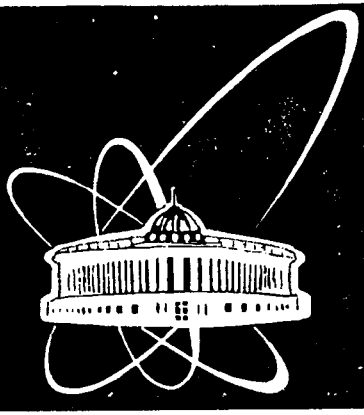

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SU(2)-MONOPOLE: INTERBASIS EXPANSION

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1 Introduction

A charge–dyon system with the $SU(2)$–monopole in the space $\mathbb{R}^5$ is described by the equation [1]

$$\frac{1}{2m}\left(-i\hbar\frac{\partial}{\partial x_j}-\hbar A_j^a\hat{T}_a\right)^2\psi+\frac{\hbar^2}{2mr^2}\hat{T}^2\psi-\frac{e^2}{r}\psi=\epsilon\psi \tag{1}$$

where $j=0,1,2,3,4$; $a=1,2,3$. The operators $\hat{T}_a$ are the generators of the $SU(2)$ group and satisfy the commutation relations

$$[\hat{T}_a,\hat{T}_b]=i\epsilon_{abc}\hat{T}_c\ .$$

The triplet of five–dimensional vectors $\vec{A}^a$ is given by the expressions

$$\begin{aligned}\vec{A}^1 &= \frac{1}{r(r+x_0)}(0,-x_4,-x_3,x_2,x_1)\\ \vec{A}^2 &= \frac{1}{r(r+x_0)}(0,x_3,-x_4,-x_1,x_2)\\ \vec{A}^3 &= \frac{1}{r(r+x_0)}(0,x_2,-x_1,x_4,-x_3)\ .\end{aligned}$$

Every term of the triplet A_j^a coincides with the vector potential of the 5D Dirac monopole [2] with a unit topological charge and the line of singularity along the nonpositive part of the x_0–axis. The vectors A_j^a are orthogonal to each other

$$A_j^aA_j^b=\frac{1}{r^2}\frac{(r-x_0)}{(r+x_0)}\delta_{ab}$$

and also to the vector $\vec{x}=(x_0,x_1,x_2,x_3,x_4)$.

The eigenvalues of the energy ($N=0,1,2...$)

$$\epsilon_N^T=-\frac{me^4}{2\hbar^2(\frac{N}{2}+2)^2} \tag{2}$$

for fixed T are degenerated with multiplicity [3]

$$\begin{aligned}g_N^T=\frac{1}{12}(2T+1)^2&\left(\frac{N}{2}-T+1\right)\left(\frac{N}{2}-T+2\right)\\ &\left\{\left(\frac{N}{2}-T+2\right)\left(\frac{N}{2}-T+3\right)+2T(N+5)\right\}\ .\end{aligned}$$

For $T=0$ and $N=2n$ (even) the r.h.s. of the last formula is equal to $(n+1)(n+2)^2(n+3)/12$, i.e., to the degeneracy of pure Coulomb levels.

The article is organized as follows: In Section 2, we describe the hyperspherical and parabolic bases for the charge–dyon system with the $SU(2)$–monopole in a way adapted to the introduction of interbasis expansions. In Section 3, we prove an additional orthogonality property for hyperspherical radial wave functions of the given hypermomentum λ. In Section 4, by using the property of biorthogonality of the hyperspherical basis, we calculate the coefficients of interbasis expansion between hyperspherical and parabolic bases.

2 Hyperspherical and parabolic bases

The variables in Eq.(1) are separated in the hyperspherical and parabolic coordinates.

Let us introduce in $\mathbb{R}^5$ the hyperspherical coordinates $r \in [0,\infty)$, $\theta \in [0,\pi]$, $\alpha \in [0,2\pi)$, $\beta \in [0,\pi]$, $\gamma \in [0,4\pi)$ according to

$$\begin{aligned} x_0 &= r\cos\theta \\ x_1 + ix_2 &= r\sin\theta\cos\frac{\beta}{2}e^{i\frac{\alpha+\gamma}{2}} \\ x_3 + ix_4 &= r\sin\theta\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}} . \end{aligned}$$

Since

$$iA_j^a\frac{\partial}{\partial x_j} = \frac{2}{r(r+x_0)}\hat{L}_a,$$

where

$$\begin{aligned} \hat{L}_1 &= \frac{i}{2}\left[D_{41}(x) + D_{32}(x)\right] \\ \hat{L}_2 &= \frac{i}{2}\left[D_{13}(x) + D_{42}(x)\right] \\ \hat{L}_3 &= \frac{i}{2}\left[D_{12}(x) + D_{34}(x)\right] \end{aligned}$$

and

$$D_{ij}(x) = -x_i\frac{\partial}{\partial x_j} + x_j\frac{\partial}{\partial x_i} .$$

Eq.(1) in the hyperspherical coordinates assumes the form

$$\left(\Delta_{r\theta} - \frac{\hat{L}^2}{r^2\sin^2\theta/2} - \frac{\hat{j}^2}{r^2\cos^2\theta/2}\right)\psi + \frac{2m}{\hbar^2}\left(\epsilon + \frac{e^2}{r}\right)\psi = 0 . \qquad (3)$$

Here

$$\Delta_{r\theta} = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^3\theta}\frac{\partial}{\partial\theta}\left(\sin^3\theta\frac{\partial}{\partial\theta}\right)$$

and $\hat{J}_a = \hat{L}_a + \hat{T}_a$. Emphasize that

$$[\hat{L}_a, \hat{L}_b] = i\epsilon_{abc}\hat{L}_c, \quad [\hat{J}_a, \hat{J}_b] = i\epsilon_{abc}\hat{J}_c\,.$$

The solution of Eq.(3) is of the form [3]

$$\psi^{sph} = R_{N\lambda}(r)Z_{\lambda LJ}(\theta)G^{JM}_{LTm'l'}(\alpha,\beta,\gamma;\alpha_T,\beta_T,\gamma_T) \tag{4}$$

where G are the eigenfunctions of $\hat{L}^2$, $\hat{T}^2$ and $\hat{J}^2$ with the eigenvalues $L(L+1)$. $T(T+1)$ and $J(J+1)$; α_T, β_T and γ_T are the coordinates of the space group of $SU(2)$ and have the form

$$G = \sqrt{\frac{(2L+1)(2T+1)}{4\pi^4}}\sum_{M=m+t}(JM|L,m\,;T,t\,)\,D^L_{mm'}(\alpha,\beta,\gamma)D^T_{tl'}(\alpha_T,\beta_T,\gamma_T)\,.$$

Here $(JM|L,m\,;T,t\,)$ are the Clebsch—Gordan coefficients, and $D^L_{mm'}$ and $D^T_{tl'}$ are the Wigner functions.

The functions $Z_{\lambda LJ}(\theta)$ and $R_{N\lambda}(r)$ normalized by the conditions

$$\int_0^\pi \sin^3\theta Z_{\lambda' LJ}(\theta)Z_{\lambda LJ}(\theta)d\theta = \delta_{\lambda'\lambda}$$

$$\int_0^\infty r^4 R_{N'\lambda}(r)R_{N\lambda}(r)dr = \delta_{N'N} \tag{5}$$

are given by the formulae

$$Z_{\lambda LJ}(\theta) = N^\lambda_{JLT}(1-\cos\theta)^L(1+\cos\theta)^J P^{(2L+1,2J+1)}_{\lambda-L-J}(\cos\theta) \tag{6}$$

$$R_{N\lambda}(r) = C_{N\lambda}e^{-\kappa r}(2\kappa r)^\lambda F\left(-\frac{N}{2}+\lambda;2\lambda+4;2\kappa r\right). \tag{7}$$

Here $P^{(\alpha,\beta)}_n(x)$ are the Jacobi polynomials; $\kappa = 2/r_0(N+4)$, $r_0 = \hbar^2/me^2$ is the Bohr radius. The normalization constants N^λ_{JLT} and $C_{N\lambda}$ equal

$$C^\lambda_{LJT} = \sqrt{\frac{(2\lambda+3)(\lambda-J-L)!\Gamma(\lambda+J+L+3)}{2^{2J+2L+3}\Gamma(\lambda+J-L+2)\Gamma(\lambda-J+L+2)}}$$

$$C_{N\lambda} = \frac{32}{(N+4)^3}\frac{1}{(2\lambda+3)!}\sqrt{\frac{(\frac{N}{2}+\lambda+3)!}{r_0^5(\frac{N}{2}-\lambda)!}}\,.$$

The quantum numbers run over the values $|L-T| \leq J \leq L+T$; $\lambda = L+J, L+J+1, ..., N/2$.

In the parabolic coordinates

$$\begin{aligned} x_0 &= \frac{1}{2}(\xi - \eta) \\ x_1 + ix_2 &= \sqrt{\xi\eta}\cos\frac{\beta}{2}e^{i\frac{\alpha+\gamma}{2}} \\ x_3 + ix_4 &= \sqrt{\xi\eta}\sin\frac{\beta}{2}e^{i\frac{\alpha-\gamma}{2}} \end{aligned}$$

where $\xi, \eta \in [0, \infty)$, upon the substitution

$$\psi^{par} = f_1(\xi) f_2(\eta) G^{JM}_{LTm't'}(\alpha, \beta, \gamma; \alpha_T, \beta_T, \gamma_T)$$

the variables in Eq.(1) are separated, which results in the system of equations

$$\begin{aligned} \frac{1}{\xi}\frac{d}{d\xi}\left(\xi^2\frac{df_1}{d\xi}\right) + \left[\frac{m\epsilon}{2\hbar^2} - \frac{1}{\xi}J(J+1) + \beta_1\right] f_1 = 0 \\ \frac{1}{\eta}\frac{d}{d\eta}\left(\eta^2\frac{df_2}{d\eta}\right) + \left[\frac{m\epsilon}{2\hbar^2} - \frac{1}{\eta}L(L+1) + \beta_2\right] f_2 = 0 \end{aligned}$$

where

$$\beta_1 + \beta_2 = \frac{me^2}{\hbar^2} \,. \tag{8}$$

At $T = 0$ (i.e. $J = L$), these equations coincide with the equations for a five dimensional Coulomb problem in the parabolic coordinate [4], and consequently,

$$\psi^{par} = \kappa^3\sqrt{2r_0} f_{n_1 J}(\xi) f_{n_2 L}(\eta) G^{JM}_{LTm't'}(\alpha, \beta, \gamma; \alpha_T, \beta_T, \gamma_T) \tag{9}$$

where

$$f_{pq}(x) = \frac{1}{(2q+1)!}\sqrt{\frac{(p+2q+1)!}{p!}}\exp\left(-\frac{\kappa x}{2}\right)(\kappa x)^q F(-p; 2q+2; \kappa x) \,.$$

Here n_1 and n_2 are non-negative integers

$$n_1 = -J - 1 + \frac{\beta_1}{\kappa}, \quad n_2 = -L - 1 + \frac{\beta_2}{\kappa}$$

from which and (2), (8) it follows that the parabolic quantum numbers n_1, n_2, J and L are connected with the principal quantum number N as follows:

$$N = 2(n_1 + n_2 + J + L) \,.$$

3 Biorthogonality of the radial wave functions

We shall prove that along with the condition (5) the radial wave functions $R_{N\lambda}(r)$ satisfy the following "additional" orthogonality condition:

$$J_{\lambda\lambda'} = \int_0^\infty r^2 R_{N\lambda'}(r) R_{N\lambda}(r) dr = \frac{16}{r_0^2 (N+4)^2} \frac{1}{2\lambda+3} \delta_{\lambda\lambda'} \cdot \tag{10}$$

This new relation shall prove useful when dealing with the interbasis expansions in the next Section. The proof of the formula (10) is as follows.

In the integral appearing in (10), we replace $R_{N\lambda}(r)$ and $R_{N\lambda'}(r)$ by their expressions (7). Then, we take the confluent hypergeometric function in (7) as an finite sum

$$F\left(-\frac{N}{2}+\lambda; 2\lambda+4; 2\kappa r\right) = \sum_{s=0}^{\frac{N}{2}-\lambda} \frac{\left(-\frac{N}{2}+\lambda\right)_s}{(2\lambda+4)_s} \frac{(2\kappa r)^s}{s!}$$

and perform the integration term by term with the help of the formula [5]

$$\int_0^\infty e^{-\lambda x} x^\nu F(\alpha, \gamma; kx)\, dx = \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}} {}_2F_1\left(\alpha, \nu+1, \gamma; \frac{k}{\lambda}\right). \tag{11}$$

By using

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{12}$$

we arrive at

$$J_{\lambda\lambda'} = \frac{16}{r_0^2(N+4)^2} \frac{\Gamma(\lambda+\lambda'+3)}{(2\lambda+3)} \sqrt{\frac{\left(\frac{N}{2}+\lambda+3\right)!}{\left(\frac{N}{2}-\lambda\right)!\left(\frac{N}{2}-\lambda'\right)!\left(\frac{N}{2}+\lambda'+3\right)!}}$$
$$\sum_{s=0}^{\frac{N}{2}-\lambda} \frac{\left(-\frac{N}{2}+\lambda\right)_s}{s!} \frac{(\lambda+\lambda'+3)_s}{(2\lambda+4)_s} \frac{\Gamma\left(\frac{N}{2}-\lambda-s+1\right)}{\Gamma(\lambda-\lambda'-s+1)} \cdot \tag{13}$$

By introducing formula [6]

$$\frac{\Gamma(z)}{\Gamma(z-n)} = (-1)^n \frac{\Gamma(-z+n+1)}{\Gamma(-z+1)} \tag{14}$$

into (13), the sum over s can be expressed in terms of the ${}_2F_1$ Gauss hypergeometric function of argument 1. We thus obtain

$$J_{\lambda\lambda'} = \frac{16}{r_0^2(N+4)^2} \frac{1}{\lambda+\lambda'+3} \sqrt{\frac{\left(\frac{N}{2}-\lambda\right)!\left(\frac{N}{2}+\lambda+3\right)!}{\left(\frac{N}{2}-\lambda'\right)!\left(\frac{N}{2}+\lambda'+3\right)!}}$$
$$\frac{1}{\Gamma(\lambda-\lambda'+1)\Gamma(\lambda'-\lambda+1)} \cdot \tag{15}$$

Equation (10) then easily follows from (15) since $[\Gamma(\lambda-\lambda'+1)\Gamma(\lambda'-\lambda+1)]^{-1} = \delta_{\lambda\lambda'}$.

The result provided by formula (10) generalizes the one for the hydrogen atom [6]. Such unusual orthogonality properties are connected with the accidental degeneracies of the energy spectrum for the charge-dyon system with the $SU(2)$-monopole.

4 Interbasis expansion

The connection between hyperspherical $(r, \theta, \alpha, \beta, \gamma)$ and parabolic $(\xi, \eta, \alpha, \beta, \gamma)$ coordinates is

$$\xi = r(1 + \cos\theta), \qquad \eta = r(1 - \cos\theta)\,. \tag{16}$$

Now, we can write, for fixed value energy, the parabolic bound states (9) as a coherent quantum mixture of the hyperspherical bound states (4)

$$\psi^{par} = \sum_{\lambda=T}^{N/2} W^{\lambda}_{n_1 n_2 JL} \psi^{sph}. \tag{17}$$

By virtue of Eq.(16), the left-hand side of (17) can be rewritten in hyperspherical coordinates. Then, by substituting $\theta = 0$ in the so-obtained equation and by taking into account that

$$P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_n}{n!}$$

we get an equation that depends only on the variable r. Thus, we can use the orthogonality relation (10) on the hypermomentum quantum numbers λ. This yields

$$W^{\lambda}_{n_1 n_2 JL} = \frac{1}{(2J+1)!(2\lambda+3)!} E^{n_1 n_2}_{\lambda JL} K^{n n_1}_{\lambda JL} \tag{18}$$

where

$$E^{n_1 n_2}_{\lambda JL} = \sqrt{(2\lambda+3)(\lambda-J-L)!\left(\frac{N}{2}+\lambda+3\right)!}$$
$$\left[\frac{\Gamma(\lambda+J-L+2)(n_1+2J+1)!(n_2+2L+1)!}{(n_1)!(n_2)!\left(\frac{N}{2}-\lambda\right)!\Gamma(\lambda-J+L+2)\Gamma(\lambda+J+L+3)}\right]^{1/2} \tag{19}$$

$$K^{n n_1}_{\lambda JL} = \int_0^{\infty} e^{-x} x^{\lambda+J+L+2} F(-n_1, 2J+2; x) F\left(-\frac{N}{2}+\lambda, 2\lambda+4; x\right) dx\,. \tag{20}$$

To calculate the integral $K^{nn_1}_{\lambda JL}$, it is sufficient to write the confluent hypergeometric function $F(-n_1, 2J+2; x)$ as a series, integrate according to (11) and use the formula (12) for the summation of the hypergeometric function ${}_2F_1$. We thus obtain

$$K^{nn_1}_{\lambda JL} = \frac{(2\lambda+3)!\left(\frac{N}{2}-J-L\right)!\Gamma(\lambda+J+L+3)}{(\lambda-J-L)!\left(\frac{N}{2}+\lambda+3\right)!}$$

$${}_3F_2\left\{\begin{matrix} -n_1, -\lambda+J+L, \lambda+J+L+3 \\ 2J+2, -\frac{N}{2}+J+L \end{matrix}\middle| 1\right\}. \qquad (21)$$

The introduction of (19) and (21) into (18) gives

$$W^{\lambda}_{n_1 n_2 JL} = \left[\frac{(2\lambda+3)\Gamma(\lambda+J+L+3)(n_1+2J+1)!(n_2+2L+1)!}{(n_1)!(n_2)!(\lambda-J-L)!\left(\frac{N}{2}-\lambda\right)!\left(\frac{N}{2}+\lambda+3\right)!}\right]^{1/2}$$

$$\frac{\left(\frac{N}{2}-J-L\right)!}{(2J+1)!}\sqrt{\frac{\Gamma(\lambda+J-L+2)}{\Gamma(\lambda-J+L+2)}}{}_3F_2\left\{\begin{matrix} -n_1, -\lambda+J+L, \lambda+J+L+3 \\ 2J+2, -\frac{N}{2}+J+L \end{matrix}\middle| 1\right\}. \qquad (22)$$

The next step is to show that the interbasis coefficients (22) are the Clebsch-Gordan coefficients for the group $SU(2)$. It is known that the Clebsch-Gordan coefficient can be written as [7]

$$C^{c\gamma}_{a\alpha;b\beta} = (-1)^{a-\alpha}\delta_{\gamma,\alpha+\beta}\frac{(a+b-\gamma)!(b+c-\alpha)!}{\sqrt{(b-\beta)!(b+\beta)!}}$$

$$\left[\frac{(2c+1)(a+\alpha)!(c+\gamma)!}{(a-\alpha)!(c-\gamma)!(a+b+c+1)!(a+b-c)!(a-b+c)!(b-a+c)!}\right]^{1/2}$$

$${}_3F_2\left\{\begin{matrix} -a-b-c-1, -a+\alpha, -c+\gamma \\ -a-b+\gamma, -b-c+\alpha \end{matrix}\middle| 1\right\}. \qquad (23)$$

By using the formula [8]

$${}_3F_2\left\{\begin{matrix} s, s', -N \\ t', 1-N-t \end{matrix}\middle| 1\right\} = \frac{(t+s)_N}{(t)_N}{}_3F_2\left\{\begin{matrix} s, t'-s', -N \\ t', t+s \end{matrix}\middle| 1\right\}$$

equation (23) can be rewritten in the form

$$C^{c\gamma}_{a\alpha;b\beta} = \left[\frac{(2c+1)(b-a+c)!(a+\alpha)!(b+\beta)!(c+\gamma)!}{(b-\beta)!(c-\gamma)!(a+b-c)!(a-b+c)!(a+b+c+1)!}\right]^{1/2}$$

$$\delta_{\gamma,\alpha+\beta}\frac{(-1)^{a-\alpha}}{\sqrt{(a-\alpha)!}}\frac{(a+b-\gamma)!}{(b-a+\gamma)!}{}_3F_2\left\{\begin{matrix} -a+\alpha, c+\gamma+1, -c+\gamma \\ \gamma-a-b, b-a+\gamma+1 \end{matrix}\middle| 1\right\}. \qquad (24)$$

By comparing (24) and (22), we finally obtain the desired representation

$$W^{\lambda}_{n_1 n_2 JL} = (-1)^{n_1} C^{\lambda+1, J+L+1}_{\frac{N-2J+2L+2}{4}, I+\frac{n_2-n_1+1}{2}; \frac{N+2J-2L+2}{4}, J+\frac{n_1-n_2+1}{2}} \ . \qquad (25)$$

At $T = 0$ (i.e. $J = L$) formula (25) turns into the formula for the five-dimensional Coulomb problem [4], as would be expected.

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SU(2)-монополь: межбазисное разложение

Проведено исследование нерелятивистской квантово-механической пятимерной системы заряд—дион с SU(2)-монополем. Переменные в уравнении Шредингера для этой системы разделяются в гиперсферических и параболических координатах. Полностью решена проблема межбазисного разложения волновых функций. Найдено, что коэффициенты разложения параболического базиса по гиперсферическому выражаются через коэффициенты Клебша—Гордана для группы SU(2).

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SU(2)-Monopole: Interbasis Expansion

This article deals with a nonrelativistic quantum mechanical study of a charge—dyon system with the SU(2)-monopole in five dimensions. The Schrödinger equation for this system is separable in the hyperspherical and parabolic coordinates. The problem of interbasis expansion of the wave functions is completely solved. The coefficients for the expansion of the parabolic basis in terms of the hyperspherical basis can be expressed through the Clebsch—Gordan coefficients of the SU(2) group.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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