Tr. J. of Physics
21 (1997) , 515 - 524.

Proceedings of the Barut Memorial
Conference on Group Theory in Physics

On the Kepler-Coulomb Problem in the Three-dimensional Space With Constant Positive Curvature

G.S. POGOSYAN and A.N. SISSAKIAN
Joint Institute for Nuclear Research,
Dubna, Moscow Region, 141980-RUSSIA

Abstract

The Schrödinger equation is analysed for the Kepler-Coulomb problem in the three-dimensional space with constant positive curvature. The representation of the elliptic basis as a superposition over hyperspherical states is obtained. The "parabolic" system of coordinates on the three-dimensional sphere is determined which is a particular case of elliptic coordinates rather than an independent system of coordinates as in the flat space.

1. Introduction

The Kepler-Coulomb problem in the spaces with constant curvature has a long history and was first considered by Schrödinger fifty years ago in the paper [1]. He used the factorization method to solve the Schrödinger equation in hyperspherical coordinates and to find the energy spectrum for the harmonic potential being an analog of the Coulomb potential on the three- dimensional sphere and showed that like in the case of flat space there occurs complete degeneracy of energy levels in orbital and azimuthal quantum numbers. Later, Higgs [3], Leemon [4] and Kurochkin and Otchik [2] have shown for the space with positive constant curvature and Bogush, Kurochkin and Otchik [5] for the space with negative constant curvature that the degeneracy of the spectrum of the Coulomb problem is caused, as in flat space, by an additional integral of motion: Runge-Lenz's vector. From the point of view of path integrals the Kepler-Coulomb problem for N-dimensional spaces with constant curvature has been investigated in detail by Barut, Inomata and Junker [10], Grosche [11] and Grosche, Pogosyan and Sissakian [12, 13].

The Schrödinger equation for the Kepler-Coulomb problem on the three-dimensional sphere $S_3 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$, has the following form $(\hbar = m = 1)$

$$H\Psi = \left\{ -\Delta_{LB} - \frac{\alpha}{R} \frac{x_3}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \right\} \Psi = E\Psi, \tag{1}$$

where $x_\mu(\mu = 1, 2, 3, 4)$ are the Cartesian coordinates in the four-dimensional ambient Euclidean space E_4 and $\Delta_{LB} = -\frac{1}{2R^2}(\mathbf{L}^2 + \mathbf{N}^2)$. The operators L_i and $N_i (i = 1, 2, 3)$ are the group $O(4)$ generators

$$L_i = -i\epsilon_{ijk} x_j \frac{\partial}{\partial x_k} \qquad , N_i = -i\epsilon_{ijk} \left(x_4 \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial x} \right) \tag{2}$$

As has been shown in refs. [3, 2], alongside with the square angular momentum L^2, the Hamiltonian (1) commutes with an additional integral of motion

$$\mathbf{A} = \frac{1}{2R} \{ [\mathbf{L}, \mathbf{N}] - [\mathbf{N}, \mathbf{L}] \} + \frac{\alpha \mathbf{x}}{\sqrt{x_1^2 + x_2^2 + x_3^2}}, \mathbf{x} = (x_1, x_2, x_3). \tag{3}$$

The operators L_i and A_i obey the following commutation relations

$$[L_i, L_j] = i\epsilon_{ijk} L_k, \qquad [L_i, A_j] = i\epsilon_{ijk} A_k, \qquad [A_i, A_j] = -2i\epsilon_{ijk} \left(H - \frac{L^2}{R^2} \right) L_k, \tag{4}$$

which are of the nonlinear nature and, consequently, do not form the finite-dimensional Lie algebra. This is an essential difference of the Coulomb problem on the sphere form the flat case when the group $O(4)$ is given in the form of a group of "hidden symmetry".

It is known that there exists a one-to-one correspondence between the sets of independent symmetry operators of the Schrödinger equation, both in the flat space [22] and the space with constant positive curvature, and the orthogonal systems of coordinates admitting separation of variables in this equation. Thus, for instance the Schrödinger equation for a hydrogen atom in the flat space is separated in four systems of coordinates [22]: spherical, sphero-conical, parabolic and prolate spheroidal, and the corresponding wave functions are eigenfunctions of the complete set of independent operators constructed out of the components of the angular momentum operator and Runge- Lenz vector.

Olevsky [16] has found that variables in the Helmholtz or Schrödinger equation on the three-dimensional sphere can be separated only in six orthogonal systems of coordinates whereas in the Euclidean three-dimensional space their number is 11.

Until recently it has been considered that variables in the Schrödinger equation (1) for a Coulomb problem on S_3 are separated only in two systems of coordinates [6, 15]: hyperspherical and sphero-conical, which is connected only with conservation of the square of the orbital momentum, and the presence of an additional integral of motion does not lead to separation in additional, with respect to them, systems of coordinates.

For the first time, one of the elliptic systems (ellipso-cylindrical II) of coordinates in the three-dimensional space with constant curvature was used in separating variables in the Schrödinger equation for the problem of two-Coulomb centers [7]. Later, in ref. [12] for the case of three-dimensional space with constant positive curvature there have been determined a rotated elliptic system of coordinates corresponding in the flat limit to a prolate spheroidal system of coordinates (when the center of the system of coordinates lies in one of the foci of the system) and admitting separation of variables in the Schrödinger equation for the Coulomb problem.

The aim of the present paper is first to construct an elliptic basis of the Coulomb problem for the three-dimensional space with constant positive curvature and, second, to find a , "lacking" **parabolic** system of coordinates which can diagonalize the Runge-Lenz operator.

The paper is organized as follows. In the first section we give some known results concerning the hyperspherical basis of the Coulomb problem on S_3. The second section is mainly the study of elliptic coordinates and determination of elliptic basis for the Coulomb problem.

2. Hyperspherical basis

Let us remind the basic results concerning the hyperspherical basis of the Coulomb problem on the sphere. The hyperspherical system of coordinates on S_3 is connected with the Cartesian coordinates x_μ of the enveloping Euclidean space E_4 by

$$\begin{array}{ll} x_1 = R\sin\chi \sin\theta\cos\phi, & x_2 = R\sin\chi\sin\theta\sin\phi, \\ x_3 = R\sin\chi\cos\theta, & x_4 = R\cos\chi, \end{array} \tag{5}$$

where $0 \leq \chi \leq \pi, 0 \leq \theta \leq \pi, 0 \leq \phi < 2\pi$. The solution of the Schrödinger equation (1) for the Coulomb problem in the hyperspherical system of coordinates has the from [6, 18]

$$\Psi_{nlm}(\chi,\theta,\phi;R) = S_{nl}(\chi;R)Y_{lm}(\theta,\varphi) \tag{6}$$

where $n \in \mathbf{N}$ is the principal quantum number, $l = 0, 1, ...n$ is the orbital quantum number and $Y_{lm}(\theta,\phi)$ is an ordinary spherical function on the sphere S_2. The quasiradial wave function $S_{nl}(\chi;R)$ orthonormalised in the interval $\chi \in [0,\pi]$ can be represented as follows:

$$\begin{aligned} S_{nl}(\chi;R) &= (i)^{n-1-1}2^n|\Gamma(l+1-i\sigma)|e^{\frac{\pi\sigma}{2}}\sqrt{\frac{(n^2+\sigma^2)(n-l-1)!}{2\pi n(n+l)!}} \\ &\cdot\ (\sin\chi)^{n-1}e^{-\chi\cdot\sigma}\ P_{n-l-1}^{(-n+i\sigma,-n-i\sigma)}(i\cot\chi) \quad . \end{aligned} \tag{7}$$

The parameter $\sigma = \alpha R/n$ and $P_n^{\alpha,\beta}(x)$ are the Jacobi polynomials. For the energy spectrum we have formula [1]

$$E_n(R) = \frac{n^2-1}{2R^2} - \frac{\alpha^2}{2n^2} \tag{8}$$

In the flat space limit when $R \to \infty$ and $\chi \to 0$ so that $R.\chi \to r$ where r is the vector radius in E_3, the quasiradial wave function (7) turns into, for $n << R$ and $n \sim pR$ (p is a constant), the radial wave function of a hydrogen atom for the cases of discrete and continuous spectra, respectively, [18].

3. Elliptic basis

3.1 Prolate elliptic coordinate system

The prolate elliptic coordinate in the algebraic from is given by [12]

$$\begin{aligned}
x_1^2 &= R^2\frac{(\rho_1-a_2)(\rho_2-a_2)}{(a_3-a_2)(a_1-a_2)}\cos{}^2\phi,\\
x_2^2 &= R^2\frac{(\rho_1-a_2)(\rho_2-a_2)}{(a_3-a_2)(a_1-a_2)}\sin{}^2\phi,\\
x_3^2 &= R^2\frac{(\rho_1-a_1)(\rho_2-a_1)}{(a_2-a_1)(a_3-a_1)},\\
x_4^2 &= R^2\frac{(\rho_1-a_3)(\rho_2-a_3)}{(a_2-a_3)(a_1-a_3)}.
\end{aligned} \tag{9}$$

where $(a_1 \le \rho_1 \le a_2 \le \rho_2 \le a_3, 0 \le \phi < 2\pi)$. In terms of the Jacobi elliptic functions we have [22, 12]

$$\begin{aligned}
x_1 &= \mathrm{Rcn}(\mu,k)\mathrm{cn}(\nu,k')\cos\phi, & -K \le \mu \le K,\\
x_2 &= \mathrm{Rcn}(\mu,k)\mathrm{cn}(\nu,k')\sin\phi, & -K' \le \nu \le K'\\
x_3 &= \mathrm{Rsn}(\mu,k)\mathrm{dn}(\nu,k'),\\
x_4 &= \mathrm{Rdn}(\mu,k)\mathrm{sn}(\nu,k').
\end{aligned} \tag{10}$$

where

$$k'^2 = \frac{a_2-a_1}{a_3-a_1} = \sin^2 f, \qquad k'^2 = \frac{a_3-a_2}{a_3-a_1} = \cos{}^2 f.$$

The Jacobi elliptic functions in the variables μ and ν have moduli k and k', respectively, and $2fR$ is the interfocal distance on the upper hemisphere; K and K' are the complete elliptic integrals. The Laplace-Beltrami operator has the following form:

$$\Delta_{LB} = \frac{1}{R^2}\left[\frac{1}{k^2\mathrm{cn}^2\mu + k'^2\mathrm{cn}^2\nu}\left(\frac{\partial^2}{\partial\mu^2} - \frac{\mathrm{sn}\mu\mathrm{dn}\mu}{\mathrm{cn}\mu}\frac{\partial}{\partial\mu} + \frac{\partial^2}{\partial\nu^2} - \frac{\mathrm{sn}\nu\mathrm{dn}\nu}{\mathrm{cn}\nu}\frac{\partial}{\partial\nu}\right) + \frac{1}{\mathrm{cn}^2\mu\mathrm{cn}^2\nu}\frac{\partial^2}{\partial\phi^2}\right] \quad (11)$$

In the flat space limit the elliptic system turns into an ordinary prolate spheroidal system of coordinates. Indeed, passing to the geodesic coordinates on the sphere, according to refs. [3, 17],

$$u_i = R\frac{x_i}{x_4} = \frac{x_i}{\sqrt{1-(x_1^2+x_2^2+x_3^2)/R^2}}, (i = 1, 2, 3),$$

assuming

$$\frac{R^2}{a_3 - a_1} = \frac{d^2}{a_2 - a_1}, \quad (12)$$

and changing a variable $\rho_1 = a_1 + (a_2 - a_1)\eta^2, \rho_2 = a_1 + (a_2 - a_1)\xi^2$, in the limit $R \sim a_3 \to \infty$ we have

$$\begin{aligned} u_1 &= d\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi, \qquad -1 \le \eta \le 1, \\ u_2 &= d\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi, \qquad 1 \le \xi < \infty, \\ u_3 &= d\xi\eta. \end{aligned} \quad (13)$$

thus arriving at the ordinary prolate spheroidal system of coordinates in E_3 where $2d$ is the interfocal distance.

The variables in the Schrödinger equation for the Coulomb problem (1) are not separated in the system of coordinates (9) or (10). Therefore, in ref. [12] the authors have introduced the rotated elliptic system of coordinates

$$\begin{aligned} x_1 &= R\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k')\cos\phi, \\ x_2 &= R\mathrm{cn}(\mu,k)\mathrm{cn}(\nu,k')\sin\phi, \\ x_3 &= R\left[k'\mathrm{sn}(\mu,k)\mathrm{dn}(\nu,k') + k\mathrm{dn}(\mu,k)\mathrm{sn}(\nu,k')\right], \\ x_4 &= R\left[k'\mathrm{dn}(\mu,k)\mathrm{sn}(\nu,k') - k\mathrm{sn}(\mu,k)dn(\nu,k')\right]. \end{aligned} \quad (14)$$

connected with (10) by the transformation f coordinate x_μ rotation on the three-dimensional sphere $x_\mu \mapsto R(f)x_\mu$ where the matrix R(f) has the form

$$R(f) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos f & \sin f \\ 0 & 0 & -\sin f & \cos f \end{pmatrix}.$$

The elliptic coordinates (10) and (14) are the one-parametric coordinate systems depending on k. In thi cases of $k \to 0$ and $k \to 1$, the elliptic coordinates transform into the hyperspherical coordinates [22, 14].

The next interesting patricular case is the coordinate system (14)

$$\begin{aligned} x_1 &= R\mathrm{cn}\mu\mathrm{cn}\nu\cos\phi, \\ x_2 &= R\mathrm{cn}\mu\mathrm{cn}\nu\sin\phi, \\ x_3 &= \frac{R}{\sqrt{2}}(\mathrm{sn}\mu\mathrm{dn}\nu + \mathrm{dn}\mu\mathrm{sn}\nu), \\ x_4 &= \frac{R}{\sqrt{2}}(\mathrm{dn}\mu\mathrm{sn}\nu - \mathrm{sn}\mu\mathrm{dn}\nu), \end{aligned} \tag{15}$$

with $k = k' = 1/\sqrt{2}(\mathrm{or} f = \pi/4)$ for all Jacobi elliptic functions.

From eq. (15) we can obtain that

$$\begin{aligned} \mathrm{sn}\mu &= \frac{1}{\sqrt{2}}\left\{\left(1+\frac{x_3}{R}\right)^{1/2}\left(1-\frac{x_4}{R}\right)^{1/2} - \left(1-\frac{x_3}{R}\right)^{1/2}\left(1+\frac{x_4}{R}\right)^{1/2}\right\} \\ \sqrt{2}\mathrm{dn}\nu &= \frac{1}{\sqrt{2}}\left\{\left(1+\frac{x_3}{R}\right)^{1/2}\left(1-\frac{x_4}{R}\right)^{1/2} + \left(1+\frac{x_3}{R}\right)^{1/2}\left(1-\frac{x_4}{R}\right)^{1/2}\right\}. \end{aligned}$$

Then, for large R we have

$$\mathrm{sn}\mu \to -1 + \frac{r(1+\cos\theta)}{2R} = -1 + \frac{u}{2R}\sqrt{2}\mathrm{dn}\nu \to 1 + \frac{r(1-\cos\theta)}{2R} = 1 + \frac{v}{2R} \tag{16}$$

and the rotated elliptic coordinates (15) in the limit $R \to \infty$ transform into

$$u_1 = \sqrt{uv}\cos\phi, \qquad u_2 = \sqrt{uv}\sin\phi, \qquad u_3 = \frac{u-v}{2}, \tag{17}$$

which are the flat space parabolic coordinates [19]. Thus, the rotated elliptic system of coordinates (15) at particular values of the parameter $k = k' = 1/\sqrt{2}$ plays the role of a "parabolic" system of coordinates on the three- dimensional sphere.

3.2 Separation of variables and integrals of motion

In the elliptic system of coordinates (15) the Coulomb potential has the form

$$V = -\frac{\alpha}{R}\frac{x_3}{\sqrt{x_1^2+x_2^2+x_3^2}} = -\frac{\alpha}{R}\frac{k'\mathrm{dn}\nu\mathrm{sn}\nu - k\mathrm{dn}\mu\mathrm{sn}\mu}{k^2\mathrm{cn}^2\mu + k'^2\mathrm{cn}^2\nu} \tag{18}$$

Choosing the wave function as

$$\Psi(\mu,\nu,\phi;R) = \psi_1(\mu;R)\psi_2(\nu;R)\frac{e^{im\phi}}{\sqrt{2\pi}} \tag{19}$$

and substituting it into the Schrödinger equation (1), after separation of variables we arrive at two ordinary differential equations

$$\frac{d^2\psi_1}{d\mu^2} - \frac{\mathrm{sn}\mu\mathrm{dn}\mu}{\mathrm{cn}\mu}\frac{d\psi_1}{d\mu} + \left\{2R^2k^2E\mathrm{cn}^2\mu - \frac{k'^2m^2}{\mathrm{cn}^2\mu} - 2\alpha Rk\mathrm{sn}\mu\mathrm{dn}\mu\right\}\psi_1 = -\lambda\psi_1 \tag{20}$$

$$\frac{d^2\psi_2}{d\nu^2} - \frac{\mathrm{sn}\nu\mathrm{dn}\nu}{\mathrm{cn}\nu}\frac{d\psi_2}{d\nu} + \left\{2R^2k'^2E\mathrm{cn}^2\nu - \frac{k^2m^2}{\mathrm{cn}^2\nu} - 2\alpha Rk'sn\nu\mathrm{dn}\nu\right\}\psi_2 = \lambda\psi_2 \tag{21}$$

where $\lambda \equiv \lambda(k;R)$ is the Coulomb elliptic separation constant. Eliminating from equations (20) and (21) energy E we arrive at the operator

$$\Lambda = \frac{1}{k^2\mathrm{cn}^2\mu + k'2\mathrm{cn}^2\nu}\left\{k^2\mathrm{cn}^2\mu\left[\frac{\partial^2}{\partial\nu^2} - \frac{\mathrm{sn}\nu\mathrm{dn}\nu}{\mathrm{cn}\nu}\frac{d}{d\nu}\right] - k'^2\mathrm{cn}^2\nu\left[\frac{\partial^2}{\partial\mu^2} - \frac{\mathrm{sn}\mu\mathrm{dn}\mu}{\mathrm{cn}\mu}\frac{d}{d\mu}\right]\right\} - \frac{k'^2\mathrm{cn}^2\nu - k^2\mathrm{cn}^2\mu}{\mathrm{cn}^2\mu\mathrm{cn}^2\nu}\frac{\partial^2}{\partial\phi^2} + 2\alpha Rkk'\frac{k\mathrm{sn}\nu\mathrm{dn}\nu\mathrm{cn}^2\mu + k'\mathrm{sn}\mu\mathrm{dn}\mu\mathrm{cn}^2\nu}{k^2\mathrm{cn}^2\mu + k'^2\mathrm{cn}^2\nu}$$

whose eigenvalues are the elliptic separation constant $\lambda(k;R)$ and the eigenfunction is the wave function (19). Passing to the coordinates x_μ we obtain that

$$\Lambda = (k^2 - k'^2)\mathbf{L}^2 + 2Rkk'A_3 = \cos 2f\mathbf{L}^2 + R\sin 2fA_3 \tag{22}$$

Thus, the elliptic integral of motion is not an independent symmetry operator of the Schrödinger equation and is a linear combination of the operators of the square angular momentum and the third component of the Runge-Lenz vector and depending on k as on the parameter.

Equations (20) or (21) can be reduced, by substituting the variable, to the Heun-type equation [20] with four singularities [7,8]. A direct solution of these equations is a complicated mathematical problem, and it cannot be derived in a closed form, i.e., as classical polynomials. As has been shown in paper [8], it can be chosen as a series over hypergeometric functions whose coefficients satisfy the three-term recurrence formulae.

We look for an elliptic basis of the Coulomb problem (19) as a superposition of hyperspherical bases (at a given energy) which could serve as an eigenfunction of the elliptic integral of motion and angular momentum projection L_3.

Let us write down the expansion we are interested in

$$\Psi_{nqm}(\mu,\nu,\phi;R) = \sum_{l=|m|}^{n-1} W_{nqm}^{l}(R)\Psi_{nlm}(\chi,\theta,\phi;R). \tag{23}$$

where the elliptic and hyperspherical bases are eigenfunctions of the operators H, L_3 and $\Lambda, \mathbf{L}^2$, respectively,

$$\Lambda\Psi_{nqm}(\mu,\nu,\phi;R) = \lambda_q(k;R)\Psi_{nqm}(\mu,\nu,\phi;R) \tag{24}$$

$$\mathbf{L}^2\Psi_{nlm}(\chi,\theta,\phi;R) = l(l+1)\Psi_{nlm}(\chi,\theta,\phi;R) \tag{25}$$

and the quantum number $q = 0, 1, ..., n-1$ labels the eigenvalues of the elliptic separation constant. Substituting expansion (23) into eq. (24), using eq. (25) and the orthogonality property of the hyperspherical function Ψ_{nlm}, we arrive at the system of homogeneous equations

$$\{\lambda_q(k;R) - (k^2 - k'^2)l(l+1)\}W_{nqm}^{l}(R) = Rkk' \sum_{l'=|m|}^{n-1} A_{ll'}W_{nqm}^{l'}(R) \tag{26}$$

where

$$A_{ll'} = R^3 \int_0^{\pi} \sin^2\chi d\chi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} \Psi_{n'lm}^{*} A_z \Psi_{nlm} d\phi \tag{27}$$

Let us write down the Runge-Lenz operator in the hyperspherical system of coordinates

$$\begin{aligned} A_z = & \frac{1}{R}\left\{ \cot\chi \cos\theta \left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \right) \right. \\ & \left. + \left(\sin\theta\frac{\partial}{\partial\theta} + \cos\theta \right)\frac{\partial}{\partial\chi} + \alpha R\cos\theta \right\} \end{aligned} \tag{28}$$

Now substituting eq. (28) into eq.(27), using the recurrence relations for the spherical harmonics $Y_{lm}(\theta,\phi)$[21] and using the results of ref. [4] we have

$$A_{ll'} = -\frac{1}{R}\left\{\sqrt{\frac{(n-l)(n+1)(l+i\sigma)(l-i\sigma)}{(2l-l)(2l+1)}}\,\delta_{l',l-1}\right.$$
$$\left.+\sqrt{\frac{(n-l-1)(n+l+1)(l+1+i\sigma)(l+1-i\sigma)}{(2l+1)(2l+3)}}\,\delta_{l',l+1}\right\} \tag{29}$$

Finally, substituting eq. (31) into eq. (26) we arrive at the three-term recurrence relations for the expansion coefficients $W_l \equiv W^l_{nqm}(R)$

$$\sqrt{\frac{[n^2-(l+1)^2][(l+1)^2+\sigma^2]}{(2l+1)(2l+3)}}W_{l+1} + \frac{1}{kk'}[\lambda_q(k;R)-(k^2-k'^2)l(l+1)]W_l$$
$$+\sqrt{\frac{(n^2-l^2)(l^2+\sigma^2)}{(2l-1)(2l+1)}}W_{l-1} = 0 \tag{30}$$

which have to be solved together with the normalization condition $\sum|W^l_{nqm}|^2 = 1$. The recurrence relations obtained are the system of $(n-|m|-1))$ linear homogeneous equations, and admissible values of the elliptic separation constant are determined from the condition for the relevant determinant to be zero.

At $k = k' = 1/\sqrt{2}$ the recurrence formula is simplified and turns into the recurrence formula for the recurrence coefficients for the "parabolic" basis over the spherical one. In contrast with the flat space, they are not expressed through the Clebsch-Gordan coefficients for the group $O(4)$[19].

4. Conclusion

We have shown that the presence of an analog of the Runge-Lenz vector for the Coulomb problem in the space with constant positive curvature leads, like in the flat space, to the separation of variables in an attitional coordinate system, the elliptic one. Also, a "parabolic" system of coordinates diagonalizing the A_3 component of the Runge-Lenz vector is found on the three-dimensional sphere. In contrast with the flat space, the **"parabolic"** system of coordinates on the three-dimensional sphere is not an independent system of coordinates but rather a particular case of the elliptic one.

Acknowledgements

One of them (G.S.P.) thanks the Organizing Committee for the invitation to the Barut Memorial Conferenc and for warm hospitality.

References

[1] E. Schrödinger, Proc. of the Royal Irish.Acad. **A46**, 93, 183, (1940).

[2] Y.A. Kurochkin, V.S.Otchik, DAN BSSR, **XXIII**, 987, (1979).

[3] P.W. Higgs *J. Phys.*, **A12**, 309, (1979).

[4] H.I. Leemon. *J. Phys.*, **A12**, 489, (1979).

[5] A.A. Bogush, Y.A. Kurochkin, V.S.Otchik DAN BSSR, **XXIV**, 19, (1980).

[6] A.A. Bogush, V.S. Θtchik, V.MbRed'kov, Vestnik AN BSSR, **3**, 56, (1983).

[7] V.S. Otchik, DAN BSSR, **XXV**, 420, (1991).

[8] V.S. Otchik, Proceedings "Symmetry Methods in Physics" Eds. A.N. Sissakyan, G.S. Pogosyan and S.I.Vinitsky, (1994).

[9] A.O. Barut and R. Wilson, *Phys.Lett.*, **A110**, 351, (1985).

[10] A.O. Barut, A. Inomata and G. Junker, J.Phys., **A20**, 6271, 1987; **A23**, 1179, (1990).

[11] C. Grosche, *Ann. Phys.*, (N.Y.), **204**, 208, (1990).

[12] C. Grosche, G.S. Pogosyan and A.N. Sissakian, Fortschritte der Physik, **43(6)**, 523, 1995.

[13] C. Grosche, G.S. Pogosyan and A.N. Sissakian, DESY Report DESY 95 - 181, September, 1995;

[14] C. Grosche, Kh.G. Karayan, G.S. Pogosyan and A.N. Sissakian, DESY Report DESY 95 -218, November 1995.

[15] Ya.A. Granovsky, A.S. Zedanov, I.M. Luzenko, Teor.Mat.Fiz. **91**, 207, 396, (1992).

[16] M.P. Olevskii, Math. Sb. **27**, 379-426, (1950).

[17] A.A. Izmest'ev, G.S. Pogosyan, A.N. Sissakian and P. Winternitz, Preprint CRM-2327, Montreal, December 1995; Preprint JINR E2-96-39, Dubna, 1996.

[18] S.I. Vinitskii, L.G. Mardoyan, G.S. Pogosyan, A.H. Sissakian, and Strizh. Phys. At. Nucl. **56** (3), 321-327, (1993).

[19] L.D. Landau, E.M. Lifshitz, *Quantum Mechanics* (Pergamon Press, Oxford, 1977).

[20] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. Tricomi, *Higher Transcendental Functions* (McGraw-Hill, New York, 1953), Vols. I and II.

[21] D.A. Varshalovich, A.N. Moskalev, and V.K. Khersonskii, *Quantum Theory of Angular Momentum*(World Scientific, Singapore, 1988).

[22] E.G. Kalnins, W. Miller, Jr. and P. Winternitz, *SIAM J.Appl. Math.*, **30**, 630, (1976).