INÖNÜ - WIGNER CONTRACTIONS FOR INTERBASES EXPANSIONS ON 2 AND 3 - DIMENSIONAL SPHERES

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Abstract

Basis function for representations of rotation groups, corresponding to different chains of subgroups, are realized by different hyperspherical functions. They are related amongst each other by interbases expansion coefficients. The asymptotic limits of these interbases expansions are obtained when the corresponding rotation groups O(3) and O(4) are contracted to the Euclidean groups E(2) and E(3).

1 Introduction

A recent article [1] was devoted to the relation between the separation of variables in the Laplace-Beltrami operator on an n-sphere S_n and the Laplace operator on the Euclidean space E_n. The connection between the two spaces is studied using the theory of Lie algebra and Lie group contractions [2]. The two groups related by the contraction procedure are the isometry groups of the two spaces, i.e. the rotation group O(n+1) and the Euclidean group E(n), respectively.

Let us consider the Laplace-Beltrami equation

$$\Delta_{LB}\Psi(u_1, u_2 ..., u_n) = -\lambda\Psi(u_1, u_2, ..., u_n) \tag{1}$$

on some homogeneous space M. We denote the isometry group of M and its Lie algebra G and L, respectively. Separated eigenfunction of Δ_{LB} can be characterized as the common eigenfunctions of a complete set of commuting operators Y_a, $a = 1, 2, .., n$

$$Y_a\Psi = -\lambda_a\Psi, \qquad [Y_a, Y_b] = 0, \qquad \Psi = f_1(u_1)f_2(u_2)....f_n(u_n). \tag{2}$$

The set of operators $\{Y_a, a = 1, 2, .., n\}$ includes the Laplace-Beltrami operator and consists of second order operators in the enveloping algebra of L. The simplest types of coordinates are obtained if all operators Y_a in the set are Casimir operators of subalgebras of L. Different types of coordinate systems and different types of eigenfunctions Ψ correspond to different types of subalgebra chains $L \supset L_1 \supset L_2$..... The corresponding coordinates are called subgroup type coordinates.

Smorodinsky, Vilenkin and collaborators introduced a graphical method, the "method of trees", for characterizing different subgroup type coordinates. The method is explained in the original articles [3. 4] and subsequent book [5]. The relation between tree diagrams and subgroup (or subalgebra) diagrams is presented in ref. [1].

The contractions considered in [1] were "analytical contractions". The contraction parameter is the radius of the sphere R. It is introduced into the generators of the Lie algebra L, hence also into the operators $\{Y_a\}$ and thus into eigenfunction Ψ and eigenvalues λ_a. The effect of taking the limit $R \to \infty$. when S_n goes to E_n. was studied for coordinates, operators, eigenfunctions and eigenvalues.

In physical application one makes use of concrete basis functions of representations and one needs other objects as well. Thus, in nuclear physics. or any many-body theory more generally, it is often necessary to expand different bases in terms of each other. The different bases are related by unitary transformations. The matrix elements of transformations between different bases for the group O(n+1). i.e. between different types of hyperspherical (or polyspherical) functions. are called "T-coefficients" and have been calculated explicitly [6].

The purpose of this article is to study the contractions of the interbases expansions and thus of the T - coefficients, in the limit $R \to \infty$.

2 Contractions of interbases expansions for O(3)

In the case of the S_2 - sphere only two trees exist (Fig.1). They are topologically equivalent and correspond to equivalent subgroup chains O(3)⊃O(2). However, in the 3-body problem, one chain privileges particles (1,2), the other particles (2,3). The spherical functions $Y_{lm}(\theta_1, \theta_2)$ corresponding to these trees are connected by the transformation

$$Y_{lm_1}\left(\frac{\pi}{2} - \theta'_1, \theta'_2\right) = \sum_{m_2=-l}^{l} D^l_{m_2,m_1}\left(\frac{\pi}{2}, \frac{\pi}{2}, 0\right) Y_{lm_2}(\theta_1, \theta_2) \tag{3}$$

where $D^l_{m_2,m_1}(\alpha, \beta, \gamma) = e^{-im_2\alpha} d^l_{m_2,m_1}(\beta) e^{-im_1\gamma}$ is the Wigner function [7] and the angles in both sides of the expansions are connected by the relations

$$\begin{aligned} u_0 &= R\cos\theta_1 &&= R\cos\theta_1'\cos\theta_2' \\ u_1 &= R\sin\theta_1\cos\theta_2 &&= R\cos\theta_1'\sin\theta_2' \\ u_2 &= R\sin\theta_1\sin\theta_2 &&= R\sin\theta_1' \end{aligned}$$

We use an integral representation for the function $d^l_{m_2,m_1}(\pi/2)$

$$d^l_{m_2,m_1}(\frac{\pi}{2}) = (-1)^{\frac{l-m_1}{2}}\frac{2^l}{\pi}\left\{\frac{(l+m_2)!(l-m_2)!}{(l+m_1)!(l-m_1)!}\right\}^{1/2}$$

$$\int_0^\pi (\sin\alpha)^{l-m_1}(\cos\alpha)^{l+m_1}e^{2im_2\alpha}d\alpha,$$

and the formulas [8]

$$\cos(2n\alpha) = T_n(\cos 2\alpha), \qquad \sin(2n\alpha) = \sin 2\alpha \cdot U_{n-1}(\cos 2\alpha),$$

where $T_l(x)$ and $U_l(x)$ are Tchebyshev polynomials of the first and second kind. After integrating over α, we obtain a representation of the Wigner D functions in terms of the hypergeometrical function ${}_3F_2$ (of argument 1):

$$D^l_{m_2,m_1}(\frac{\pi}{2},\frac{\pi}{2},0) = \frac{(-1)^{\frac{l+m_2-m_1}{2}}}{\sqrt{\pi}J!}\sqrt{(l+m_2)!(l-m_2)!} \tag{4}$$

$$\begin{cases} \left\{\frac{\Gamma(\frac{l+m_1+1}{2})\Gamma(\frac{l-m_1+1}{2})}{\Gamma(\frac{l+m_1+2}{2})\Gamma(\frac{l-m_1+2}{2})}\right\}^{\frac{1}{2}} {}_3F_2\left(\begin{matrix}-m_2, m_2, \frac{l+m_1+1}{2} \\ \frac{1}{2}, l+1\end{matrix}\middle| 1\right) & (l-m_1) - \text{even} \\ \frac{2il}{(l+1)}\left\{\frac{\Gamma(\frac{l+m_1+2}{2})\Gamma(\frac{l-m_1+2}{2})}{\Gamma(\frac{l+m_1+1}{2})\Gamma(\frac{l-m_1+1}{2})}\right\}^{\frac{1}{2}} {}_3F_2\left(\begin{matrix}1-m_2, m_2+1, \frac{l+m_1+2}{2} \\ \frac{3}{2}, l+2\end{matrix}\middle| 1\right) & (l-m_1) - \text{odd} \end{cases}$$

Consider now the contraction limit $R \to \infty$ in the expansion (3). For large R we put

$$l \sim kR, \qquad m_1 \sim k_1 R, \qquad \theta_1 \sim \frac{r}{R}, \qquad \theta_1' \sim \frac{y}{R}, \qquad \theta_2' \sim \frac{x}{R},$$

where $k^2 = k_1^2 + k_2^2$, and have

$$\lim_{R\to\infty}\frac{1}{\sqrt{R}}Y_{lm_2}(\theta_1,\theta_2) = (-1)^{\frac{m_2+|m_2|}{2}}\sqrt{k}J_{|m_2|}(kr)\frac{e^{im_2\theta_2}}{\sqrt{2\pi}},$$

$$\lim_{R\to\infty}(-1)^{-\frac{l-|m_1|}{2}}Y_{lm_1}\left(\frac{\pi}{2}-\theta_1',\theta_2'\right) = \sqrt{\frac{k}{k_2}\frac{e^{ik_1x}}{\pi}}\begin{cases}\cos k_2 y,\ (l-|m_1|) - \text{even}, \\ -i\sin k_2 y,\ (l-|m_1|) - \text{odd},\end{cases}$$

Using known asymptotic formulas for the ${}_3F_2$ functions and Γ - functions in eq. (4) we obtain:

$$\lim_{R\to\infty}(-1)^{-\frac{l-|m_1|}{2}}\sqrt{R}\,D^l_{m_2,m_1}(\frac{\pi}{2},\frac{\pi}{2},0) = (-1)^{\frac{m_2}{2}}\sqrt{\frac{2}{\pi k}}$$

$$\times\left\{\begin{array}{ll}\left(\frac{k^2}{k_2}\right)^{\frac{1}{4}}\ {}_2F_1\left(-m_2,m_2;\ \frac{1}{2};\ \frac{k+k_1}{2k}\right) & (l-m_1)-\text{even},\\ \\ -im_2\ \left(\frac{k_2}{k^2}\right)^{\frac{1}{4}}\ {}_2F_1\left(-m_2+1,m_2+1;\ \frac{3}{2};\ \frac{k+k_1}{2k}\right) & (l-m_1)-\text{odd}.\end{array}\right.$$

$$= (-1)^{\frac{3m_2}{2}}\sqrt{\frac{2}{\pi k_2}}\left\{\begin{array}{ll}\cos m_2\phi, & (l-m_1)-\text{even},\\ \\ i\sin m_2\phi, & (l-m_1)-\text{odd},\end{array}\right. \tag{5}$$

Multiplying the interbases expansion (3) by the factor $(-1)^{-\frac{l-|m_1|}{2}}$ and taking the contraction limit $R\to\infty$ we obtain ($\theta\equiv\theta_2$, $m\equiv m_2$)

$$e^{ik_1x}\left\{\begin{array}{c}\cos k_2y\\ \sin k_2y\end{array}\right\} = \sum_{m=-\infty}^{\infty}(i)^{|m|}\left\{\begin{array}{c}\cos m\varphi\\ -\sin m\varphi\end{array}\right\}J_{|m|}(kr)\,e^{im\theta} \tag{6}$$

or in exponential form

$$e^{ikr\cos(\theta-\varphi)} = \sum_{m=-\infty}^{\infty}(i)^m\,J_m(kr)\,e^{im(\theta-\varphi)} \tag{7}$$

The inverse expansion is

$$J_m(kr)e^{im\theta} = \frac{(-i)^m}{2\pi}\int_0^{2\pi}e^{im\varphi-ikr\cos(\theta-\varphi)}\,d\varphi. \tag{8}$$

For $\theta=0$ the two last formulas are equivalent to well known formulas in the theory of Bessel functions [8], namely expansions of plane waves in terms of cylindrical ones and vice versa.

3 Contractions on the interbases expansions for O(4)

Let us consider now the interbases expansions for the hyperspherical functions on the three-dimensional sphere S_3. There are only three (see Fig.2.) elementary notrivial transformations between trees for S_3.

1. First we consider the expansion between trees on Fig.2(a) [6]

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \sum_{l=|m|}^{J} (i)^{l-|m|}(-1)^{\frac{J-|m|-n}{2}} C^{l,|m|}_{\frac{J}{2},\frac{|m|+n}{2};\frac{J}{2},\frac{|m|-n}{2}} \Psi_{Jlm}(\theta_1',\theta_2',\theta_3), \quad (9)$$

where

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2 = R\cos\theta_1' \\ u_1 &= R\cos\theta_1\sin\theta_2 = R\sin\theta_1'\cos\theta_2' \\ u_2 &= R\sin\theta_1\cos\theta_3 = R\sin\theta_1'\sin\theta_2'\cos\theta_3 \\ u_3 &= R\sin\theta_1\sin\theta_3 = R\sin\theta_1'\sin\theta_2'\sin\theta_3, \end{aligned}$$

$C^{l,\gamma}_{a,\alpha;b,\beta}$ - are the Clebsch-Gordan coefficients for the SU(2) group and the corresponding hyperspherical functions have the form:

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{2J+2}}{2\pi}\sqrt{\frac{\left(\frac{J+|m|+|n|}{2}\right)!\left(\frac{J-|m|-|n|}{2}\right)!}{\left(\frac{J+|m|-|n|}{2}\right)!\left(\frac{J-|m|+|n|}{2}\right)!}}\, e^{in\theta_2}e^{im\theta_3}$$

$$\times(\sin\theta_1)^{|m|}(\cos\theta_1)^{|n|}\, P^{(|m|,|n|)}_{\frac{J-|m|-|n|}{2}}(\cos 2\theta_1), \quad (10)$$

$$\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}$$

$$\times(\sin\theta_1')^l\, P^{(l+\frac{1}{2},l+\frac{1}{2})}_{J-l}(\cos\theta_1')\, Y_{lm}(\theta_2',\theta_3), \quad (11)$$

where $P^{(\alpha,\beta)}_n(x)$ are Jacobi polynomials. In the contraction limit $R\to\infty$ and

$$\theta_1' \to \frac{r}{R} \qquad \theta_1 \to \frac{\rho}{R}, \qquad \theta_2 \to \frac{x_1}{R}, \qquad J \sim kR, \qquad n \sim k_1 R,$$

where $r = \sqrt{x_1^2+\rho^2} = \sqrt{x_1^2+x_2^2+x_3^2}$, $k = \sqrt{k_1^2+p^2} = \sqrt{k_1^2+k_2^2+k_3^2}$, we have [1]

$$\lim_{R\to\infty}\frac{1}{\sqrt{R}}\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \Phi_{kk_1m}(x_1,\rho,\theta_3) = \sqrt{\frac{k}{\pi}}\, J_{|m|}(p\rho)e^{ik_1x_1}\frac{e^{im\theta_3}}{\sqrt{2\pi}}, \quad (12)$$

and

$$\lim_{R\to\infty}\frac{1}{R}\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \Phi_{klm}(r,\theta_2',\theta_3) = \sqrt{\frac{k}{r}}\, J_{l+\frac{1}{2}}(kr)\, Y_{lm}(\theta_2',\theta_3). \quad (13)$$

Taking the Clebsch-Gordan coefficients in the form

$$C^{l,|m|}_{\frac{J}{2},\frac{|m|+n}{2};\frac{J}{2},\frac{|m|-n}{2}} = (-1)^{\frac{J-|m|-n}{2}} \frac{(J)!}{(|m|)!} \sqrt{\frac{(2l+1)(l+|m|)!}{(J-l)!(J+l+1)!(l-|m|)!}}$$

$$\times \sqrt{\frac{(\frac{J+|m|-|n|}{2})!(\frac{J-|m|+|n|}{2})!}{(\frac{J+|m|+|n|}{2})!(\frac{J-|m|-|n|}{2})!}}\ {}_3F_2\left\{\begin{matrix} -\frac{J-n-|m|}{2}, -l, l+1, \\ -J, |m|+1. \end{matrix}\middle| 1\right\}. \tag{14}$$

in the contraction limit $R\to\infty$, we get

$$\lim_{R\to\infty}\sqrt{R}(-1)^{\frac{J-|m|-n}{2}} C^{l,|m|}_{\frac{J}{2},\frac{|m|+|n|}{2};\frac{J}{2},\frac{|m|-|n|}{2}} = W^{l}_{k|m|}(\cos\phi) = \sqrt{\frac{(2l+1)(l+|m|)!}{k(l-|m|)!}}$$

$$\frac{(\sin\phi)^{|m|}}{2^{|m|}|m|!}{}_2F_1\left(-l+|m|, l+|m|+1; |m|+1; \frac{1-\cos\phi}{2}\right) = \sqrt{\frac{2}{k}}\ \mathcal{P}^{|m|}_l(\cos\phi), \tag{15}$$

where

$$\mathcal{P}^{|m|}_l(x) = \sqrt{\frac{(2l+1)(l-|m|)!}{2(l+|m|)!}}\ P^{|m|}_l(x)$$

are the orthonormalized Legendre polynomials and $\cos\phi = p/k$. Thus the interbases expansion (9) transforms to the expansion between the cylindrical and spherical bases for the Helmholtz equation

$$\Phi_{kk_1m}(x_1,\rho,\theta_3) = \sum_{l=|m|}^{\infty} W^{l}_{k|m|}(\cos\phi)\,\Phi_{klm}(r,\theta_2',\theta_3), \tag{16}$$

We use the formula

$$\int_0^{\pi} W^{l}_{k|m|}(\cos\phi)\, W^{*l'}_{k|m|}(\cos\phi)\,\sin\phi d\phi = 2\,\delta_{l,l'} \tag{17}$$

to obtain the inverse expansion

$$\Phi_{klm}(r,\theta_2',\theta_3) = \frac{1}{2}\int_0^{\pi} W^{*l}_{k,|m|}(\cos\phi)\Phi_{kk_1m}(x_1,\rho,\theta_3)\sin\phi\, d\phi. \tag{18}$$

Putting the exact form of the functions (12)-(13) and interbases coefficients (15) into the expansions (16) and (18), we obtain

$$\frac{1}{\sqrt{2\pi}}\, e^{ikr\cos\phi\cos\theta_2'}\, J_{|m|}(p\rho) = \sum_{l=|m|}^{\infty}(i)^{l+m}\,\frac{1}{\sqrt{kr}}\, J_{l+\frac{1}{2}}(kr)\ \mathcal{P}^{|m|}_l(\cos\phi)\,\mathcal{P}^{|m|}_l(\cos\theta_2') \tag{19}$$

$$\frac{1}{\sqrt{kr}} J_{l+\frac{1}{2}}(kr)\mathcal{P}_l^{|m|}(\cos\theta_2') = \frac{(-i)^{l+m}}{\sqrt{2\pi}} \int_0^{\pi} e^{ikr\cos\phi\cos\theta_2'} J_{|m|}(p\rho)\mathcal{P}_l^{|m|}(\cos\phi)\sin\phi\, d\phi$$

The last two expansions coincide with well known formulas in the theory of the Bessel functions [8].

2. The second expansion is:

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \sum_{l=|m|}^{J} T_{Jnm}^{l}\Psi_{Jlm}(\theta_1',\theta_2',\theta_3). \tag{20}$$

where

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2 &&= R\cos\theta_1' \\ u_1 &= R\cos\theta_1\sin\theta_2\cos\theta_3 &&= R\sin\theta_1'\cos\theta_2'\cos\theta_3 \\ u_2 &= R\cos\theta_1\sin\theta_2\sin\theta_3 &&= R\sin\theta_1'\cos\theta_2'\sin\theta_3 \\ u_3 &= R\sin\theta_1 &&= R\sin\theta_1'\sin\theta_2', \end{aligned}$$

[see Fig.2(b)]. The hyperspherical wavefunctions corresponding to these two trees are

$$\Psi_{Jnm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{(2J+1)(J+n+1)!(J-n)!}}{2^{n+1}\Gamma(J+\frac{3}{2})}$$

$$\times(\cos\theta_1)^n P_{J-n}^{(n+\frac{1}{2},n+\frac{1}{2})}(\sin\theta_1)\, Y_{nm}(\theta_2,\theta_3),$$

$$\Psi_{Jlm}(\theta_1',\theta_2',\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}$$

$$\times(\sin\theta_1')^l P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\cos\theta_1')\, Y_{lm}(\frac{\pi}{2}-\theta_2',\theta_3).$$

The interbases coefficients T_{Jnm}^{l} have the following form [6]

$$T_{jnm}^{l} = \left[\frac{1+(-1)^{J-n+l-m}}{2}\right](-1)^{\frac{J-n+l-m}{2}} \frac{2^{l+n-2m}}{|m|!} \frac{\Gamma(\frac{J-n-l+|m|}{2}+1)}{\Gamma(\frac{J+n+l-|m|}{2}+1)}$$

$$\sqrt{\frac{(2l+1)(2n+1)(n+|m|)!(l+|m|)!(J-l)!(J-n)!}{(n-|m|)!(l-|m|)!(J+n+1)!(J+l+1)!}}$$

$${}_4F_3\left\{\begin{matrix} -\frac{n-|m|}{2}, -\frac{n-|m|-1}{2}, -\frac{l-|m|}{2}, -\frac{l-|m|-1}{2} \\ |m|+1, -\frac{J+n+l-|m|}{2}, \frac{J-n-l+|m|}{2}+1 \end{matrix}\middle| 1\right\}.$$

In the contraction limit $R \to \infty$ and

$$\theta_1 \sim \frac{x_3}{R}, \qquad \theta_2 \sim \frac{\rho}{R}, \qquad \theta_1' \sim \frac{r}{R}, \qquad J \sim kR, \quad n \sim pR,$$

where $r = \sqrt{\rho^2 + x_3^2}$ and $k = \sqrt{p^2 + k_3^2}$, we obtain [1]

$$\begin{aligned}\lim_{R\to\infty} \frac{(-1)^{-\frac{J-n}{2}}}{\sqrt{R}} \Psi_{Jnm}(\theta_1,\theta_2,\theta_3) &= \Phi_{kpm}(\rho, x_3, \theta_3) \\ &= \sqrt{k}\, J_m(p\rho)\, \frac{e^{im\theta_3}}{\pi} \begin{cases} \cos k_3x_3, & J-n \text{ - even} \\ i\sin k_3 x_3, & J-n \text{ - odd}\end{cases}\end{aligned}$$

$$\lim_{R\to\infty} \frac{1}{R} \Psi_{Jlm}(\theta_1', \theta_2', \theta_3) = \Phi_{klm}(r, \theta_2', \theta_3) = \sqrt{\frac{k}{r}}\, J_{l+\frac{1}{2}}(kr)\, Y_{lm}(\frac{\pi}{2} - \theta_2', \theta_3).$$

For the contractions of interbases coefficients T^l_{Jnm} we get

$$\begin{aligned}\lim_{R\to\infty} (-1)^{-\frac{J-n}{2}} \sqrt{R}\, T^l_{Jnm} = W^l_{k|m|}(\cos\phi) = \frac{(-1)^{\frac{l-m}{2}}}{|m|!} \sqrt{\frac{(2l+1)(l+|m|)!}{2k(l-|m|)!}} \\ (\cot\phi)^{|m|+\frac{1}{2}} (\sin\phi)^l\, {}_2F_1\left(-\frac{l-|m|}{2}, -\frac{l-|m|-1}{2}; |m|+1; -\cot^2\phi\right) \\ = (-1)^{\frac{l+|m|}{2}} \sqrt{\frac{2}{k}} (\cot\phi)^{\frac{1}{2}}\, \mathcal{P}_l^{|m|}(\sin\phi), \qquad (21)\end{aligned}$$

where $\cos\phi = p/k$. The interbasis expansion in (20) transforms to the expansion between the cylindrical and spherical bases for the Helmholtz equation

$$\frac{J_m(p\rho)}{\sqrt{\pi}} \begin{Bmatrix} \cos k_3 x_3 \\ i \sin k_3 x_3 \end{Bmatrix} = \sum_l \frac{(-1)^{\frac{l-|m|}{2}}}{\sqrt{kr}} J_{l+\frac{1}{2}}(kr)(\cot\phi)^{\frac{1}{2}} \mathcal{P}_l^{|m|}(\sin\phi) \mathcal{P}_l^{|m|}(\sin\theta_2') \quad (22)$$

where the top line on the left hand side corresponds to a summation over $l = |m|, |m|+2, |m|+4, \ldots$ and the bottom one to a summation over $l = |m|+1, |m|+3, \ldots$ on the right hand side. The E(3) expansion (22) is related to the expansion (19) by the substitution $k_1 = k\cos\phi \to k_3$, $x_1 = r\cos\theta_2' \to x_3$, $\phi \to \pi/2 - \phi$ and $\theta_2' \to \pi/2 - \theta_2'$.

3. The third expansion is:

$$\Psi_{Jlm}(\theta_1,\theta_2,\theta_3) = \sum_{n=-(J-|m|)}^{(J-|m|)} (-i)^{l-|m|}\, C^{l,|m|}_{\frac{J}{2},\frac{|m|-n}{2};\frac{J}{2},\frac{|m|+n}{2}}\, \Psi_{Jmn}(\theta_1',\theta_2',\theta_3) \qquad (23)$$

where n has the same parity as $J-|m|$ and

$$\begin{aligned} u_0 &= R\cos\theta_1\cos\theta_2\cos\theta_3 = R\cos\theta_1'\cos\theta_3' \\ u_1 &= R\cos\theta_1\cos\theta_2\sin\theta_3 = R\cos\theta_1'\sin\theta_3' \\ u_2 &= R\cos\theta_1\sin\theta_2 = R\sin\theta_1'\cos\theta_2' \\ u_3 &= R\sin\theta_1 = R\sin\theta_1'\sin\theta_2', \end{aligned}$$

[see Fig.2(c)]. The corresponding hyperspherical function is:

$$\Psi_{Jlm}(\theta_1,\theta_2,\theta_3) = \frac{\sqrt{(2J+1)(J+l+1)!(J-l)!}}{2^{l+1}\Gamma(J+\frac{3}{2})}$$

$$(\cos\theta_1)^l\, P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\sin\theta_1)\, Y_{lm}\left(\frac{\pi}{2}-\theta_2,\theta_3\right),$$

and the wave function $\Psi_{Jmn}(\theta_1',\theta_2',\theta_3')$ is given by (10) (with n replaced by m).

The contraction in this case (see Fig.2(c) and eq. (24) below) will involve 3 quantum numbers J, l and m. Eq. (14)expressinf Clebsch-Gordan coefficients in terms of the ${}_3F_2$ function is not convenient for taking this limit. Instead, we use the following integral representation [6]

$$C^{l,|m|}_{\frac{J}{2},\frac{|m|-n}{2};\frac{J}{2},\frac{|m|+n}{2}} = (-i)^{l-|m|}(-1)^{\frac{J-|m|+n}{2}} \left\{ \frac{(l+|m|)!(\frac{J-|m|-n}{2})!(\frac{J-|m|+n}{2})!}{(\frac{J+|m|-n}{2})!(\frac{J+|m|+n}{2})!(l-|m|)!} \right\}^{1/2}$$

$$\frac{\sqrt{(2l+1)(J-l)!(J+l+1)!}}{2^{l+|m|+2}\Gamma(J+3/2)}\, \frac{1}{\sqrt{\pi}} \int_0^{2\pi} (\sin\phi)^{l-|m|}\, P_{J-l}^{(l+\frac{1}{2},l+\frac{1}{2})}(\cos\phi) e^{in\phi} d\phi$$

and the formulas [8]

$$P_n^{(\alpha,\alpha)}(\cos\phi) = \frac{\Gamma(\alpha+n+1)}{\Gamma(\alpha+1)n!}$$

$$\times \begin{cases} {}_2F_1(-\frac{n}{2},\frac{n+1}{2}+\alpha;\alpha+1;\sin^2\phi) & n-\text{even}, \\ \cos\phi\, {}_2F_1(-\frac{n-1}{2},\frac{n}{2}+\alpha+1;\alpha+1;\sin^2\phi) & n-\text{odd}. \end{cases}$$

After integrating over ϕ, we obtain a representation of the Clebsch-Gordan coefficients in terms of the hypergeometrical function ${}_4F_3$ (of argument 1):

$$C^{l,|m|}_{\frac{J}{2},\frac{|m|-n}{2};\frac{J}{2},\frac{|m|+n}{2}} = (i)^{l-|m|}(-i)^{\frac{J-|m|+2n}{2}}\frac{\sqrt{2l+1}}{2^{2l}}\sqrt{\frac{(J+l+1)!(\frac{J-|m|-n}{2})!(\frac{J+|m|-n}{2})!}{(J-l)!(\frac{J-|m|+n}{2})!(\frac{J+|m|+n}{2})!}}$$

$$\begin{cases} \dfrac{\sqrt{(l-|m|)!(l+|m|)!}}{\Gamma(\frac{l+|m|-n+2}{2})\Gamma(\frac{l-|m|-n+2}{2})}\dfrac{\Gamma(\frac{J-l+1}{2})}{\Gamma(\frac{J+l+3}{2})}\,{}_4F_3\left(\begin{matrix} -\frac{n}{2}, \frac{1-n}{2}, \frac{J+l+2}{2}, \frac{l-J}{2} \\ \frac{1}{2}, \frac{l-|m|-n+2}{2}, \frac{l+|m|-n+2}{2} \end{matrix}\middle| 1\right) & (J-l) - \text{even} \\ \dfrac{-in\sqrt{(l-|m|)!(l+|m|)!}}{\Gamma(\frac{l+|m|-n+3}{2})\Gamma(\frac{l-|m|-n+5}{2})}\dfrac{\Gamma(\frac{J-l}{2})}{\Gamma(\frac{J+l+2}{2})}\,{}_4F_3\left(\begin{matrix} \frac{1-n}{2}, \frac{2-n}{2}, \frac{J+l+3}{2}, \frac{l-J+1}{2} \\ \frac{3}{2}, \frac{l-|m|-n+3}{2}, \frac{l+|m|-n+3}{2} \end{matrix}\middle| 1\right) & (J-l) - \text{odd} \end{cases}$$

(To our knowledge, this expression is new). In the contraction limit $R \to \infty$ and

$$\theta_1 \sim \frac{x_3}{R}, \quad \theta_2 \sim \frac{x_2}{R}, \quad \theta_3 \sim \frac{x_1}{R}, \quad \theta_1' \sim \frac{\rho}{R}, \quad J \sim kR, \quad l \sim pR, \quad m \sim k_1 R. \tag{24}$$

we get

$$\lim_{R\to\infty}(-1)^{-\frac{J-|m|}{2}}\Psi_{Jlm}(\theta_1,\theta_2,\theta_3) = \sqrt{\frac{2kp}{\pi k_2 k_3}}\frac{e^{ik_1x_1}}{\pi}$$

$$\begin{cases} \cos k_2x_2 \cos k_3x_3 & (J-|m|) - \text{even}, \quad (l-|m|) - \text{even}, \\ -i\sin k_2x_2 \cos k_3x_3 & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{even}, \\ -i\cos k_2x_2 \sin k_3x_3 & (J-|m|) - \text{even}, \quad (l-|m|) - \text{odd}, \\ -\sin k_2x_2 \sin k_3x_3 & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{odd}, \end{cases} \tag{25}$$

$$\lim_{R\to\infty}(-i)^{l-|m|}2(-1)^{-\frac{J-|m|}{2}}\sqrt{R}\,C^{l,|m|}_{\frac{J}{2},\frac{|m|-n}{2};\frac{J}{2},\frac{|m|+n}{2}} = \sqrt{\frac{8p}{(k^2-k_1^2)\pi}}(\sin 2\phi)^{-\frac{1}{2}}$$

$$\times\begin{cases} \cos n\phi, & (J-|m|) - \text{even}, \quad (l-|m|) - \text{even}, \\ -i\sin n\phi, & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{even}, \\ -i\sin n\phi, & (J-|m|) - \text{even}, \quad (l-|m|) - \text{odd}, \\ -\cos n\phi, & (J-|m|) - \text{odd}, \quad (l-|m|) - \text{odd}, \end{cases} \tag{26}$$

where $\cos\phi = (p^2-k_1^2)/(k^2-k_1^2)$ and $k^2 = p^2 + k_3^2 = k_1^2 + k_2^2 + k_3^2$. Substituting the formulas (12), (25) and (26) into the expansion (23) we have

$$\begin{Bmatrix} \cos k_2x_2 \cos k_3x_3 \\ \sin k_2x_2 \cos k_3x_3 \\ \cos k_2x_2 \sin k_3x_3 \\ \sin k_2x_2 \sin k_3x_3 \end{Bmatrix} = \sum_{n=-\infty}^{\infty}\begin{Bmatrix} \cos n\phi \\ \sin n\phi \\ \sin n\phi \\ \cos n\phi \end{Bmatrix} J_{|n|}(q\rho)e^{in\theta_2'}$$

where

$$\tan\theta_2' = \frac{x_3}{x_2}, \qquad q^2 = k_2^2 + k_3^2, \qquad \rho^2 = x_2^2 + x_3^2, \qquad \cos^2\phi = \frac{k_2^2}{k_2^2 + k_3^2}$$

Thus the interbasis expansion (23) transforms to the expansion between Cartesian and cylindrical bases for the Helmholtz equation on E_2.

4 Conclusions

In this article we rectricted ourselves to subgroup type coordinates only and moreover to the lowest dimensional spheres S_2 and S_3. Two earlier articles were devoted to contractions of separated basis functions that correspond to nonsubgroup type coordinates, in particular elliptic coordinates on S_2 and on the hyperboloid H_2 [9, 10]. It would also be possible to obtain interbases expansions for other types of bases, though so far this has not been done.

We mention that though the two different trees on S_2 are topologically equivalent, the contraction to E_2 destroys the symmetry between them. Indeed, one system of coordinates on S_2 goes into polar coordinates on E_2, the other into Cartesian ones. Correspondingly, contracted interbasis expansions give relations between plane and cylindrical waves namely eq. (7) and (8). These expansion formulas are of course well known.

Five types of trees exist for S_3. Only two of them correspond to mutually nonisomorphic subgroup chains. After the contraction, we obtain expansions between functions separated in spherical, cylindrical and Cartesian coordinates, respectively. We have presented only 3 of the 10 possible interbasis expansions and their contractions. The others are either obtained by composing the 3 elementary ones, or correspond to transitions within an O(3) subgroup. The transition between spherical and Cartesian basis in E(3) is obtained by composing the Cartesian to cylindrical and cylindrical to spherical ones.

An article on constractions of interbases expansions on S_n for n arbitrary, is in progress.

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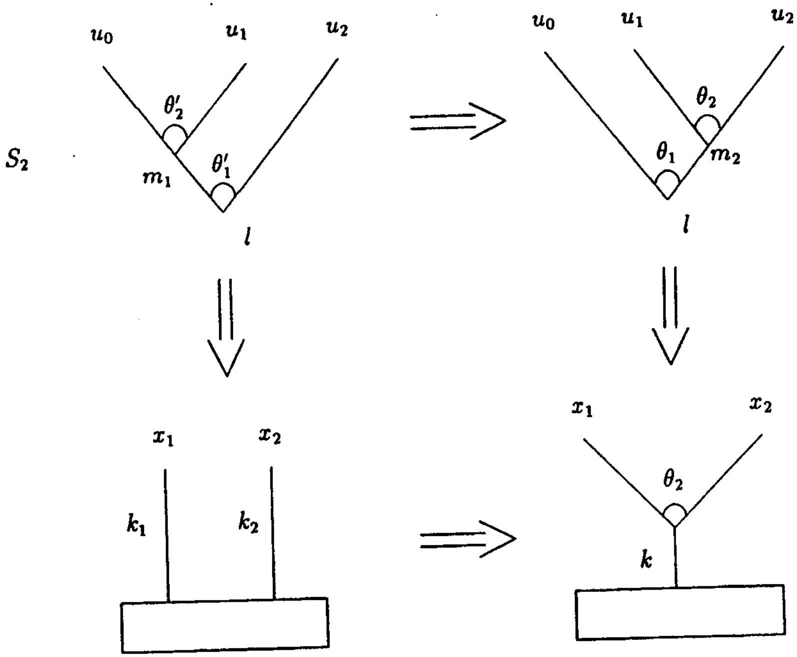


Figure 1. Interbasis expansions contracted from $O(3)$ to $E(2)$.

Figure 2. Elementary interbasis expansions contracted from $O(4)$ to $E(3)$.

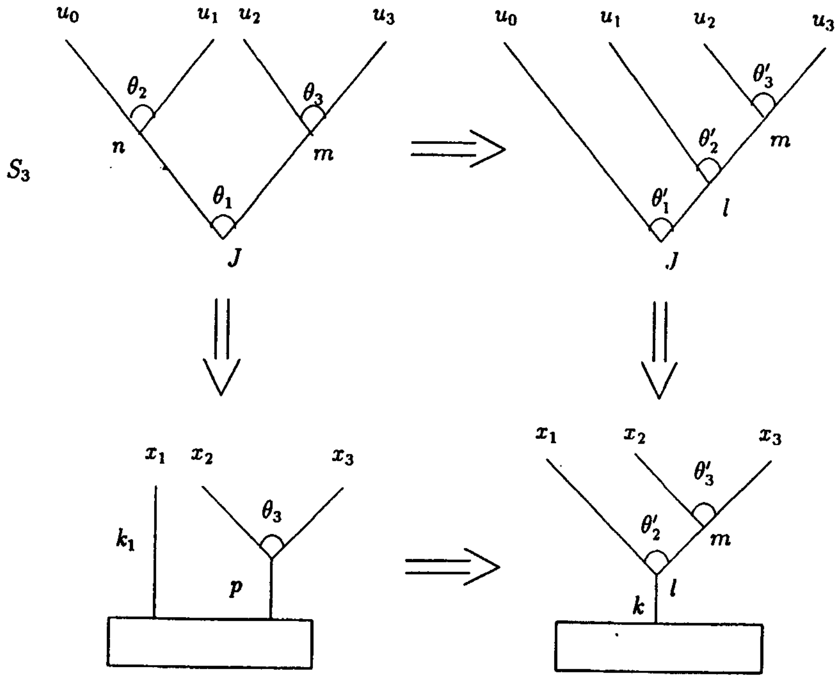
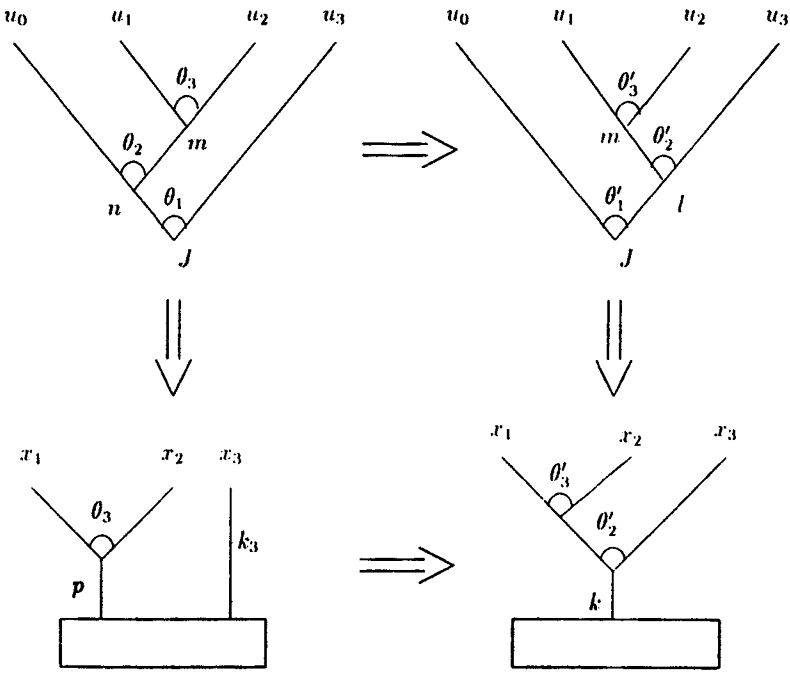


Figure 2a.


u_0
u_1
u_2
u_3
θ_3
θ_2
m
n
θ_1
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u_0
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θ'_3
m
θ'_2
θ'_1
l
J
x_1
x_2
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x_3
θ'_3
θ'_2
k

Figure 2b.

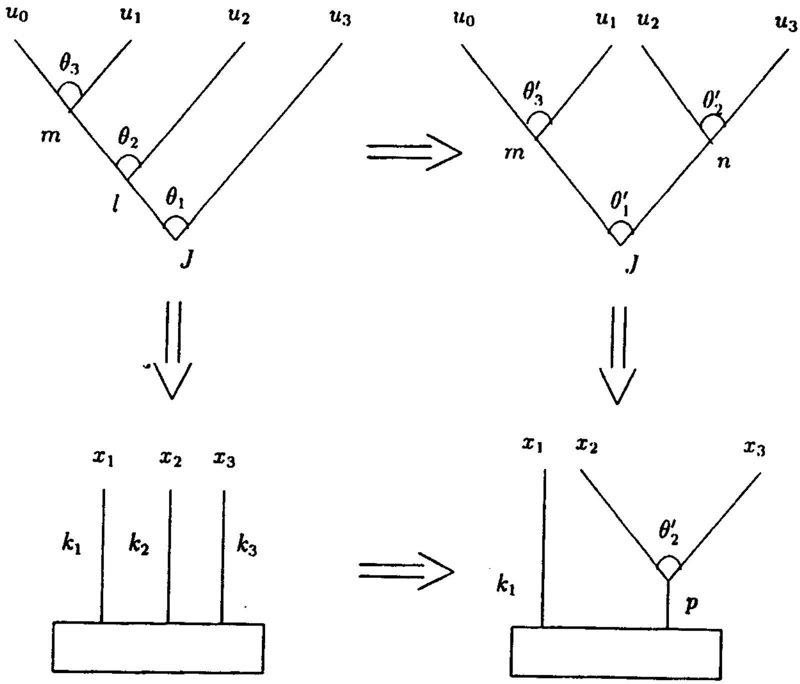

Figure 2c.

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