2.9}$$

and set $\kappa = 1$ after all calculations. In this case, any power of $[(\zeta^{-1} - 1)x^2]$ in Eq. (2.8) we can obtain by differentiations with respect to κ. The remaining polynomial $(-g x^4)^k$ has the standard perturbative form, therefore, we have a possibility to apply to calculations the standard diagram technique with modified propagator

$$\Delta = \frac{1}{1 + \kappa(\zeta^{-1} - 1)}. \tag{2.10}$$

For $\kappa = 1$, one finds $\Delta = \zeta$.

The terms of the VPT expansion can be written down in the form

$$W_n = \sum_{k=0}^{n} \frac{1}{(n-k)!} \left(-\frac{\partial}{\partial \kappa} \right)^{n-k} \omega_k, \tag{2.11}$$

where the coefficients

$$\omega_k = \frac{1}{k!} \int dx \, (-g x^4)^k \exp\left(-x \Delta^{-1} x \right) \tag{2.12}$$

are given by the standard diagrams of perturbation theory with the propagator (2.10).

Consider a structure of the VPT term (2.11). First of all, note that the differentiation with respect to parameter κ gives the additional factor $(1 - \zeta)$

$$\frac{1}{m!} \left(-\frac{\partial}{\partial \kappa} \right)^m \Delta(\kappa = 1) = (1 - \zeta)^m \Delta(\kappa = 1). \tag{2.13}$$

Secondly, it is easy to see that in this model the number of internal lines (L) in any diagram (here, all diagrams are vacuum diagrams) equal to the double number of vertices (V): $L = 2V$. The internal line corresponds to propagator and leads to the factor ζ, and the vertex gives the factor g. Thus, schematically, one can write down

$$W_n \sim (g\zeta^2)^n + (1-\zeta)(g\zeta^2)^{n-1} + \cdots + (1-\zeta)^{n-1}(g\zeta^2) + (1-\zeta)^n. \tag{2.14}$$

N	0	1	2	3	4	6	8
C	1.14	2.64	3.56	5.46	6.12	871	11.33
D(g = 10) %	2.76	4.83	0.26	0.73	0.038	0.006	0.0012
D(g = 1000) %	5.01	6.52	0.56	1.13	0.089	0.017	0.0033

Table 1: *The relative error* $D(g) = |W_{\text{theor.}}(g)/W_{\text{exper.}}(g) - 1|$.

From (2.14), we can see that if the value of $(1-\zeta)$ will be proportional to $(g\zeta^2)$, the expression W_n will contain the common factor $(1-\zeta)^n$. So, let the parameter ζ obeys to equation

$$1 - \zeta = C g \zeta^2 \tag{2.15}$$

with some positive constant C. We see from (2.15) that for all values of the initial coupling constant g the new expansion parameter $a = 1 - \zeta$ obeys the inequality: $0 \leq a < 1$. The remained parameter C is independent on the value of the coupling constant g and can be found by different ways. For example, if we consider the first non-trivial order $W^{(1)}(g) = W_0(g) + W_1(g)$ and use so-called "fastest apparent convergence", from point of view of which an absolute value of the last calculated term in the expansion should be minimal or vanishes, and require that $W_1 = 0$ we find $C = 3/2$. In this case, we have the approximation $W(g)$ by the expression $W^{(1)}(g)$ (with $W_1 = 0$) with an accuracy better then six percent for all interval of g. In particular, at $g \to \infty$, the relative error of approximation is about 5.1%. Similar results can be obtained if one uses the principle of minimum sensitivity, or a normalization at some "experimental" value $W(g_0) = W_{\text{experim.}}$. If one includes to our consideration the next orders of the VPT expansion we will obtain a best approximation of $W(g)$.

In Table 1 we can see a dependence of the parameter C of the order expansion and the relative error $D(g) = |W_{\text{theor.}}(g)/W_{\text{exper.}}(g) - 1|$. Here, to find the parameter C one makes use the normalization condition $\min|W(g_0) - W_{\text{exper.}}| = \min|W(g_0) - W_{\text{exact}}(g_0)|$ at $g_0 = 1$.

3 Variational perturbation theory in QCD

To explain the basic idea of the method in the QCD case, let us first consider the pure Yang–Mills theory (quarks can be included without prob-

lems). The Lagrangian density has the form

$$L_{YM} = -\frac{1}{4}(F_{\mu\nu})^2 - \frac{1}{2}g\,F_{\mu\nu}[A_\mu \times A_\nu] + \frac{1}{4}g^2[A_\mu \times A_\nu]^2 + L_{g.f.} + L_{F.P.}$$
$$= L_0(A) + g\,L_3(A) + g^2\,L_4(A) \quad , \tag{3.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $L_{g.f.}$ and $L_{F.P.}$ are gauge fixing and Faddeev–Popov terms.

The $L_3(A)$ generates the three–gluon and ghost–gluon–ghost vertices. This interaction is the Yukawa type interaction. The term $L_4(A)$ generates the four–gluon vertices. Let us introduce the $\chi_{\mu\nu}$ field and transform the term $L_4(A)$ to the Yukawa type diagrams

$$\exp\left\{i\frac{g^2}{4}\int dx\,[A_\mu \times A_\nu]^2\right\} = \int D\chi \exp\left\{-\frac{i}{2}\int dx\chi^2_{\mu\nu} + i\frac{g}{\sqrt{2}}\int dx\chi_{\mu\nu}[A_\mu \times A_\nu]\right\}. \tag{3.2}$$

The action functional can be written in the form

$$S = S_0(\chi) + S(A,\chi) + S^{Yuk.}_{YM}(A), \tag{3.3}$$

where

$$S(A,\chi) = \frac{1}{2}\int dx\,dy\; A^a_\mu(x)\left[D^{-1}(x,y|\chi)\right]^{ab}_{\mu\nu} A^b_\nu(y) \tag{3.4}$$

and the gluon propagator $D(x,y|\chi)$ in the χ–field is defined as

$$\left[D^{-1}(x,y|\chi)\right]^{ab}_{\mu\nu} = \left[-\partial^2 g_{\mu\nu}\delta_{ab} + g\sqrt{2}f_{abc}\chi^c_{\mu\nu} + \text{gauge terms}\right]\delta(x-y). \tag{3.5}$$

The Green's functions can be written as

$$G(\cdots) = \langle G_{Yuk.}(\cdots|\chi)\rangle\,, \tag{3.6}$$

where

$$G_{Yuk.}(\cdots|\chi) = \int DA[\cdots]\exp\left\{i\left[S(A,\chi) + S^{Yuk.}_{YM}(A)\right]\right\}, \tag{3.7}$$

and

$$\langle\cdots\rangle = \int D\chi[\cdots]\exp\left[i\,S_0(\chi)\right]. \tag{3.8}$$

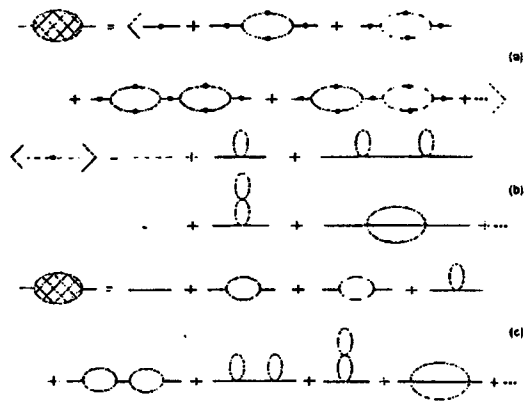

Figure 1: *The perturbation expansion of the full gluon propagator by using the χ-transformation. The gluon line with point corresponds to the function $D(\chi)$.*

The Green's functions $G_{Yuk.}(\cdots|\chi)$ contain only the Yukawa type diagrams appearing inside the brackets $\langle\cdots\rangle$ with the gluon propagator $D(x,y|\chi)$. In Fig. 1 (a), the full gauge propagator is shown. The expansion $D(x,y|\chi)$ in perturbation theory generates the four-gluon graphs [Fig. 1 (b)] that are added to the Yukawa diagrams, and in this case we obtain the standard perturbation expansion [Fig. 1 (c)].

Let us rewrite the Lagrangian in the form

$$\begin{aligned} L(A,\chi) &= L_0(A,\chi) + L_I(A,\chi)\,, \\ L_0(A,\chi) &= \zeta^{-1}\,L(A,\chi) + \xi^{-1}\,L(\chi)\,, \\ L_I(A,\chi) &= \eta\left[g\,L_{YM}^{Yuk.}(A) - (\zeta^{-1}-1)\,L(A,\chi) - (\xi^{-1}-1)\,L(\chi)\right], \end{aligned} \tag{3.9}$$

where ζ and ξ are the parameters of variational type. The original quantity $L(A,\chi)$ does not depend on ζ and ξ. Therefore, the freedom in choosing ζ and ξ can be used to improve the series properties . In the variational perturbation series a new action of interaction is used for constructing the expansion. It is clear that if the parameters $0 < \zeta < 1$ and $0 < \xi < 1$, we "strengthen" the new free Lagrangian and, at the same time, "weaken" the Lagrangian of interaction. After all calculations we put $\eta = 1$. This parameter will be also written in the propagator $D(x,y|\chi)$

in the combination with the coupling constant. The VPT series for the Green's function is given by

$$
\begin{aligned}
G(\cdots) &= \sum_n G_n(\cdots), \\
G_n(\cdots) &= \frac{1}{n!}\eta^n \int D\chi DA[\cdots][iS_I(A,\chi)]^n \exp[iS_0(A,\chi)]. \qquad (3.10)\\
&= (i\eta)^n \sum_{k=0}^{n} \frac{1}{(n-k)!\,k!} \int D\chi DA[\cdots]\left[g S_{YM}^{Yuk}(A)\right]^k \\
&\quad \left[(\zeta^{-1}-1)S(A,\chi) + (\xi^{-1}-1)S(\chi)\right]^{n-k} \exp[iS_0(A,\chi)].
\end{aligned}
$$

We redefine the $L_0(A,\chi)$ for convenience of calculations as follows:

$$
L_0(A,\chi) \Rightarrow L_0'(A,\chi) = [1+\kappa(\zeta^{-1}-1)]L(A,\chi)+[1+\kappa(\xi^{-1}-1)]L(\chi). \quad (3.11)
$$

In this case, any power of $[(\zeta^{-1}-1)S(A,\chi) + (\xi^{-1}-1)S(\chi)]$ in (3.10) can be obtained by the corresponding number of differentiation of the expression $\exp[iS_0'(A,\chi,\kappa)]$ with respect to κ. After all calculations we set $\kappa = 1$.

From Eqs. (3.10) and (3.11) we have

$$
G_n = \eta^n \sum_{k=0}^{n} \frac{1}{(n-k)!}\left(-\frac{\partial}{\partial\kappa}\right)^{n-k} \langle g_k(\kappa)\rangle \quad , \qquad (3.12)
$$

where the functions

$$
\begin{aligned}
g_k(\kappa) &= \frac{i^k}{k!}\int DA\ [\cdots]\left[g\, S_{YM}^{Yuk}(A)\right]^k \\
&\quad \exp\left\{i[1+\kappa(\zeta^{-1}-1)]\int dx\ L(A,\chi)\right\} \qquad (3.13)
\end{aligned}
$$

correspond to the Yukawa diagrams of the Yang-Mills theory with gluon propagator $[1+\kappa(\zeta^{-1}-1)]^{-1}D(x,y|\chi) \to \zeta D(x,y|\chi)$ for $\kappa = 1$. The propagator of χ-field includes the factor $[1+\kappa(\xi^{-1}-1)]^{-1}$ transformed into ξ for $\kappa = 1$.

The operator of differentiation $(-\partial/\partial\kappa)^l/l!$ gives the factor $(1-\zeta)^l$ for the gluon propagator and $(1-\xi)^l$ for the propagator of the χ-field.

It is easy to verify that the Nth order of the VPT series contains the Nth order of a perturbation series with the correction $O(g^{N+1})$, therefore, the VPT expansion does not contradict the perturbative results obtained for the small coupling constant.

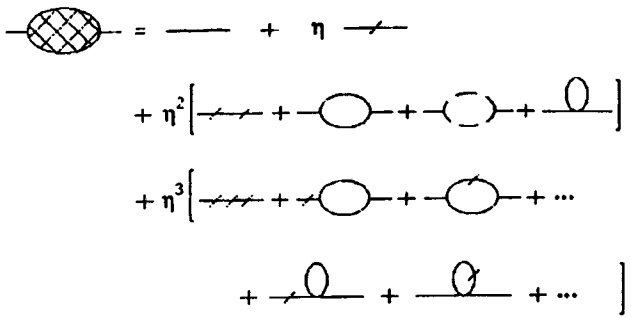

Figure 2: *The diagrams for the VPT expansion of the full gluon propagator.*

The diagrams corresponding to the new expansion of the full gluon propagator are shown in Fig. 2. The gluon line with slash signifies the differentiation over κ and contains the factor $(1-\zeta)$. If this line arises due to the χ–field propagator, the corresponding factor is $(1-\xi)$.

The outline of the VPT expansion structure can be written as

$$1 + \eta(1-\zeta) + \eta^2\left[(1-\zeta)^2 + g^2\zeta^3 + g^2\xi\right] \qquad (3.14)$$
$$+ \eta^3\left[(1-\zeta)^3 + g^2\zeta^3(1-\zeta) + g^2\xi(1-\zeta) + g^2\xi(1-\xi)\right] + \cdots$$

The construction of expansions for the Green's functions corresponding to three-, four-gluon, ghost-gluon-ghost vertices are introduced in similar manner.

If we choose $\xi = \zeta^3$ and $(1-\zeta)^2 = C\lambda\zeta^3$, where C is a positive constant, we obtain that the nth order term of our series contains the factor $(1-\zeta)^n$ and the expansion parameter $a = (1-\zeta) < 1$ for all values of the initial coupling constant. Now, one can perform the renormalization procedure and define the renormalization constants a power series of a [18, 19].

Consider the connection between the perturbative and non-perturbative regimes of the running coupling constant $\alpha_s(Q^2)$. To fix the parameter C we will use non-perturbative information from meson spectroscopy and derive $\alpha_s(Q^2)$ in the perturbative region at large Q^2. In other words, we will find the connection between the universal tension σ in the linear part of

the quark-antiquark static potential $V_{\rm lin}(r) = \sigma r$, which can be determined from meson spectroscopy, and the description of high energy physics. If, as usual, we assume that the quark potential in momentum space can be written as $V(q^2) = -16\pi\alpha_s(q^2)/3q^2$, where $\alpha_s(q^2)$ describes both large and small momentum, and that $\alpha_s(q^2)$ has the singular infrared asymptotics $\alpha_s(q^2) \sim q^{-2}$, we obtain, by taking the three-dimensional Fourier transform, the large-distance linear potential in coordinate space. The corresponding singular infrared behaviour of $\lambda = \alpha_s/(4\pi)$ conforms to the asymptotics of the β-function: $\beta(\lambda) \to -\lambda$ for a large coupling constant.

In the framework of this approach consider the functions $\beta^{(2)}$, $\beta^{(3)}$, $\beta^{(4)}$ and $\beta^{(5)}$ that are obtained if we take into consideration the terms $O(a^2)$, $O(a^3)$, $O(a^4)$ and $O(a^5)$ in the corresponding renormalization constant Z_λ. As has been shown [19], the values of $-\beta^{(k)}(\lambda)/\lambda$ as functions of the coupling constant for parameters $C_2 = 0.977$, $C_3 = 4.1$, $C_4 = 10.4$ and $C_5 = 21.5$ go to 1 at sufficiently large λ. The increase of C_k with the order of the expansion is explained by the necessity to compensate the high order contribution. A similar situation takes place also in zero- and one-dimensional models. The behaviour of the functions $-\beta^{(k)}(\lambda)/\lambda$ gives evidence for the convergence of the results, in accordance with the phenomenon of induced convergence. At large coupling, $-\beta^{(k)}(\lambda)/\lambda \simeq 1$, which corresponds to $\alpha_s(Q^2) \sim Q^{-2}$ at small Q^2.

The value of the coefficient σ in the linear part of the quark-antiquark static potential $V_{\rm lin}(r) = \sigma r$ is $\sigma \simeq 0.15 \div 0.20$ GeV^2. At a small value of Q^2 the corresponding behaviour of $\alpha_s(Q^2)$ is $\alpha_s(Q^2) \simeq 3\sigma/2Q^2$. Here we will use this equation at a certain normalization point Q_0 and the value $\sigma = 0.1768$ GeV^2 which has been obtained in [24]. The renormalization group method gives the following equation for the Q^2-evolution of the expansion parameter a:

$$Q^2 = Q_0^2 \exp[\,\phi(a, N_f) - \phi(a_0, N_f^0)\,] \tag{3.15}$$

with

$$\phi(a, N_f) = \int^{\lambda} \frac{d\lambda}{\beta(\lambda)}\,. \tag{3.16}$$

In an appropriate region of the momentum, the value of $\sigma(Q^2) = 2/3\,Q^2\,\alpha_s(Q^2)$ is almost independent of the choice of Q_0 and lies in the interval $0.15 \div 0.20$ GeV^2. This result agrees with the phenomenology of meson spectroscopy. Thus, we have found all the parameters and can now

consider the behaviour of the effective coupling constant at large Q^2. For example, we find $\alpha_{\rm eff}(m_Z) = 0.126$. It should be stressed that we have obtained this result by evolution of the effective coupling starting from a very low energy scale. Taking into account this fact the value of $\alpha_{\rm eff}(m_Z)$ obtained in such a way seems to be quite reasonable.

4 Renormalon representation and τ decay

In this section we will concentrate on a description of the inclusive decay of the τ lepton taking into account renormalon contributions (for detais, see [25]). Consider the Adler D-function $D(Q^2) = -Q^2 d\Pi/dQ^2$ corresponding to the vector hadronic correlator in the massless case. The two-loop perturbative approximation is given by $D(t,\lambda) = 1 + 4\lambda(\mu^2)$, where $t = Q^2/\mu^2$. Standard renormalization group improvement leads to the substitution $\lambda(\mu^2) \to \overline{\lambda}(t,\lambda)$, which implies a summation of the leading logarithmic contributions. However, due to the ghost pole of the running coupling at $Q^2 = \Lambda^2_{QCD}$ this substitution breaks the analytic properties of the D-function in the complex $q^2 = -Q^2$ plane, namely that the D-function should only have a cut on the positive real q^2 axis. We may correct this feature by noting that the above solution of the renormalization group equation is not unique. The general solution is a function of the running coupling with the asymptotic behaviour $1 + 4\lambda$, for small λ. To maintain the analytic properties [2] of the D-function we can write it as the dispersion integral of $R(s) = (1/\pi)\mathrm{Im}\Pi(s + i\epsilon)$, and use RG improvement on the integrand rather than D itself. This method leads to $D(t,\lambda) = 1 + 4\lambda_{\rm eff}(t,\lambda)$ with $\tau = s/Q^2$. The Borel representation of $\lambda_{\rm eff}(t,\lambda)$ has the form

$$\lambda_{\rm eff}(t,\lambda) = \int_0^\infty db\, \mathrm{e}^{-b/\overline{\lambda}(t,\lambda)}\, B(b)\,, \qquad (4.1)$$

with $B(b) = \Gamma(1 + b\beta_0)\,\Gamma(1 - b\beta_0)$. Here $\beta_0 = 11 - 2/3N_f$ is the first coefficient of the β-function, and N_f is the number of active flavours. Thus, in the Borel plane there are singularities at $b\beta_0 = -1, -2, \ldots$ and $b\beta_0 = 1, 2, \ldots$ corresponding to ultraviolet and infrared (IR) renormalons respectively.

The first IR singularity at $b\beta_0 = 1$ is probably absent since there is no corresponding operator in the operator product expansion. Although

[2] Recently, in [26, 27], it has been shown that requiring the correct analytic properties for the running coupling leads to the non-perturbative power corrections of the form $\exp(-1/(\overline{\lambda}(Q^2)\beta_0))$.

this issue is not currently settled, it seems reasonable to assume that the first IR renormalon occurs at $b = 2/\beta_0$, and we would like to use this property of the operator product expansion as an additional constraint on the choice of solution to the renormalization group equation. This can be simply achieved (by judicious integration by parts), and as result we obtain the following expression for $\lambda_{\rm eff}$:

$$\lambda_{\rm eff}(t,\lambda) = \int_0^\infty d\tau\, \omega(\tau) \frac{\bar{\lambda}(kt,\lambda)}{1 + \bar{\lambda}(kt,\lambda)\beta_0 \ln \tau}\,. \tag{4.2}$$

in which the factor k reflects the renormalization scheme ambiguity and the function $\omega(\tau) = 2\tau/(1+\tau)^3$ describes the distribution of virtuality usually associated with renormalon chains. The function $B(b)$ in the Borel transform of (4.2) has the form

$$B(b) = \Gamma(1 + b\beta_0)\,\Gamma(2 - b\beta_0)\,. \tag{4.3}$$

Thus in this representation for $\lambda_{\rm eff}$ the positions of all ultraviolet singularities remain unchanged, but the first IR renormalon singularity at $b = 1/\beta_0$ is absent.

In order to render Eq. (4.2) integrable we must combine this method with the nonperturbative a-expansion in which from the beginning the running coupling has no ghost pole. Separating the QCD contribution to R_τ-ratio as Δ_τ and writing $R_\tau = R_\tau^0(1 + \Delta_\tau)$, where R_τ^0 is the well-known electroweak factor, we obtain the expression [25]

$$\Delta_\tau = 48 \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(\frac{s}{M_\tau^2}\right)^2 \left(1 - \frac{s}{M_\tau^2}\right) \tilde{\lambda}(ks)\,, \tag{4.4}$$

in which the factor k again parametrizes the renormalization scheme and $\tilde{\lambda} = a^2(1 + 3a)/C$. In what follows we shall use the $\overline{MS}$ scheme, in which $k = \exp(-5/3)$. Note that the renormalon representation obtained for the coupling modifies the polynomial in the integral so that the maximum now occurs near $s = (2/3)M_\tau^2$.

Taking as input the experimental value of $R_\tau^{\rm exp} = 3.56 \pm 0.03$ [28], three active quark flavours and the variational parameter $C = 4.1$, we find $\alpha_s(M_\tau^2) = 0.339 \pm 0.015$ which differs significantly from that obtained ($\alpha_s(M_\tau^2) = 0.40$ in leading order [22]) without the renormalon-inspired representation for the coupling. The method, applying the matching procedure in the physical s-channel and using standard heavy quark masses, leads to $R_Z = 20.90 \pm 0.03$, which agrees well with experimental data.

5 Conclusion

In these lectures we have considered an approach to quantum field theory – the method of variational perturbation theory. The original action functional is rewritten using some variational addition and an expansion in the effective interaction is made. Therefore, in contrast to many nonperturbative approaches, in the VPT the quantity under consideration from the very beginning is written in the form of a series which makes it possible to calculate the needed corrections. The VPT method thereby allows for the possibility of determining the degree to which the principal contribution found variationally using some variational principle adequately reflects the problem in question and determining the region of applicability of the results obtained.

The possibility of performing calculations using this approach is based on the fact that the VPT, like standard perturbation theory, uses only Gaussian functional quadratures. Here, of course, the VPT series possesses a different structure and, in addition, some of the Feynman rules are modified at the level of the propagators and vertices. The form of diagrams themselves does not change, which is very important technically. The diagrams contributing to the Nth order of the VPT expansion are of the same form as those contributing to the Nth order of ordinary perturbation theory.

The variational parameters arising in the VPT method allow the convergence properties of the VPT series to be controlled. In [8, 12] has been shown that in the case of the anharmonic variational procedure for the scalar φ^4 model there is a finite region of parameter values in which the VPT series converges for all positive values of the coupling constant. For the harmonic variational procedure there are indications that VPT series can be also converge on the sense of so-called induced convergence, by fine-tuning the variational parameters from order to order. Note also, that a possibility of constructing Leibnitz series in field models is interesting, because, in this case, the first few terms of the series can be used to obtain two-sided estimates of the sum of the series, and existence of variational parameters makes it possible to narrow these estimates the maximum amount in a given order of VPT (see [23]).

Here, we have mainly concentrated upon the application of the method to quantum chromodynamics (see also [29]), where the VPT idea leads to an expansion with a new small expansion parameter. This parameter

obeys an equation whose solution is always smaller than unity for any value of the coupling constant. Therefore, while remaining within the limits of applicability of this expansion it is possible to deal with considerably lower energies than in the case of perturbation theory. An important feature of this approach is the fact that for sufficiently small value of the running coupling constant $\bar{\alpha}_s$ it reproduces perturbative predictions. Therefore, all the high-energy physics is preserved in the VPT method. In going to lower energies, where standard perturbation theory ceases to be valid, $\bar{\alpha}_s \simeq 1$, the VPT running expansion parameter $\bar{a}$ remains small and we do not find ourselves outside the region of applicability of the method.

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